Stepping into high-order interpolation in space

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Abstract

We review some concepts and basic features of high-order polynomial interpolation of fields in space. The interpolation, from weights, of Lagrange and Hermite types for physical fields, intended as differential forms, is analyzed on an interval.

Keywords: Lagrange and Hermite interpolation, Lebesgue constant, differential forms

MSC Classification: 65D05

1 Introduction

Polynomial interpolation is a key aspect in numerical analysis, used in very classical settings such as reconstructing a physical field from measures or replacing a field expression by a simpler one, defining quadrature formulas to compute integrals. It consists in relying on a discrete set of known values (field measures) to construct a polynomial which allows then to obtain new data (new measures) for intermediate positions of the independent variables or probes in a domain. Mathematical and numerical research in polynomial interpolation has been in continuous evolution since the 17th century. Some key publications, to explore different directions, are, e.g., [1], [2], [3], [4], [5], [6], [7], and the multitude of references therein. In this short note, we step into the one dimensional framework and elucidate the relation between the discrete representation of the spatial domain (an interval $I \subset \mathbb{R}$) and the physical field we wish to reconstruct (intended as differential k-form, $k = 0$ or $k = 1$). The choice of the degrees of freedom (or type of probes) we rely on, and resulting in measures at points or sub-intervals, is such that (i) they are physically meaningful for the considered field, (ii) they are supported by the selected discrete representation (points or sub-intervals) of the interval, and (iii) they reflect the global regularity properties we wish to enforce on the reconstructed field ($\mathcal{C}^0(I)$ or $\mathcal{C}^1(I)$). We work in an interval I for simplicity and because most of the basic machinery of polynomial interpolation can be described in one dimension of space.

We start in Section 2 by presenting the sampling, namely the measures we interpolate and their link with the field under consideration. We then consider, in Section 3, 0-forms and how to interpolate them by a global polynomial, either with low regularity (Lagrange type) or with higher regularity (Hermite type), starting from weights at points in I. We present, in Section 4, similar tools to interpolate 1-forms by a global polynomial, starting from averages on sub-intervals in I. This note ends with some concluding remarks. The Hermite type polynomial interpolation of 1-forms here presented is new and its complete validation is in progress.

2 The sampling

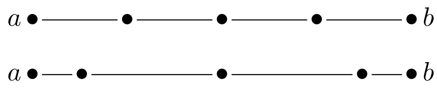
In engineering and science, one often has a number N_r of real data $\{\alpha_i\}$, obtained by sampling or experimentation, which refer to a physical field f for a limited number of values $\{s_i\}$ of the independent geometrical variable. We thus have a set of real-valued linear functionals μ_i, called degrees of freedom, acting on f with the property of determining uniquely an approximation f_δ of f in a finite dimensional space $\mathcal{P}$, once the values $\mu_i(f) = \langle f, s_i\rangle$ are known (we say that the set $\{\mu_i\}$ is unisolvent for the space $\mathcal{P}$). Here, $\langle f, s_i\rangle$ is a real number, giving the measure or result of the experimentation for the value s_i of the variable or probe on the field f. The data $\{\alpha_i\}$ are hence measures $\langle f, s_i\rangle$ for a selected set of geometrical supports s_i. Interpolation stands for generating a new value, say α^*, to approximate the measure of the field f at an intermediate value, say s^*, of the variable, by using a reconstruction f_δ of f, i.e., $\alpha^* = \langle f_\delta, s^*\rangle \approx \langle f, s^*\rangle$. In polynomial interpolation, the field f_δ is a polynomial, expressed on a selected basis $\{\varphi_i\}$, i.e., $f_\delta = \sum_i \beta_i \varphi_i$ with the real coefficients β_i determined in order to have $\langle f_\delta, s_i\rangle = \alpha_i$. We have $\beta_i = \alpha_i$, for any i, only if the polynomials φ_i are the cardinal functions, that is, if they verify the condition $\mu_\ell(\varphi_i) = \langle \varphi_i, s_\ell\rangle = \delta_{\ell,i}$, with $\delta_{\cdot,\cdot}$ the Kronecker symbol.

Note that the sampling type has to be physically meaningful for the field f, namely we rely on weights that are values at nodes if f is a 0-form, such as for example, a scalar potential, or circulations along lines if f is a 1-form, such as an electric field, or fluxes through surfaces if f is a 2-form, such as a surface flow rate (see the approach to forms described in [5], [8]). In one dimension, we involve linear functionals associated with points x and segments e with the following meaning:

$$\begin{array}{lll} \mu_x : f \longrightarrow \langle f, x\rangle = f(x)\,, & 0\text{-form}\, f\,, & 0\text{-simplex}\ x \in I, \\ \mu_e : w \longrightarrow \langle w, e\rangle = \int_e w\,, & 1\text{-form}\, w\,, & 1\text{-simplex}\ e \subset I. \end{array}$$

When we use values $\mu_s(z)$ where $\dim(s)$ is equal to the order k of the form z and which involves only z, we say that we consider the polynomial interpolation of Lagrange-type. If we introduce more information at each probe s, namely if we consider not only the value $\mu_s(z)$ but also $\mu_s(z')$, with $z' = \partial_x z$, the first derivative of z (in a sense that will be precised when dealing with 1-forms), then we say that we consider a polynomial interpolation of Hermite type.

The discrete realization of the interval I, as a finite set of nodes or edges, depends on whether the values α_i are associated with geometrical objects s_i of dimension $k = 0$ or $k = 1$. In both nodal or edge cases, the distribution of the samplings (probes) can be uniform, as in the case of equally-spaced points and corresponding edge decomposition of I, or not (see below for a simple visualization in $I = [a, b]$).



(uniform sampling)

(non uniform sampling)

We go through some known concepts about the polynomial interpolation of scalar values over an interval $I \subset \mathbb{R}$. We thus describe how the field nature as k-form, $0 \le k \le 1$, influences the interpolating process in a 1-dimensional domain I.

3 Polynomial interpolation of 0-forms in $\mathbb{R}$

We denote by $\mathbb{P}_r(I)$ the space of polynomials in $n = 1$ variable of degree at most r and set $N_r = \dim(\mathbb{P}_r(I)) = r + 1$. We introduce the set of integers $\mathcal{S} = \{1, ..., N_r\}$. Hereon, $\mathcal{C}^0(I)$ is the space of continuous scalar functions in I, and $\mathcal{C}^1(I)$ the subspace of functions with continuous (standard) first derivative in I.

3.1 The Lagrange case

Let $\{\mu_i\}$ be the set of N_r linear functionals that associate with a scalar function f (a 0-form in I) its values at N_r distinct points $\{x_i\}$ in I. Given $\{y_i\}$, a finite set of real values, we wish to construct a polynomial function, denoted by $\Pi_r f$, that interpolates the $\{y_i\}$ at the $\{x_i\}$. Therefore, we look for $\Pi_r f \in \mathbb{P}_r(I)$ such that $\mu_i(\Pi_r f) = y_i = \mu_i(f)$, with $x_i \neq x_j$ for $i \neq j$, with $i, j \in \mathcal{S}$.

Proposition 1. *Given $N_r = r + 1$ reals $\{y_i\}$, there exists a unique polynomial $\Pi_r f \in \mathbb{P}_r(I)$ that interpolates $\{y_i\}$ at the $\{x_i\}$, i.e., $\Pi_r f(x_i) = y_i$, $\forall\, i \in \mathcal{S}$.*

Proof of Proposition 1. Existence is proved by construction, namely by checking that $\Pi_r f(x) = \sum_{i=1}^{N_r} y_i\, \varphi_i(x)$ verifies the interpolation conditions. Here,

$$\varphi_i(x) = \prod_{j \in \mathcal{S} \setminus \{i\}} \frac{x - x_j}{x_i - x_j}\,,$$

and $\{\varphi_i\}$ is the basis of $\mathbb{P}_r(I)$ of Lagrangian polynomials associated with the x_i. Uniqueness is proved by the fact that, if there were two of such polynomials, say P and Q, their difference $(P - Q)$ would be a polynomial of degree $\le r$ with $N_r = r + 1$ zeros in I, so it would be necessarily zero on I. □

Thank to Proposition 1, we have a reconstruction operator $p_r^0 : \mathbb{R}^{N_r} \to \mathbb{P}_r(I)$ which associates, with a vector $\mathbf{y} \in \mathbb{R}^{N_r}$, a unique polynomial $p_r^0(\mathbf{y}) = \Pi_r f$ in $\mathbb{P}_r(I)$.

We note that $\{\varphi_k\}$ is the basis of $\mathbb{P}_r(I)$ in duality with the evaluations at the $\{x_k\}$, namely $\mu_j(\varphi_k) = \varphi_k(x_j) = \delta_{j,k}$, where $\delta_{\cdot,\cdot}$ is the Kronecker symbol. To compute the function φ_k with a general technique, we proceed as follows: (i) choose a basis $\{\psi_\ell\}$ in $\mathbb{P}_r(I)$; (ii) build up the generalised Vandermonde matrix V of size N_r with entries $(V)_{j,\ell} = \mu_j(\psi_\ell)$; (iii) write $\varphi_k(x) = \sum_{\ell=1}^{N_r} c_\ell^k\, \psi_\ell(x)$; (iv) find the vector $\mathbf{c}^k$ of coefficient c_ℓ^k by solving $V\,\mathbf{c}^k = \mathbf{e}_k$, with $\mathbf{e}_k$ the k-th column of the identity matrix defined in $\mathbb{R}^{N_r}$.

If we look behind, we see that a discretization procedure has occurred when we have replaced (1) the interval I by a finite set of points $\{x_i\}$ (this is a geometrical step), (2) the 0-form f by a linear combination of the polynomials 0-forms $\{\psi_\ell\}$ (this is a functional step). Aspects (1) and (2) are merged when we compute the entries of the matrix V, namely $\psi_\ell(x_j)$. Moreover, the way we build up the entries of V (by evaluating the ψ_ℓ at the x_j) reflects the nature of the interpolated data $\{y_i\}$ considered here, namely evaluations of f at the $\{x_i\}$ (and this comes from physics). So, the kind of physical field f determines the type of data $\{y_i\}$, in the sense that the data have to be physically meaningful for f. Then, the type of data determines the way we represent the physical domain at the discrete level.

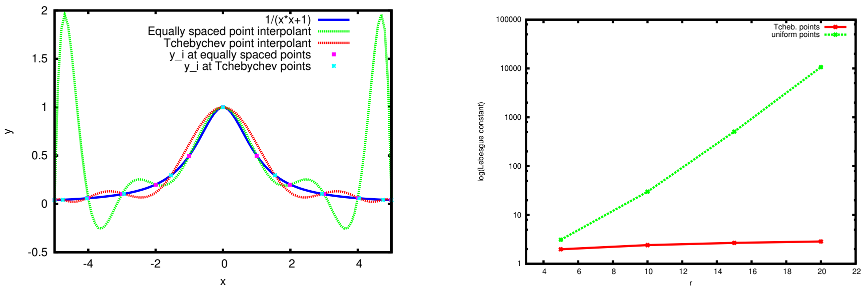


Fig. 1 Left: The function named "Witch of Agnesi" $f(x) = 1/(1+x^2)$ over $I = [-5, 5]$ introduced by Maria Gaetana Agnesi (in blue) and the polynomial function $\Pi_{10}f$ interpolating f at a uniform distribution of points $\{x_i\}$ in I (in green). Right: The Lebesgue constant (shown in log-scale with respect to the polynomial degree r) behaves as $c\,e^r$ if the $\{x_i\}$ are equally spaced points in I and as $c\,\ln(r)$ if not, as it occurs with Tchebychev point distribution, respectively.

Up to now, we have not indicated how to select neither the basis $\{\psi_\ell\}$ nor the set of points $\{x_i\}$. This is a numerical step which may have drawbacks on the quality of the interpolation if things are not done with care. These drawbacks are more visible when the polynomial degree r is high, say $r > 4$. When we solve the system with matrix V for the construction of the cardinal basis (step (iv)), the condition number cond(V) of V matters and depends on the basis $\{\psi_\ell\}$. Numerical results show that it is better to use bases composed by orthogonal polynomials, such as Legendre or Tchebychev polynomials, rather than the classical canonical basis where $\psi_\ell = x^{\ell-1}$. On the other hand, the Weierstrass approximation theorem states that every continuous function defined on a closed interval I can be approximated as closely as desired by a polynomial function. However, as shown by Runge in 1901, the approximation of the "versiera di Agnesi" f by $\Pi_r f$ may give bad results, namely $\lim_{r\to+\infty} ||f - \Pi_r f|| \neq 0$, with $||.||$ the sup-norm on $\mathcal{C}^0(I)$. Wild oscillations, known as Runge's phenomenon, in the interpolating polynomial $\Pi_r f$ appear at the extremity of the interval (see, for $r = 10$, Fig. 1, left), when $\Pi_r f$ is constructed over a uniform distribution of points in I. Taking non-uniform distributions of points, as for example the zeroes of the Tchebychev or Legendre polynomials in I, things improve. It is thus natural to ask whether there is a quantity that indicates how to select the interpolating grid $\{x_i\}$. Indeed, it turns out that the distribution of points $\{x_i\}$ (the probes) has to be selected in order to minimize (as much as possible) the Lebesgue's constant Λ_r. This constant appears to bound the norm $||\Pi_r||$ of the interpolating operator Π_r in the interpolation error estimation as follows.

Proposition 2. *There exists a constant* Λ_r *such that* $||f-\Pi_r f|| \leq (1+\Lambda_r)\, ||f-f_r^*||$ *where* $||g|| = \sup_{x\in I} |g(x)|$ *and* $||f-f_r^*|| = \inf_{g\in\mathbb{P}_r(I)} ||f-g||$*, where* f_r^* *denotes the best-fit polynomial of degree* r *for* f *with respect to* $||.||$*.*

Proof of Proposition 2. Since

$$||\Pi_r|| = \sup_{||g||=1} ||\Pi_r g|| = \sup_{||g||=1} \max_{x\in I} |\sum_i g(x_i)\,\varphi_i(x)| \leq \max_{x\in I} \sum_i |\varphi_i(x)| = \Lambda_r,$$

we can write $||f-\Pi_r f|| = ||f - f_r^* + f_r^* - \Pi_r f|| = ||f - f_r^* + \Pi_r f^* - \Pi_r f||$, hence

$$\begin{aligned} ||f-\Pi_r f|| &\leq ||f-f_r^*|| + ||\Pi_r(f-f_r^*)|| \\ &\leq (1+||\Pi_r||)\,||f-f_r^*|| \quad \leq (1+\Lambda_r)\,||f-f_r^*||, \end{aligned}$$

which is the desired estimation. □

If Λ_r grows faster in r than the error $||f-f_r^*||$ dies away, convergence for $||f-\Pi_r f||$ is impossible to attain when $r \to +\infty$. The inequality in Proposition 2 suggests that, when the Lebesgue constant does not grow too fast, we can find an approximation of a function f on the interval I, that is almost as good as the best-fit polynomial f_r^*, by just taking $\Pi_r f$, which is generally much easier to compute than f_r^*. If we look behind once again, we see that the expression of Λ_r involves evaluations (same kind of data as the y_i) over the whole domain I of the cardinal functions φ_i forming the basis of $\mathbb{P}_r(I)$.

To ensure the stability of the interpolating problem when increasing the polynomial degree r, it is crucial to keep under control the growth of Λ_r. Indeed, the role of condition number for the interpolation problem is played by Λ_r as follows. If $\{\tilde{y}_i\}$ are perturbations of the data $\{y_i\}$ with $\max_i |y_i-\tilde{y}_i| \leq \epsilon$, then $||\Pi_r f - \Pi_r \tilde{f}|| \leq \epsilon\,\Lambda_r$, where $\Pi_r f$ (resp., $\Pi_r \tilde{f}$) interpolates the $\{y_i\}$ (resp. $\{\tilde{y}_i\}$). In fact, let $\Pi_r f(x) = \sum_i y_i\,\varphi_i(x)$ and $\Pi_r \tilde{f}(x) = \sum_i \tilde{y}_i\,\varphi_i(x)$, then

$$\begin{aligned} ||\Pi_r f - \Pi_r \tilde{f}|| &= \max_{x\in I} |\textstyle\sum_i (y_i-\tilde{y}_i)\,\varphi_i(x)| \\ &\leq (\max_i |y_i-\tilde{y}_i|)\,(\max_{x\in I} \textstyle\sum_i |\varphi_i(x)|) \;\leq \epsilon\,\Lambda_r. \end{aligned}$$

Hence, small changes on y_i yield small changes on $\Pi_r f$ only if Λ_r is small.

To compute an approximated value Λ_h of $\Lambda_r = \max_{x\in I} \sum_{i=1}^{N_r} |\varphi_i(x)\,|$ with a general technique we can proceed as follows: (1) replace the interval I by $S = \{z_q\}$, a finite set of points $z_q \in I$ with $\text{card}(S) \gg N_r$ (note that $\{z_q\}$ is a discrete representation of I of nature similar to $\{x_i\}$), (2) compute $\Lambda_h = \max_{z_q \in S} \sum_{i=1}^{N_r} |\varphi_i(z_q)|$ (note that $\Lambda_h = 1$ if $S \equiv \{x_i\}$). Numerical results show that Λ_h does not depend on the basis $\{\psi_\ell\}$ previously selected to have small values of $\text{cond}(V)$, but depends heavily on the distribution of points x_i in I. Behaviors shown in Fig. 1, right, are in agreement with the fact that $\Lambda_r \sim \frac{2^{r+1}}{e\, r\, \ln r}$ when equally spaced points are used, whereas $\Lambda_r \sim \ln(r+1)$ with non-uniformly distributed points as the Tchebychev ones.

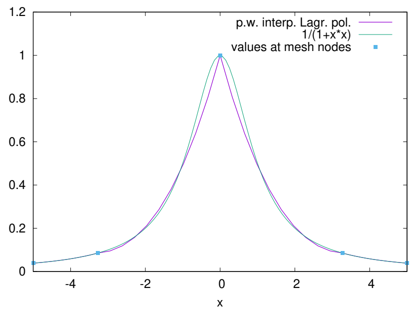


Fig. 2 The piece-wise Lagrange polynomial of degree $r = 3$ over $I = [-5, 5]$, interpolating $f(x) = 1/(1 + x^2)$ at equally-spaced nodes inside each sub-interval. The sub-interval extremities coincide with the $N_r = 5$ nodes of Tchebychev in I.

Remark 1. *The quality of the interpolation improves if we decompose the interval I into non-overlapping sub-intervals of maximal diameter h and, in each sub-interval, we interpolate a smaller number of data by a polynomial of low degree (this is what is meant by the piece-wise interpolation). In Fig. 2, the global interval $I = [-5, 5]$ is divided into $N_a = 4$ sub-intervals $I_\ell = [x_\ell, x_{\ell+1}]$, with $x_1 = -5$, $x_5 = 5$ and $x_{j+1} = 5\cos((2j-1)\pi/6)$, $j = 1, 2, 3$, the Tchebychev nodes in I; each I_ℓ contains $m = 3$ segments, separated by $(m-1)$ points $\diamond$. The point sampling is schematized below:*

$$\overset{x_1}{\bullet} \;\diamond\; \diamond\; \overset{x_2}{\bullet} \quad \diamond \quad \diamond \quad \overset{x_3}{\bullet} \quad \diamond \quad \diamond \quad \overset{x_4}{\bullet} \;\diamond\; \diamond\; \overset{x_5}{\bullet}$$

Small groups of data (i.e., blocs of 4 at the points $\bullet \diamond \diamond \bullet$, in the example above) are interpolated in each sub-interval I_ℓ. By gluing together the polynomials $(f_h)_{|I_\ell} = \Pi_m(f_{|I_\ell})$, we get a global representation f_h of f in I. In the Lagrange case, without additional conditions at the points x_i, the function f_h is $\mathcal{C}^0(I)$ at most (as in Fig. 2).

For an extension of all the previous concepts around the nodal polynomial interpolation of a scalar function in a domain $\Omega \in \mathbb{R}^n$, $n > 1$, we refer, for example, to [9], [10], [7], and the references therein.

3.2 The Hermite case

Polynomial interpolation can be generalized to include the knowledge of the first derivatives of a function f at the nodes $\{x_i\}$. Let $\{\mu_i\}$ and $\{\nu_i\}$ be two sets of N_r linear functionals that associate with a scalar function f (a 0-form in I) its values and the values of its first derivative $f' = \partial_x f$ at N_r distinct points $\{x_i\}$ in I. Hence we have

$$\mu_i : f \longmapsto \langle f, x_i \rangle = f(x_i)\,, \qquad \nu_i : f \longmapsto \langle f', x_i \rangle = f'(x_i)\,, \qquad i = 1, ..., N_r. \quad (1)$$

Let $\{y_i\}$ and $\{d_i\}$ be two sets of N_r real values at distinct points $\{x_i\}$ in I. So, at each point x_i, we have two values, and globally we have $2\,N_r$ data. We wish to define a polynomial $H_{2r+1}f \in \mathbb{P}_{2r+1}(I)$ verifying the interpolation conditions $(H_{2r+1}f)(x_i) = y_i$ and $(H_{2r+1}f)'(x_i) = d_i$, with $x_i \neq x_j$ for $i \neq j$, with $i, j \in \mathcal{S}$. To this purpose, we introduce, respectively, the polynomials:

$$b_{0,i}(x) = (1 - 2\,\varphi_i'(x_i)(x - x_i))\,\varphi_i^2(x), \qquad b_{1,i}(x) = (x - x_i)\,\varphi_i^2(x),$$

where $\{\varphi_i\}$ is the basis of $\mathbb{P}_r(I)$ verifying $\varphi_k(x_i) = \delta_{i,k}$ already involved to define the Lagrange interpolation polynomial $\Pi_r f$. The polynomials $b_{0,i}$ and $b_{1,i}$ verify, respectively, the conditions $\mu_i(b_{0,k}) = \delta_{i,k}$, $\nu_i(b_{0,k}) = 0$, and $\mu_i(b_{1,k}) = 0$, $\nu_i(b_{1,k}) = \delta_{i,k}$. Then, the polynomial $(H_{2r+1}f)(x) = \sum_{i=1}^{N_r} y_i\, b_{0,i}(x) + \sum_{i=1}^{N_r} d_i\, b_{1,i}(x)$ belongs to $\mathbb{P}_{2r+1}(I)$ and verifies the interpolation conditions for the values y_i and d_i.

Proposition 3. *Given two sets $\{y_i\}$, $\{d_i\}$, of N_r real values, there exists a unique polynomial $H_{2r+1}f \in \mathbb{P}_{2r+1}(I)$ such that $(H_{2r+1}f)(x_i) = y_i$ and $(H_{2r+1}f)'(x_i) = d_i$, $\forall\, i \in \mathcal{S}$.*

Proof of Proposition 3. For all integers $i \in \mathcal{S}$, let us set

$$\phi_i(x) = \varphi_i(x)^2$$

It is easy to see that $\phi_k(x_i) = \delta_{k,i}$. If we differentiate $\phi_i(x)$ we obtain

$$\phi_i'(x) = 2\varphi_i'(x)\varphi_i(x)$$

and so we have $\phi_i'(x_k) = 0$ if $k \neq i$.

Let us introduce the set $\mathcal{H} = \{h \in \mathbb{P}_{2r+1}(I),\ h(x_i) = y_i,\ h'(x_i) = d_i,\ \forall\, i \in \mathcal{S}\}$. For the existence step, we have to prove that $\mathcal{H} \neq \emptyset$. To this purpose, we check that the polynomial $H_{2r+1}f(x) = \sum_{i=1}^{N_r} [(1 - \phi_i'(x_i)\,(x - x_i))\, y_i + (x - x_i)\, d_i]\, \phi_i(x)$ verifies the conditions to be in $\mathcal{H}$. Clearly, $H_{2r+1}f \in \mathbb{P}_{2r+1}(I)$. Moreover, for all $k \in \mathcal{S}$, we have

$$\begin{aligned} H_{2r+1}f(x_k) &= \textstyle\sum_{i=1}^{N_r} [(1 - \phi_i'(x_i)\,(x_k - x_i))\, y_i + (x_k - x_i)\, d_i]\, \phi_i(x_k) \\ &\overset{i=k}{=} [(1 - \phi_k'(x_k)\,(x_k - x_k))\, y_k + (x_k - x_k)\, d_k] = y_k. \end{aligned}$$

If we differentiate $H_{2r+1}f(x)$ we have

$$(H_{2r+1}f)'(x) = \sum_{i=1}^{N_r} [(1 - \phi_i'(x_i)\,(x - x_i))\, y_i + (x - x_i)\, d_i]\, \phi_i'(x) + \sum_{i=1}^{N_r} [-\phi_i'(x_i)\, y_i + d_i]\, \phi_i(x).$$

For all $k \in \mathcal{S}$, we hence obtain

$$\begin{aligned}(H_{2r+1}f)'(x_k) &= \textstyle\sum_{i=1}^{N_r}[(1-\phi_i'(x_i)\,(x_k-x_i))\,y_i + (x_k-x_i)\,d_i]\,\phi_i'(x_k)\\ &\quad + \textstyle\sum_{i=1}^{N_r}[-\phi_i'(x_i)\,y_i +\, d_i]\,\phi_i(x_k)\\ &\overset{i=k}{=} [(1-\phi_k'(x_k)\,(x_k-x_k))\,y_k + (x_k-x_k)\,d_k]\,\phi_k'(x_k) + [-\phi_k'(x_k)\,y_k +\, d_k]\\ &= y_k\,\phi_k'(x_k) - \phi_k'(x_k)\,y_k + d_k = d_k.\end{aligned}$$

So existence holds true. We now turn to uniqueness. Let us assume that there exists another polynomial $Q \in \mathcal{H}$. Therefore, $Q \in \mathbb{P}_{2r+1}(I)$ and

$$Q(x_i) = y_i = (H_{2r+1}f)(x_i) \quad \text{ and } \quad Q'(x_i) = d_i = (H_{2r+1}f)'(x_i), \qquad \forall\, i \in \mathcal{S}$$

so, the polynomial $(Q - H_{2r+1}f)$ has roots x_i of multiplicity 2, since $(Q-H_{2r+1}f)(x_i) = 0$ and $(Q-H_{2r+1}f)'(x_i) = 0$, $\forall\, i \in \mathcal{S}$. Hence $W = \prod_{i\in\mathcal{S}}(x-x_i)^2$ divides $(Q-H_{2r+1}f)$, because any root x_i gives a linear factor $(x-x_i)$ of $(Q-H_{2r+1}f)$ and all the roots x_i have multiplicity 2. To sum up, we have $Q \in \mathcal{H}$ iff $(Q-H_{2r+1}f) = R\,W$ for some R. As $\deg(W) = 2r+2$, then $\deg(Q-H_{2r+1}f) \geq 2r+2$ if $R \neq 0$, that is a contradiction, being Q and $H_{2r+1}f$ both in $\mathbb{P}_{2r+1}(I)$. Therefore, $R = 0$ and uniqueness holds true. □

In the Lagrange case, we consider the value of a 0-form f at N_r points x_i. In the Hermite case, we also evaluate a differential operator of first order on f, here f', at the same points x_i. Thank to Proposition 3, we have a reconstruction operator $q^0_{2r+1} : \mathbb{R}^{2\,N_r} \to \mathbb{P}_{2\,r+1}(I)$ which associates, with $(\mathbf{y},\mathbf{d}) \in \mathbb{R}^{2\,N_r}$, a unique polynomial in $\mathbb{P}_{2\,r+1}(I)$ that is $q^0_r(\,(\mathbf{y},\mathbf{d})\,) = H_{2r+1}f$.

To compute the cardinal functions $\varphi^H_{2i-1} = b_{0,i}$, $\varphi^H_{2i} = b_{1,i}$ with a general technique, we proceed as follows: (i) choose a basis $\{\psi_\ell\}$ in $\mathbb{P}_{2r+1}(I)$; (ii) build up the generalised Vandermonde matrix V of size $2N_r$ with entries $(V)_{2j-1,\ell} = \psi_\ell(x_j)$ and $(V)_{2j,\ell} = \psi_\ell'(x_j)$; (iii) write $\varphi^H_k(x) = \sum_{\ell=1}^{2N_r} c^k_\ell\,\psi_\ell(x)$; (iv) find the vector $\mathbf{c}^k$ of coefficient c^k_ℓ by solving $V\,\mathbf{c}^k = \mathbf{e}_k$, with $\mathbf{e}_k$ the k-th column of the identity matrix defined in $\mathbb{R}^{2N_r}$.

Can we still speak of Lebesgue's constant, of Runge's phenomenon ? The answer is affirmative, and we comment on this. We recall that on $\mathcal{C}^0(I)$, the usual norm is $||f|| = \sup_{x\in I}|f(x)|$ and on $\mathcal{C}^1(I)$ it is, for example, $||f||_1 = \max(||f||, ||f'||)$ (and $\mathcal{C}^1(I)$ is a Banach space with respect to $||.||_1$). The norm to use on f changes depending on the space f belongs to (and this is an analytic step). To make the difference with the Lagrange interpolator, we note that

$$\begin{aligned} \Pi_r &: \mathcal{C}^0(I) \to \mathcal{C}^0(I), & \Pi_r f &= \textstyle\sum_{i\in\mathcal{S}} f(x_i)\,\varphi_i(x),\\ K_{2r+1} &: \mathcal{C}^0(I) \to \mathcal{C}^0(I), & K_{2r+1} f &= \textstyle\sum_{i\in\mathcal{S}} f(x_i)\,b_{0,i}(x),\\ H_{2r+1} &: \mathcal{C}^1(I) \to \mathcal{C}^1(I), & H_{2r+1} f &= \textstyle\sum_{i\in\mathcal{S}} f(x_i)\,b_{0,i}(x) + \sum_{i\in\mathcal{S}} f'(x_i)\,b_{1,i}(x).\end{aligned} \tag{2}$$

We have seen that $||\Pi_r|| = \sup_{||g||=1}||\Pi_r g|| \leq \Lambda_r$ and now it can be proved the following.

Proposition 4. *Let $I = [a, b]$ and $q^*_{2r} \in \mathbb{P}_{2r}(I)$ be the best-fit polynomial for f' with respect to $||.||$. There exist reals $\Upsilon^K_r, \Upsilon^H_r$ such that*

$$||f - H_{2r+1}f|| \leq [(b-a)\,(1+\Upsilon^K_r) + \Upsilon^H_r]\,||f' - q^*_{2r}||.$$

Proof of Proposition 4. The condition $||g|| = 1$ yields $|g(x)| \leq 1$, for all $x \in I$. So

$$||K_{2r+1}|| = \sup_{||g||=1} ||K_{2r+1}g|| \leq \sup_{x\in I} \sum_{i=1}^{N_r} |b_{0,i}(x)| := \Upsilon^K_r,$$

and if we set $\Upsilon^H_r := \sup_{x\in I} \sum_{i=1}^{N_r} |b_{1,i}(x)|$, then we have that

$$\begin{aligned}
||f - H_{2r+1}f|| &= ||f - p^*_{2r+1} + p^*_{2r+1} - H_{2r+1}f|| \\
&= ||f - p^*_{2r+1} - H_{2r+1}(f - p^*_{2r+1})|| \\
&= ||f - p^*_{2r+1} - (K_{2r+1} + H_{2r+1} - K_{2r+1})(f - p^*_{2r+1})|| \\
&\leq ||f - p^*_{2r+1} - (K_{2r+1})(f - p^*_{2r+1})|| + ||(H_{2r+1} - K_{2r+1})(f - p^*_{2r+1})|| \\
&\leq (1 + ||K_{2r+1}||)\,||f - p^*_{2r+1}|| + ||(H_{2r+1} - K_{2r+1})(f - p^*_{2r+1})|| \\
&\leq (1 + \Upsilon^K_r)\,||f - p^*_{2r+1}|| + ||(H_{2r+1} - K_{2r+1})(f - p^*_{2r+1})||
\end{aligned}$$

where p^*_{2r+1} is the primitive of q^*_{2r} that is equal to $f(a)$ at $x = a$. Since $p^*_{2r+1} \in \mathbb{P}_{2r+1}(I)$, we have that $H_{2r+1}(p^*_{2r+1}) = p^*_{2r+1}$ by the uniqueness property of Proposition 3. We apply the mean value theorem on I to the function $g = (f - p^*_{2r+1})$ to bound $||g||$ in terms of $||g'||$. We can write

$$|g(x)| = |g(x) - 0| = |g(x) - g(a)| \leq |g'(c)|\,|x - a| \leq ||g'||\,(b-a) \qquad \forall\, x \in I,$$

for $c \in]x, x_i[$ or $]x_i, x[$, and this yields $||g|| \leq ||g'||\,(b-a)$ by considering $\sup_{x\in I} |g(x)|$. Moreover, we remark that

$$|(H_{2r+1} - K_{2r+1})(f - p^*_{2r+1})(x)| \leq \sum_{i=1}^{N_r} |f'(x_i) - (p^*_{2r+1})'(x_i)|\,|b_{1,i}(x)|, \qquad \forall\, x \in I\,.$$

Thus, $||(H_{2r+1} - K_{2r+1})(f - p^*_{2r+1})|| \leq ||f' - (p^*_{2r+1})'||\,\Upsilon^H_r$. Hence,

$$||f - H_{2r+1}f|| \leq (1 + \Upsilon^K_r)\,(b-a)\,||f' - q^*_{2r}|| \,+\, \Upsilon^H_r\,||f' - q^*_{2r}||,$$

and this ends the proof. □

Proposition 5. *It holds that $\Upsilon^H_r \leq ||H_{2r+1}|| \leq 2\max(\Upsilon^K_r, \Upsilon^H_r)$, where the norm $||H_{2r+1}||$ is defined as $||H_{2r+1}|| = \sup_{||g||_1=1} ||H_{2r+1}g||$.*

Proof of Proposition 5. We have $||H_{2r+1}g|| = ||\sum_i g(x_i)\,b_{0,i}(x) + \sum_i g'(x_i)\,b_{1,i}(x)||$, hence for the upper bound,

$$||H_{2r+1}g|| \;\leq ||g||\,\Upsilon^K_r + ||g'||\,\Upsilon^H_r \;\leq 2\,||g||_1\,\max(\Upsilon^K_r, \Upsilon^H_r).$$

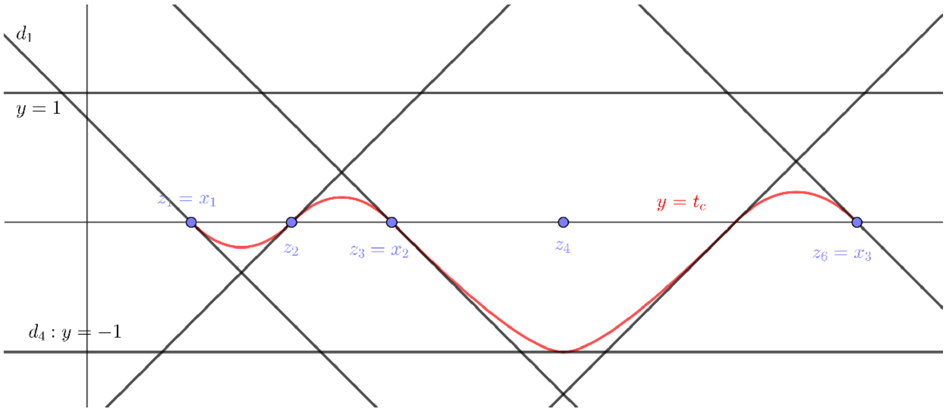


Fig. 3 The construction of the function t_c in the interval I. Points x_i are the micro-mesh in I. we extend this family to a family z_i by adding first some points z_i in order to have $\varepsilon_i = \pm 1$, alternatively (for instance we have added z_2 since we had $\varepsilon_{c,1} = -1 = \varepsilon_{c,2}$ initially), then other points z_j are added between two consecutive z_i when two consecutive lines $d_i; y = \varepsilon_i(x - z_i)$ intersect each other, at one point (x_p, y_p) with $y_p > 1$ (for instance the point z_4 here).

We now prove the lower bound. For any $c \in I$ fixed, we first construct a $\mathcal{C}^1$ function t_c on I with $\|t_c\|_1 = 1$ and such that for any i we have $t_c(x_i) = 0$ and $t'_c(x_i) = \varepsilon_{c,i} \in \{-1, 1\}$ where $\varepsilon_{c,i} = 1$ if $b_{0,i}(c) \geq 0$ and $\varepsilon_{c,i} = -1$ otherwise: start with the family $F = \big((z_i, \varepsilon_i)\big)$ of points of $I \times \{\pm 1\}$ where $z_i = x_i$ and $\varepsilon_i = \varepsilon_{c,i}$ for any i. Up to adding an element $(z = \frac{x_i + x_{i+1}}{2}, \varepsilon = -\varepsilon_{c,i})$ to F each times we have $\varepsilon_i = \varepsilon_{i+1}$ and renumbering the elements of F to keep the first components increasing, we can assume that the family F satisfies $\varepsilon_{i+1} = -\varepsilon_i$ for any i. For any i, we set d_i for the line of equation $y = \varepsilon_i(x - z_i)$ in the plane and consider the family $L = \big((z_i, d_i)\big)$. Once again, each times we have the intersection point of d_i and d_{i+1} in the plane with ordinate larger than 1 (respectively lower than -1) we add $(z = \frac{z_i + z_{i+1}}{2}, d : y = 1)$ (resp. $(z = \frac{z_i + z_{i+1}}{2}, d : y = 1)$) to L and renumber the element of L such that the z_i be still increasing. Note that we get a family L such that $d_i \cap d_{i+1}$ is in the set $|y| \leq 1$ for any i. We then get the graph of t_c by gluing conic section arcs tangent to d_i at its point of abscissa z_i and to d_{i+1} at its point of abscissa z_{i+1} (see Fig. 3). We get a function t_c that is $\mathcal{C}^1$ on I, that is concave or convex on each $[z_i, z_{i+1}]$ and whose graph is in the band $|y| \leq 1$. So we have $\|t_c\| \leq 1$ and since $t'_c(z_i) \in \{-1, 0, 1\}$ for any i, the concavity or convexity on each $[z_i, z_{i+1}]$ implies that $\|t'_c\| \leq 1$. We thus have

$$||H_{2r+1}|| \geq ||H_{2r+1}t_c|| \geq |(H_{2r+1}t_c)(c)| = |\sum_{i=1}^{N_r} b_{0,i}(c)t_c(x_i) + b_{1,i}(c)t'_c(x_i)| = \sum_{i=1}^{N_r} |b_{1,i}(c)|$$

and since it is true for any $c \in I$, we get $\Upsilon_r^H = \max_{x \in I} \sum_{i=1}^{N_r} |b_{1,i}(x)| \leq ||H_{2r+1}||$. The lower bound for $||H_{2r+1}||$ is thus proved. □

The quantities Υ_r^K and Υ_r^H play an important role in investigating the convergence behavior of $H_{2r+1}f$ to f for $r \to +\infty$. With Tchebychev nodes, the Hermite interpolation process converges since $\Upsilon_r^K = 1$ and $\Upsilon_r^H \le C \ln(r)/r$, with constant C independent from the degree r (see [11]). This favourable behaviour does not hold in another common interpolation setting, the equidistant case, where Υ_r^K and Υ_r^H both grow exponentially fast with r. In [12], [13], it is proved that $\Upsilon_r^K, \sim 2^{2r+1}/r^2$ and $\Upsilon_r^H \sim 2^{2r+1}/(r^2\sqrt{r})$, and numerical results confirm theoretical behaviors (see Fig. 4).

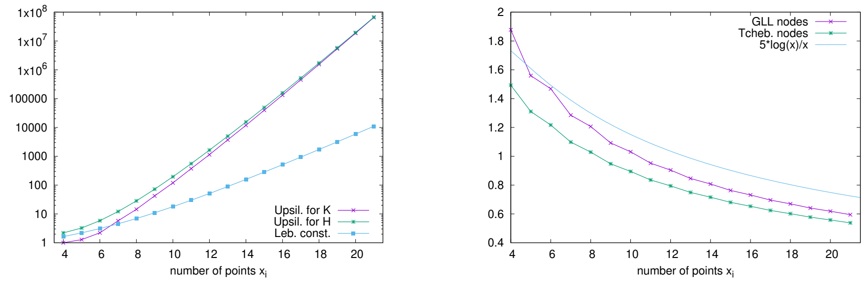


Fig. 4 Left: The behavior of the constants Λ_r, Υ_r^K, Υ_r^H for, respectively, the operators Π_r, K_{2r+1}, H_{2r+1} associated with a uniform distribution of N_r points x_i in $I = [-5, 5]$ (semi-log plot). Right: The behavior of Υ_r^H associated with the non-uniform distribution of N_r points x_i of Tchebychev and Gauss-Lobatto-Legendre in $I = [-5, 5]$.

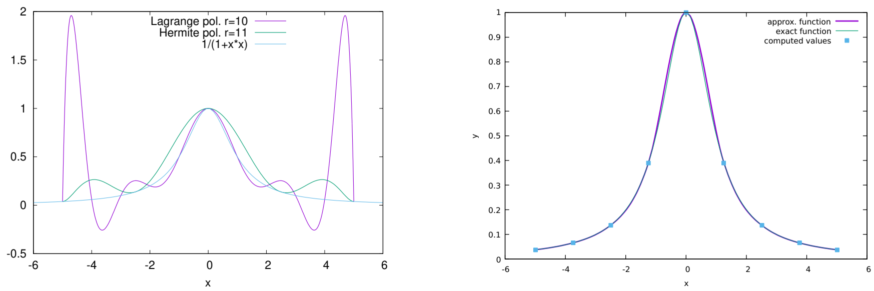


Fig. 5 Left: The function $f(x) = 1/(1+x^2)$ over $I = [-5, 5]$ (in blue), the Lagrange polynomial $\Pi_{10}f$ interpolating f at a uniform distribution of 11 points $\{x_i\}$ in I (in purple) and the Hermite polynomial $H_{11}f$ interpolating f at a uniform distribution of 5 points $\{x_i\}$ in I (in green), respectively. Right: The piece-wise Hermite polynomial of degree $m = 3$ over $I = [-5 : 5]$, interpolating $f(x) = 1/(1+x^2)$ at equally-spaced nodes inside each sub-interval. The sub-intervals have all unit length.

We have seen that non-uniform distributions have to be preferred to reduce oscillations. However, for a given degree r, the Hermite polynomial at equally-spaced points is characterised by smaller oscillations with respect to those of the Lagrange polynomial (see an example in Fig. 5, left). For the Hermite case too, the quality of the interpolation improves if we decompose the interval I into non-overlapping sub-intervals of maximal diameter h and, in each sub-interval, we interpolate a smaller number of data by a polynomial of low degree. Then, by gluing together the local polynomials we get a global representation f_h of f in I. Being I an interval of $\mathbb{R}$, all sub-intervals lie on the same straight line, and at the internal points x_i (shared by two adjacent sub-intervals), the function f_h is $\mathcal{C}^1(I)$.

4 Polynomial interpolation of 1-forms in $\mathbb{R}$

In the next two sections, we use some notation from the discrete exterior calculus and we refer to [14] for a brief ready-to-use introduction of the de Rham complex, the exterior derivative d, the Hodge operator $\star$, and the interior product $\iota_{\mathbf{e}_x}$ (being $\mathbf{e}_x$ the unit vector in the x-direction) on differential forms defined in I. It is natural to ask ourselves whether the concepts introduced in Sections 3.1 and 3.2 for 0-forms, namely scalar fields u, can be straightforwardly extended to 1-forms. In particular, what is a Hermite-type interpolation of a 1-form ? The way of interpolating 1-forms is not independent from that adopted for 0-forms, since the polynomial spaces together with the reconstruction and differential operators have to compose a commutative de Rham diagram. We thus rely on this reasoning to answer the question. Hereon, the discrete representation of I is by sub-intervals I_ℓ such that

$$I = \bigcup_{\ell=1}^{N_r-1} [x_\ell, x_{\ell+1}] = \bigcup_{\ell=1}^{N_r-1} I_\ell, \qquad \text{and} \qquad I_\ell \cap I_j \text{ contains at most one point, } \ell \neq j,$$

with $N_r = r + 1$ the number of points x_ℓ and $N_a = N_r - 1 = r$ the number of sub-intervals I_ℓ.

4.1 Interpolation in the space $\mathcal{P}_{r-1}\Lambda^1(I)$

The space $\mathcal{P}_{r-1}\Lambda^1(I)$ of polynomial 1-forms in I contains fields that read $w = f\,\mathrm{d}x$, with $f \in \mathbb{P}_{r-1}(I)$ and $\mathrm{d}x$ the elementary length. In the literature, this space is denoted by $\mathcal{P}_r^-\Lambda^1(I)$. We prefer to simplify the notation since the 'minus' superscript has effect in dimensions higher than 1 (see [15], [6] for a detailed presentation). For 1-forms $w = f\,dx$, instead of values at points, we use weights $\{a_\ell\}$ on sub-intervals $\{I_\ell\}$ in I, $\ell = 1, ..., N_a$. We hence introduce the linear functionals, named weigths (see [8], [16]), with supports the I_ℓ, defined as

$$\mu_\ell : w \longmapsto \langle w, I_\ell \rangle = \int_{x_\ell}^{x_{\ell+1}} f\,dx, \qquad \ell = 1, ..., N_a. \tag{3}$$

We construct a polynomial $\Pi^1_{r-1}w$ that interpolates the $\{a_\ell\} = \mu_\ell(w)$ on the $\{I_\ell\}$. The superscript ^{1} in $\Pi^1_{r-1}w$ is set to recall the order of forms, here 1 (but for 0-forms, we have omitted putting the superscript ^{0}).

Proposition 6. *The set $\{\mu_\ell\}$ defined in (3) is unisolvent in $\mathcal{P}_{r-1}\Lambda^1(I)$.*

Proof of Proposition 6. Let $w = f\,dx$ be a 1-form in $\mathcal{P}_{r-1}\Lambda^1(I)$, hence $f \in \mathbb{P}_{r-1}(I)$. Thank to the mean value theorem for integrals, the N_a conditions $\mu_i(w) = 0$ imply that $f(\xi_j) = 0$ at r distinct points ξ_j in I (each ξ_j in one sub-interval I_j). Thus f is identically zero on I. □

Hence, we have a reconstruction operator $p^1_{r-1} : \mathbb{R}^{N_a} \to \mathcal{P}_{r-1}\Lambda^1(I)$ which associates, with $\mathbf{v} \in \mathbb{R}^{N_a}$, a unique polynomial 1-form $p^1_{r-1}(\mathbf{v}) = \Pi^1_{r-1} f$ in $\mathcal{P}_{r-1}\Lambda^1(I)$.

It is possible to extend, to the case of 1-forms, all the concepts that have been discussed for 0-forms in Section 3.1. We recall here below the key definitions and give some references. The interpolating polynomial $\Pi^1_{r-1} w \in \mathcal{P}_{r-1}\Lambda^1(I)$ reads

$$\Pi^1_{r-1} w = \sum_{i=1}^{N_r-1} (\int_{\sigma_i} f\,dx)\,(\varphi_i\,dx), \qquad \int_{\sigma_j} \varphi_i\,dx = \delta_{i,j}$$

To compute φ_i we use a general technique, namely (i) we choose a basis $\{\psi_\ell\}$ in $\mathbb{P}_{r-1}(I)$, (ii) we build the matrix V with entries $(V)_{j,\ell} = \int_{\sigma_j} \psi_\ell\,dx$, (iii) we write $\varphi_i(x) = \sum_{\ell=1}^{N_r-1} c^i_\ell\,\psi_\ell(x)$, (iv) we find the vector $\mathbf{c}^i$ of coefficient c^k_ℓ by solving $V\,\mathbf{c}^i = \mathbf{e}_i$. Similarly to the nodal case, cond(V) is sensible to the choice of the basis $\{\psi_\ell\}$ selected to express the cardinal functions $\{\varphi_i\}$. The Runge phenomenon may occur if $\{\sigma_i\}$ is a uniform distribution of segments (sub-intervals) in I (see Fig. 3 in [17]). Moreover, estimates on the interpolation error for 1-forms are similar to those for 0-forms, but the norm changes. As explained in [18], we define the mass $|\sigma|_0$ of a 1-simplex σ as $|\sigma|_0 = \text{diam}(\sigma)$. Then, for a 1-chain $s = \sum_{j\in J} c_j\,s_j$ (namely, s is a formal linear combination of segments $s_j \subset I$ with coefficients $c_j \in \mathbb{R}$), we define the mass by $|s|_0 = \sum_{j\in J} |c_j|\,|s_j|_0$. This notion of mass allows to generalize the definition of Λ_r to the case of 1-forms (here $z = g\,dx$ and $\theta_i = \varphi_i\,dx$) as follows

$$\Lambda^1_r = \sup_{s\subset I} \sum_i |\sigma_i|_0 \,\frac{|\int_s \varphi_i\,dx|}{|s|_0} = \sum_i |\sigma_i|_0\,\|\theta_i\|_0, \qquad \text{with} \quad ||z||_0 = \sup_{s\neq 0,\, s\subset I} \frac{|\int_s g\,dx|}{|s|_0}.$$

Indeed, for 0-chain, the mass of a point x (0-simplex) can be set to $|x|_0 = 1$ and we can identify $\int_x \varphi_i dx = \varphi_i(x)$. If σ_i is an interpolation point x_i and s is a generic point x, then the definition of Λ^1_r coincides with that of the classical Lebesgue constant Λ_r recalled in Section 3.1 (indeed, $||g||_0$ is $||g||$). We can prove (see [18], Proposition 2) that $||w - \Pi^1_{r-1} w||_0 \le (1 + \Lambda^1_r)\,||w - \tilde{w}^*||_0$, with $\tilde{w}^*$ the best-fit polynomial of w in $\mathcal{P}_{r-1}\Lambda^1(I)$ with respect to $||.||_0$. This recalls very closely the estimate for 0-forms given in Proposition 2. See [18] and [19] for results on the behavior of Λ^1_r with respect to r, depending on the type (uniform or not) of the sub-intervals σ_i segments.

We denote by $\mathcal{P}_r\Lambda^0(I)$ the space $\mathbb{P}_r(I)$ of polynomial 0-forms defined on I (either globally or piece-wisely) in Section 3.1. We can arrange the spaces $\mathcal{P}_q\Lambda^k(I)$ in a complex, known as the discrete de Rham diagram, that here reads

$$\begin{array}{ccccccc}
\mathbb{R} & \longrightarrow & \mathcal{P}_r\Lambda^0(I) & \xrightarrow{\mathrm{d}} & \mathcal{P}_{r-1}\Lambda^1(I) & \longrightarrow & \{0\} \\
 & & p^0_r \uparrow & \circlearrowleft & \uparrow p^1_{r-1} & & \\
\mathbb{R} & \longrightarrow & \mathbb{R}^{N_r} & \xrightarrow{G} & \mathbb{R}^{N_a} & \longrightarrow & \{0\} \\
x & \longmapsto & (x, \cdots, x) & & & &
\end{array}$$

with N_r (resp., N_a) the number of unisolvent weights for 0-forms (resp., 1-forms). In the diagram, $\mathbb{R}^{N_r}$, $\mathbb{R}^{N_a}$ are the corresponding spaces of arrays gathering the weight values, p_r^0 and p_{r-1}^1 are the reconstruction maps from arrays to forms. Finally, G is the $N_a \times N_r$ matrix entries $G(k,k) = -1$, $G(k,k+1) = +1$, and $G(k,s) = 0$ for $s \notin \{k, k+1\}$. This sequence is (globally) exact on intervals, meaning that $\mathrm{d}\,(\mathcal{P}_r\Lambda^0(I)) = \mathcal{P}_{r-1}\Lambda^1(I)$. Moreover, the diagram is commutative ($\circlearrowleft$), in the sense that one can follow the arrows along any path between the two spaces and still obtain the same result. Indeed, it can be proved that $p_{r-1}^1(G\,\mathbf{y}) = \mathrm{d}\,(p_r^0\,\mathbf{y})$ (see, e.g., Proposition 3.3 in [17]).

If we adopt a piece-wise approach (as explained in Remark 1), by gluing together the 1-forms locally computed in each sub-interval I_ℓ, we get a global representation $w_h = g_h\,dx$ of $w = g\,dx$, with $(g_h)_{|I_\ell} \in \mathbb{P}_{m-1}(I_\ell)$, for each $\ell = 1, ..., N_a$, if I_ℓ is partitioned into m segments. Without additional conditions, the function g_h is discontinuous, when passing from I_ℓ to $I_{\ell+1}$.

4.2 Interpolation in the space $\mathcal{P}_{1,q}\Lambda^1(I)$

Let $\mathcal{P}_{1,q}\Lambda^1(I)$, for $q \geq 2$, be the space of 1-forms $w = f\,dx$ with $f \in \mathbb{P}_q(I)$, either globally or piece-wisely in I. In the second case, we ask for a global regularity one step higher than that in Section 4.1, i.e, $f \in \mathcal{C}^0(I)$. To describe a field (a 1-form) $w = f\,dx$ in $\mathcal{P}_{1,q}\Lambda^1(I)$, $q = 2r$ and $r \geq 1$, we need (either globally or piece-wisely) $2r+1$ suitable functionals μ_i, ν_j. In particular, we involve the codifferential $\delta = -\star \mathrm{d} \star$ for 1-forms (as we are in dimension 1) and the operator $\iota_{\mathbf{e}_x}$, with $\mathbf{e}_x$ the unit vector in the x-direction, known as interior product, here acting on 1-forms. We consider the following set:

$$\begin{aligned} &\mu_i : w \longmapsto \langle w, I_i\rangle = \textstyle\int_{x_i}^{x_{i+1}} w\,, && i = 1, ..., N_a,\\ &\nu_j : w \longmapsto \langle \delta w, I_j\rangle = \textstyle\int_{x_j}^{x_{j+1}} \delta w\,, && j = 1, ..., N_a,\\ &\nu_0 : w \longmapsto \langle \iota_{\mathbf{e}_x} w, x_1\rangle = f(x_1) = \beta_0\,, && \end{aligned} \tag{4}$$

with x_1 the left-hand extremity of I.

Proposition 7. *The set $\{\mu_i, \nu_j, \nu_0\}$ defined in (4) is unisolvent in $\mathcal{P}_{1,2r}\Lambda^1(I)$.*

Proof of Proposition 7. Let g be a $\mathcal{C}^1(I)$ function and let $I_\ell = [x_\ell, x_{\ell+1}]$. By using the differential Dg of g, the values $\nu_\ell(w)$, $\ell > 0$, with $w = g\,dx$, can be written as

$$-g(x_{\ell+1}) + g(x_\ell) = \int_0^1 -Dg((1-s)x_\ell + s\,x_{\ell+1})\,ds\,(x_{\ell+1} - x_\ell) = \int_{I_\ell} - \star\, d \star w\,. \tag{5}$$

Let us consider the linear vector map $L : \mathcal{P}_{1,2r}\Lambda^1(I) \to \mathbb{R}^{2N_a+1}$ (note that $N_a = r$) whose components are given by all the μ_i and ν_i. A 1-form w in its kernel has the polynomial part g of degree lower than $2r$. Furthermore, the r integrals $\big(\mu_\ell(w) = \int_{[x_\ell, x_{\ell+1}]} g\,dx\big)$ are 0 and g takes the same value α all the $r+1$ points x_ℓ. Since $\beta_0 = g(x_1) = 0$, this value $\alpha = 0$. Moreover $\int_{[x_\ell, x_{\ell+1}]} g\,dx = 0$ implies that there exists at least one $\xi_\ell \in]x_\ell, x_{\ell+1}[$ such that $g(\xi_\ell) = 0$. So g has at least $r + 1 + N_a = 2r + 1$ roots in I therefore $g = 0$. Hence, the map L is a linear isomorphism and the set of functionals in (4) is unisolvent in $\mathcal{P}_{1,2r}\Lambda^1(I)$. □

Given a basis $\{\psi_\ell\}$ of $\mathcal{P}_{1,2r}\Lambda^1(I)$, we construct the cardinal basis $\{\varphi_k\}$ associated with the μ_i and ν_j by relying on the matrix V of size $2N_a+1$, with entries

$$\begin{aligned} V_{j,\ell} &= \mu_j(\psi_\ell), && j=1,...,N_a, \\ V_{N_a+j,\ell} &= \nu_j(\psi_\ell), && j=1,...,N_a\,, \\ V_{2\,N_a+1,\ell} &= \nu_0(\psi_\ell)\,, \end{aligned}$$

for $\ell = 1,...,2N_a+1$. The matrix V is non singular since the linear functionals μ_i and ν_j are unisolvent in $\mathcal{P}_{1,2r}\Lambda^1(I)$ (see Proposition 7). Let $\{a_\ell\}$ be a set of real values $\mu_\ell(w)$ associated with the N_a segments I_ℓ defined in I by the points $\{x_\ell\}$. We also consider $\{c_j\}$, a set of real values $\nu_j(w)$ associated with the same N_a segments I_j, and $\beta_0 = \nu_0(w)$ associated with the first node x_1. Then $H^1_{2r}w \in \mathcal{P}_{1,2r}\Lambda^1(I)$ is uniquely determined as

$$H^1_{2r}w = \sum_{i=1}^{N_a} a_i\,\varphi_i + \sum_{j=1}^{N_a} c_j\,\varphi_{N_a+j} + \beta_0\,\varphi_{2\,N_a+1},$$

where $\{\varphi_k\}$ is the cardinal basis in duality with the functionals μ_i, ν_j defined in (4). We recall that the Lebesgue constant plays the role of condition number for the interpolation. Here, its expression, say Υ^1_r, is modified with respect to that of Λ^1_r. Indeed, Υ^1_r should take into account not only the $\mu_i(.)$ but also the $\nu_j(.)$, similarly to what we have seen for 0-forms (with the definition of Υ^K_r and Υ^H_r).

We hence have a reconstruction operator $q^1_{2r} : \mathbb{R}^{2N_a+1} \to \mathcal{P}_{1,2r}\Lambda^1(I)$ which associates, with a vector $(\mathbf{a},\mathbf{c},\beta_0) \in \mathbb{R}^{2N_a+1}$, a unique polynomial 1-form $q^1_{2r}(\,(\mathbf{a},\mathbf{c},\beta_0)\,) = H^1_{2r}w$ in $\mathcal{P}_{1,2r}\Lambda^1(I)$. We denote by $\mathcal{P}_{1,2r+1}\Lambda^0(I)$ the space, introduced in Section 3.2, of polynomial 0-forms in I containing scalar fields u_h such that $u_h \in \mathbb{P}_{2r+1}$, either globally in I or piece-wisely, in each I_ℓ, with global regularity 1 (that is, having u_h and $\partial_x u_h$ in $\mathcal{C}^0(I)$). We can arrange the spaces $\mathcal{P}_{1,q}\Lambda^k(I)$ in another discrete de Rham diagram, that here reads

$$\begin{array}{ccccccc} \mathbb{R} \longrightarrow & \mathcal{P}_{1,2r+1}\Lambda^0(I) & \xrightarrow{\mathrm{d}} & \mathcal{P}_{0,2r}\Lambda^1(I) & \longrightarrow \{0\} \\ & q^0_{2r+1}\uparrow & \circlearrowleft & \uparrow q^1_{2r} & \\ \mathbb{R} \longrightarrow & \mathbb{R}^{2\,N_r} & \xrightarrow{\tilde{G}} & \mathbb{R}^{2N_a+1} & \longrightarrow \{0\} \\ x \longmapsto & (x,\cdots,x) & & & \end{array}$$

with $2\,N_r$ (resp., $2N_a+1$) the number of unisolvent weights for 0-forms (resp., 1-forms) in the corresponding polynomial spaces. In the diagram, $\tilde{G}$ is the $(2N_a+1)\times 2N_r$ matrix representing the operator d, as follows

$$\tilde{G} = \begin{pmatrix} G & \\ & -G \\ & \mathbf{e}_1 \end{pmatrix}\,,$$

with the vector $\mathbf{e}_1 = (1,0,...,0) \in \mathbb{R}^{N_r}$, the matrix G as the one defined in Section 4.1, and empty blocs in $\tilde{G}$ stand for blocs of entries equal to 0. The previous sequence is (globally) exact on intervals. Moreover, the diagram is commutative (↻), namely it can be proved that $q^1_{2r}(\,(\mathbf{y},\mathbf{d})\,\tilde{G}^\top) = \mathrm{d}\,(q^0_{2r+1}\,(\mathbf{y},\mathbf{d}))$. It can be remarked that $\ker(\tilde{G}) = \{y_1\,(\mathbf{1}_{N_r}\,,\,\mathbf{0}_{N_r}\,)\}$ with y_1 the value at x_1 and $\mathbf{1}_{N_r}$ (resp., $\mathbf{0}_{N_r}$) the vector of size N_r with all entries equal to 1 (resp., 0).

Finally, the functional ν_0 can be used to impose the required regularity, in a piecewise polynomial interpolation. Let us consider two sub-intervals I_ℓ, each divided into $m = 4$ segments. We apply the reconstruction operator q^1_{2m}, here $m = 4$, in each sub-interva I_ℓ, and obtain

$$(w_h)_{|I_1} = q^1_{2m}(\,(\mathbf{a},\mathbf{c},\beta_0)\,), \qquad (w_h)_{|I_2} = q^1_{2m}(\,(\tilde{\mathbf{a}},\tilde{\mathbf{c}},\tilde{\beta}_0)\,)\,.$$

Then, $w_h = g_h\,dx$ has $g_h \in \mathcal{C}^0(I)$ iff $\tilde{\beta}_0 = \beta_0 + \sum_{j=1}^m c_j$, that is, when the value $\tilde{\beta}_0$ at the first node of the current sub-interval I_ℓ is equal to the sum of all values $\nu_j(w)$ assigned on the previous neighbouring one $I_{\ell-1}$. So, ν_0 takes part to the unisolvence in Proposition 7 and it can be used to impose the condition required for continuity.

5 Conclusions

In these pages, we have reconsidered classical knowledge on polynomial interpolation of fields on an interval $I \subset \mathbb{R}$. We have pointed out the mathematical, physical or numerical nature of the choices one makes all along the interpolation process. We have revisited polynomial interpolations of Lagrange and Hermite types for scalar fields, intended as 0-forms, together with remarks on convergence properties for $r \to +\infty$. Taking the cue from existing literature and relying on tools from the discrete exterior calculus, we have proposed a very natural extension of the Hermite case to 1-forms. The numerical validation of the presented variant and its theoretical extension to higher dimensions are in progress (see, e.g., [20] for high regularity de Rham complexes).

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