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Abstract We provide an overview of several series identities involving the Cauchy numbers and various types of harmonic numbers, all of which are closely connected to certain alternating series with zeta values (or harmonic zeta values); we then give, for each of these identities, an interpretation in terms of Ramanujan summation with the hope that this unusual but interesting interpretation of still little-known formulas could be useful for further research on the topic.

Keywords Cauchy numbers; harmonic numbers; binomial transform; series with zeta values; Ramanujan summation of series.

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1 Introduction

A decade ago, we showed how Ramanujan's method of summation of series could be used to generate a number of identities linking together Cauchy numbers (also known as Bernoulli numbers of the second kind), harmonic numbers, and values of the Riemann zeta function at positive integers [4]. This powerful method is based on a binomial transformation formula that relates the Cauchy numbers to the Ramanujan summation of series. Recently, we have obtained new results by a refinement of the same method, thanks notably to the consideration of a rather natural generalization of the harmonic numbers (see [7] for details). Some

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special cases are presented in Section 3. On the other hand, an alternative efficient method for generating such identities, based on a similar binomial transformation formula, is presented in Section 4. Reversing the procedure and combining these two methods, we are able to provide in Section 5 an interesting interpretation in terms of Ramanujan summation of almost all the identities mentioned above.

2 Reminder of some basic definitions

We first recall some basic facts about the Cauchy numbers using the notations of [4] and [7], we then introduce various types of harmonic numbers.

a) The non-alternating Cauchy numbers $\{\lambda_n\}_{n\geq 1}$ are defined explicitely by the formula

$$\lambda_n := \int_0^1 x(1-x)\cdots(n-1-x)\,dx\,.$$

Alternatively, they can be defined recursively by means of the relation

$$\sum_{k=1}^{n-1} \frac{\lambda_k}{k!\,(n-k)} = \frac{1}{n} \qquad (n\geq 2)\,.$$

The first ones are the following:

$$\lambda_1 = \frac{1}{2}\,,\ \lambda_2 = \frac{1}{6}\,,\ \lambda_3 = \frac{1}{4}\,,\ \lambda_4 = \frac{19}{30}\,,\ \lambda_5 = \frac{9}{4}\,,\ \lambda_6 = \frac{863}{84}\,,\ \text{etc.}$$

The non-alternating Cauchy numbers form a sequence of positive rational numbers which are closely linked to the Bernoulli numbers of the second kind b_n (first introduced by Jordan [8]) through the relation

$$\lambda_n = (-1)^{n-1} n!\, b_n = n!\,|b_n| \qquad (n\geq 1)\,.$$

Finally, the exponential generating function of the non-alternating Cauchy numbers is given by

$$\sum_{n=1}^{\infty} \lambda_n \frac{x^n}{n!} = 1 + \frac{x}{\ln(1-x)} \qquad (|x|<1)\,.$$

In particular, the series $\sum_{n\geq 1}\frac{\lambda_n}{n!}$ converges to 1.

b) The classical harmonic numbers $\{H_n\}_{n\geq 1}$ are defined by

$$H_n = \sum_{j=1}^{n} \frac{1}{j} = \psi(n+1) + \gamma\,,$$

where $\psi = \Gamma'/\Gamma$ denotes the digamma function and $\gamma = -\psi(1)$ is the Euler constant.

c) For any integer $k \geq 1$, the generalized harmonic numbers $\{H_n^{(k)}\}_{n\geq 1}$ are defined by $H_n^{(1)} = H_n$, and

$$H_n^{(k)} = \sum_{j=1}^{n} \frac{1}{j^k} = \frac{(-1)^{k-1}}{(k-1)!}\partial^{k-1}\psi(n+1) + \zeta(k) \qquad (k \geq 2)\,.$$

d) For any integer $k \geq 0$, the Roman harmonic numbers $\{H_{n,k}\}_{n\geq 1}$ are defined by $H_{n,0} = 1$, and

$$H_{n,k} = \sum_{n\geq j_1 \geq \cdots \geq j_k \geq 1} \frac{1}{j_1\, j_2 \cdots j_k} \qquad (k \geq 1).$$

The Roman harmonic numbers[1] can be expressed as polynomials in the generalized harmonic numbers $H_n, H_n^{(2)}, \cdots, H_n^{(k)}$ [4, Eq. (18)]. More precisely, $H_{n,1} = H_n$, and

$$H_{n,k} = \frac{1}{k!}(H_n)^k + \cdots + \frac{1}{k}H_n^{(k)} = P_k(H_n, \cdots, H_n^{(k)}) \qquad (k \geq 2)\,,$$

where P_k are the modifed Bell polynomials [4, Def. 2]. In particular,

$$H_{n,2} = \sum_{k=1}^{n} \frac{H_k}{k} = \frac{1}{2}(H_n)^2 + \frac{1}{2}H_n^{(2)} = P_2(H_n, H_n^{(2)})\,.$$

A natural generalization of the ordinary Roman harmonic numbers noted $H_{n,k}^{(r)}$ such that $H_{n,1}^{(r)} = H_n^{(r)}$ was also introduced in [7]. It is given by the following expression [7, Def. 2]:

$$H_{n,k}^{(r)} = \sum_{n\geq j_1 \geq \cdots \geq j_k \geq 1} \frac{1}{j_1\, j_2 \cdots j_k^r} \qquad (k \geq 1, r \geq 1)\,.$$

3 Overview of some known formulas

In this section, we mention a number of more or less known identities and make some comments about them.

a) The formula

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\, n} = \sum_{n=2}^{\infty} \frac{(-1)^n}{n}\,\zeta(n) = \gamma \qquad (1)$$

1. Introduced three decades ago by S. Roman, G-C. Rota and D. Loeb (see [9] for historical details).

is a classical representation of γ dating back to Mascheroni and Euler which can be slightly modified as follows:

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{(n+1)!\,n} = \sum_{n=2}^{\infty} \frac{(-1)^n}{n} \{\zeta(n) - 1\} = \gamma + \ln 2 - 1\,. \tag{2}$$

b) A non-trivial generalization of (1) is the following formula:

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\,n^2} = \frac{1}{2}\gamma^2 + \frac{1}{2}\zeta(2) + \gamma_1 - \sum_{n=2}^{\infty} \frac{(-1)^n}{n}\,\zeta(n+1)\,, \tag{3}$$

where

$$\gamma_1 = \lim_{n\to\infty} \left\{ \sum_{j=1}^{n} \frac{\ln j}{j} - \frac{1}{2}\ln^2 n \right\}$$

is the first Stieltjes constant. This intriguing identity is already known and appears in [6, 7].

c) The formula

$$\sum_{n=1}^{\infty} \frac{\lambda_n\, H_n}{n!\,n} = \zeta(2) - 1 \tag{4}$$

is a fairly known representation of $\zeta(2) = \dfrac{\pi^2}{6}$. This is in fact a particular case of the more general formula

$$\sum_{n=1}^{\infty} \frac{\lambda_n\, H_{n,k}}{n!\,n} = \zeta(k+1) - \frac{1}{k} \qquad (k \geq 1)\,,$$

which is sometimes called *Hermite's formula* [5].

d) A non-trivial generalization of (4) consists of the following formula:

$$\sum_{n=1}^{\infty} \frac{\lambda_n\, H_n^{(2)}}{n!\,n} = \zeta(3) + \{\gamma + \ln(2\pi) - 12\ln A\}\,\zeta(2) + \sum_{n=2}^{\infty} \frac{(-1)^n}{n}\,\zeta(n+2) \tag{5}$$

where

$$A = \lim_{n\to\infty} \left\{ \frac{\prod_{k=1}^{n} k^k}{n^{\frac{n^2}{2}+\frac{n}{2}+\frac{1}{12}}\, e^{-\frac{n^2}{4}}} \right\}$$

is the Glaisher-Kinkelin constant. This identity is a direct consequence of [7, Eq. (19)] and the well-known relation:

$$\zeta'(2) = (\gamma + \ln(2\pi) - 12\ln A)\,\zeta(2)\,.$$

Furthermore, formula (5) admits a kind of reciprocal which is given by [7, Eq. (18)]. More precisely, we have the following formula:

$$\sum_{n=1}^{\infty} \frac{\lambda_n H_n}{n!\, n^2} = \ln\left(\frac{A^{12}}{2\pi}\right)\zeta(2) - 1 - \sum_{n=2}^{\infty} \frac{(-1)^n}{n}\,\zeta(n+2) \tag{6}$$

which is another non-trivial generalization of (4).

Remark 1. When k is greater than 2, no explicit formula, even conjectural, appears to be known for the sum $\sum_{n=1}^{\infty} \frac{\lambda_n H_n^{(k)}}{n!\, n}$, nor for the reciprocal sum $\sum_{n=1}^{\infty} \frac{\lambda_n H_n}{n!\, n^k}$ (see however Remark 4 below for an interpretation of these sums in terms of Ramanujan summation).

4 New supplementary formulas

The following binomial formula is nothing else than a variant of [1, Prop. 1] which is an elementary but efficient tool to generate several series identities with Cauchy and harmonic numbers. For appropriate analytic functions f with moderate growth, one has the relation:

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!} \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} f(k) = \int_0^1 f(x)\, dx\,. \tag{7}$$

For instance, formulas (1) and (4) above can be easily derived from formula (7) using adequate functions f [1, Ex. 4]. We now give another new interesting identities that can also be deduced by this method.

e) Applying (7) with $f(x) = \dfrac{\psi(x+1)+\gamma}{x+1}$, and using the binomial identity [2, Eq. (9.32)]

$$\frac{H_n}{n+1} = \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} \frac{H_k}{k+1}\,,$$

allows us to deduce the following identity:

$$\sum_{n=1}^{\infty} \frac{\lambda_n H_n}{(n+1)!} = \int_0^1 f(x)\, dx = \frac{1}{2}\zeta(2) + \ln 2 - 1 + \sum_{n=3}^{\infty} \frac{(-1)^n}{n} \left\{ \sum_{k=2}^{n} (\zeta(k) - 1) \right\} \tag{8}$$

which is a refinement of a formula previously given by Boyadzhiev [1, Ex. 5]. Moreover, substracting (8) from (4) leads to the following new formula:

$$\sum_{n=1}^{\infty} \frac{\lambda_n H_n}{(n+1)!\, n} = \frac{1}{2}\zeta(2) - \ln 2 - \sum_{n=3}^{\infty} \frac{(-1)^n}{n} \left\{ \sum_{k=2}^{n} (\zeta(k) - 1) \right\}. \tag{9}$$

This last formula can be seen as a modification of (4) quite similar to (2).

f) Applying (7) with $f(x) = \dfrac{\psi(x+1)+\gamma}{x}$, and using the binomial identity [2, Eq. (5.22)]

$$H_n^{(2)} = \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} \frac{H_k}{k} \,,$$

allows us to deduce the following identity:

$$\sum_{n=1}^{\infty} \frac{\lambda_n \, H_n^{(2)}}{n!} = \int_0^1 f(x)\, dx = \zeta(2) - \sum_{n=2}^{\infty} \frac{(-1)^n}{n} \zeta(n+1)\,. \tag{10}$$

Moreover, writing $H_n^{(2)} = H_{n-1}^{(2)} + \dfrac{1}{n^2}$ and using (3) leads to another nice identity:

$$\sum_{n=1}^{\infty} \frac{\lambda_{n+1} \, H_n^{(2)}}{(n+1)!} = \frac{1}{2}\zeta(2) - \frac{1}{2}\gamma^2 - \gamma_1\,. \tag{11}$$

Remark 2. If ζ_H denotes the harmonic zeta function defined by

$$\zeta_H(s) = \sum_{n=1}^{\infty} \frac{H_n}{n^s} \qquad (\mathrm{Re}(s) > 1)\,,$$

then, by [6, Eq. (14)], we have the following identity:

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{n} \zeta_H(n) = \frac{1}{2}\zeta(2) + \frac{1}{2}\gamma^2 + \gamma_1\,.$$

As a consequence of this identity, formula (11) may also be rewritten as follows:

$$\sum_{n=1}^{\infty} \frac{\lambda_{n+1} \, H_n^{(2)}}{(n+1)!} = \zeta(2) - \sum_{n=2}^{\infty} \frac{(-1)^n}{n} \zeta_H(n)\,. \tag{11 bis}$$

Remark 3. Very recently, Young [10] has shown the following formula:

$$\sum_{n=1}^{\infty} \frac{\lambda_n \, H_n^{(k)}}{n!} = \zeta(k) - \sum_{n=2}^{\infty} \frac{(-1)^n}{n} \zeta(n+1, \underbrace{1, \ldots, 1}_{k-2}) \qquad (k \geq 2)$$

which significantly generalizes (10). However, it should be noted that this formula cannot be extended to the case $k = 1$ since the series $\sum_{n\geq 1} \frac{\lambda_n H_n}{n!}$ is divergent.

5 Interpretation as Ramanujan summation

If $\sum_{n\geq 1}^{\mathcal{R}} f(n)$ denotes the $\mathcal{R}$-sum of the series $\sum_{n\geq 1} f(n)$ (i.e. the sum of the series in the sense of Ramanujan's summation method as exposed in [3]), then, under certain appropriate conditions of growth and analyticity, we can make use of the following binomial transformation formula [7, Eq. (10)]:

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\, n} \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} k f(k) = \sum_{n\geq 1}^{\mathcal{R}} f(n)\,. \tag{12}$$

This formula allows us to give an interesting interpretation in terms of Ramanujan summation of almost all the series identities mentioned above.

a) From the binomial identities

$$\sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} = 1 \quad \text{and} \quad \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} \frac{k}{k+1} = \frac{1}{n+1}\,,$$

we derive respectively

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\, n} = \sum_{n\geq 1}^{\mathcal{R}} \frac{1}{n}\,, \tag{A}$$

and the shifted formula

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{(n+1)!\, n} = \sum_{n\geq 1}^{\mathcal{R}} \frac{1}{n+1}\,. \tag{B}$$

b) From the reciprocal binomial identities

$$\sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} H_k = \frac{1}{n} \quad \text{and} \quad \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} \frac{1}{k} = H_n\,,$$

we derive respectively the formulas

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\, n^2} = \sum_{n\geq 1}^{\mathcal{R}} \frac{H_n}{n}\,, \tag{C}$$

and

$$\sum_{n=1}^{\infty} \frac{\lambda_n H_n}{n!\, n} = \sum_{n\geq 1}^{\mathcal{R}} \frac{1}{n^2}\,. \tag{D}$$

c) From the reciprocal binomial identities

$$\sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}\frac{H_k}{k} = H_n^{(2)} \quad \text{and} \quad \sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}H_k^{(2)} = \frac{H_n}{n},$$

we derive respectively the formulas

$$\sum_{n=1}^{\infty}\frac{\lambda_n H_n^{(2)}}{n!\,n} = \sum_{n\geq 1}^{\mathcal{R}}\frac{H_n}{n^2}, \tag{E}$$

and

$$\sum_{n=1}^{\infty}\frac{\lambda_n H_n}{n!\,n^2} = \sum_{n\geq 1}^{\mathcal{R}}\frac{H_n^{(2)}}{n}. \tag{F}$$

Remark 4. a) Formulas (A) and (C) are two particular cases of the more general relation [7, Eq. (12)]:

$$\sum_{n=1}^{\infty}\frac{\lambda_n}{n!\,n^k} = \sum_{n\geq 1}^{\mathcal{R}}\frac{H_{n,k-1}}{n} \qquad (k \geq 1).$$

b) Formulas (D) and (E) are two particular cases of the more general relation [7, Eq. (11)]:

$$\sum_{n=1}^{\infty}\frac{\lambda_n H_n^{(k)}}{n!\,n} = \sum_{n\geq 1}^{\mathcal{R}}\frac{H_{n,k-1}}{n^2} \qquad (k \geq 1).$$

c) Formula (F) can also be extended thanks to the generalized Roman harmonic numbers $H_{n,k}^{(r)}$ introduced in [7]. Indeed, by [7, Eq. (12)], we have

$$\sum_{n=1}^{\infty}\frac{\lambda_n H_n}{n!\,n^k} = \sum_{n\geq 1}^{\mathcal{R}}\frac{H_{n,k-1}^{(2)}}{n} \qquad (k \geq 2).$$

d) From the the self-reciprocal binomial identity

$$\sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}\frac{H_k}{k+1} = \frac{H_n}{n+1},$$

we deduce the formula

$$\sum_{n=1}^{\infty}\frac{\lambda_n H_n}{(n+1)!\,n} = \sum_{n\geq 1}^{\mathcal{R}}\frac{H_n}{n(n+1)}. \tag{G}$$

e) Let us now apply the formula (12) to the function

$$f(x) = \frac{\psi(x+1) + \gamma - 1}{x(x-1)} \,,$$

which verifies

$$f(1) = \lim_{x \to 1} \frac{\psi(x+1) + \gamma - 1}{x(x-1)} = \zeta(2) - 1 \,,$$

and

$$f(n) = \frac{H_n - 1}{n(n-1)} \qquad (n \geq 2) \,.$$

Then, by means of the binomial identity

$$\sum_{k=2}^{n} (-1)^{k-1} \binom{n}{k} \frac{1 - H_k}{k-1} = nH_n^{(2)} - n$$

which is the reciprocal of the identity [2, Eq. (5.24)]:

$$\sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} kH_k^{(2)} = \frac{1 - H_n}{n-1} \qquad (n \geq 2) \,,$$

we deduce the formula

$$\sum_{n=1}^{\infty} \frac{\lambda_n \, H_n^{(2)}}{n!} = \zeta(2) - \sum_{n \geq 1}^{\mathcal{R}} \frac{H_n - 1}{n(n-1)} \,. \tag{H}$$

Furthermore, subtracting (C) from (H) allows us to derive the formula

$$\sum_{n=1}^{\infty} \frac{\lambda_{n+1} \, H_n^{(2)}}{(n+1)!} = \zeta(2) - \sum_{n \geq 1}^{\mathcal{R}} \frac{H_{n-1}}{n-1} \,. \tag{I}$$

Remark 5. Very recently, Young [10] has established the following relation:

$$\sum_{n=1}^{\infty} \frac{\lambda_n \, H_n^{(k+1)}}{n!} = \zeta(k+1) - \sum_{n \geq 1}^{\mathcal{R}} \frac{H_{n,k} - H_{n,k-1}}{n(n-1)} \qquad (k \geq 1)$$

which is a substantial improvement of our formula (H).

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