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Abstract We provide an overview of several series identities involving Cauchy numbers and harmonic numbers, all of which are closely linked to certain alternating series with zeta (or harmonic zeta) values; we then give, for each of them, their interpretation in terms of Ramanujan summation with the hope that this unusual interpretation of still little-known formulas should be useful for further research on the topic.

Keywords Cauchy numbers; harmonic numbers; binomial transform; series with zeta values; Ramanujan summation of series.

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1 Introduction

A decade ago, we showed how Ramanujan's method of summation of series could be used to generate a number of identities linking together Cauchy numbers (also known as Bernoulli numbers of the second kind), harmonic numbers, and values of the Riemann zeta function at positive integers [4]. Recently, we have improved this method by introducing a natural generalization of the classical Roman harmonic numbers [8] and by systematically using a binomial transformation formula that relates the Cauchy numbers to the Ramanujan summation of series (see [6] for details). In some cases, an alternative efficient method for generating such identities, based on a similar binomial transformation formula, is described in [1]. By

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combining these two methods and reversing the procedure, we are able to provide an interesting interpretation in terms of Ramanujan summation of almost all the identities presented here (see Section 5).

2 Reminder of some basic definitions

We first recall some basic facts about the Cauchy numbers (also known as Bernoulli numbers of the second kind) and introduce various types of harmonic numbers.

a) The non-alternating Cauchy numbers $\{\lambda_n\}_{n\geq 1}$ are defined explicitely by the formula

$$\lambda_n := \int_0^1 x(1-x)\cdots(n-1-x)\,dx\,.$$

Alternatively, they can be defined recursively by means of the relation

$$\sum_{k=1}^{n-1} \frac{\lambda_k}{k!\,(n-k)} = \frac{1}{n} \qquad (n\geq 2)\,.$$

The first ones are the following:

$$\lambda_1 = \frac{1}{2}\,,\ \lambda_2 = \frac{1}{6}\,,\ \lambda_3 = \frac{1}{4}\,,\ \lambda_4 = \frac{19}{30}\,,\ \lambda_5 = \frac{9}{4}\,,\ \lambda_6 = \frac{863}{84}\,,\ \text{etc.}$$

The non-alternating Cauchy numbers λ_n are closely linked to the Bernoulli numbers of the second kind b_n first introduced by Jordan [7] through the relation

$$\lambda_n = n!\,|b_n| \qquad (n\geq 1)\,.$$

Otherwise, with the current notation c_n used in [1], we have the simple relation

$$\lambda_n = (-1)^{n-1}\,c_n \qquad (n\geq 1)\,.$$

We also recall the convergence to 1 of the series $\sum_{n\geq 1}\frac{\lambda_n}{n!}$.

b) The classical harmonic numbers $\{H_n\}_{n\geq 1}$ are defined by

$$H_n = \sum_{j=1}^{n}\frac{1}{j} = \psi(n+1)+\gamma\,,$$

where ψ denotes the digamma function and $\gamma = -\psi(1)$ is the Euler constant.

c) For any integer $k\geq 1$, the generalized harmonic numbers $\{H_n^{(k)}\}_{n\geq 1}$ are defined by $H_n^{(1)} = H_n$, and

$$H_n^{(k)} = \sum_{j=1}^{n}\frac{1}{j^k} = \frac{(-1)^{k-1}}{(k-1)!}\partial^{k-1}\psi(n+1)+\zeta(k) \qquad (k\geq 2)\,.$$

d) For any integer $k \geq 0$, the Roman harmonic numbers $\{H_{n,k}\}_{n\geq 1}$ are defined by $H_{n,0} = 1$, and

$$H_{n,k} = \sum_{n\geq j_1\geq\cdots\geq j_k\geq 1} \frac{1}{j_1\, j_2 \cdots j_k} \qquad (k \geq 1).$$

The Roman harmonic numbers[1] can be expressed as polynomials in the generalized harmonic numbers $H_n, H_n^{(2)}, \cdots, H_n^{(k)}$ (see [4, Eq. (18)], [8, Eq. (29)]). More precisely, we have $H_{n,1} = H_n$, and

$$H_{n,k} = \frac{1}{k!}(H_n)^k + \cdots + \frac{1}{k}H_n^{(k)} = P_k(H_n, \cdots, H_n^{(k)}) \qquad (k \geq 2)\,,$$

where P_k are the modifed Bell polynomials [4, Def. 2]. In particular,

$$H_{n,2} = \sum_{k=1}^{n} \frac{H_k}{k} = \frac{1}{2}(H_n)^2 + \frac{1}{2}H_n^{(2)} = P_2(H_n, H_n^{(2)})\,.$$

A natural generalization of the ordinary Roman harmonic numbers noted $H_{n,k}^{(r)}$ such that $H_{n,1}^{(r)} = H_n^{(r)}$ was also introduced in [6] (see [6, Def. 2]). It is given by the following expression:

$$H_{n,k}^{(r)} = \sum_{n\geq j_1\geq\cdots\geq j_k\geq 1} \frac{1}{j_1\, j_2 \cdots j_k^r} \qquad (k \geq 1, r \geq 1)\,.$$

3 Overview of some known formulas

In this section, we mention a number of more or less known identities and make some comments about them.

a) The formula

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\, n} = \sum_{n=2}^{\infty} \frac{(-1)^n}{n}\zeta(n) = \gamma \tag{1}$$

is a classical representation of γ due to Mascheroni and Euler which can be slightly modified as follows:

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{(n+1)!\, n} = \sum_{n=2}^{\infty} \frac{(-1)^n}{n}\{\zeta(n) - 1\} = \gamma + \log 2 - 1\,. \tag{2}$$

1. Introduced three decades ago by S. Roman, G-C. Rota and D. Loeb (see [8] for historical details).

b) A non-trivial generalization of (1) is the following formula:

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\, n^2} = \frac{1}{2}\gamma^2 + \frac{1}{2}\zeta(2) + \gamma_1 - \sum_{n=2}^{\infty} \frac{(-1)^n}{n}\zeta(n+1)\,, \tag{3}$$

where γ_1 denotes the first Stieltjes constant. This intriguing identity is already known and appears in [3, 5, 6].

c) The formula

$$\sum_{n=1}^{\infty} \frac{\lambda_n\, H_n}{n!\, n} = \zeta(2) - 1 \tag{4}$$

is a fairly known representation of $\zeta(2) = \dfrac{\pi^2}{6}$ which is in fact a particular case of the more general formula

$$\sum_{n=1}^{\infty} \frac{\lambda_n\, H_{n,k}}{n!\, n} = \zeta(k+1) - \frac{1}{k} \qquad (k \geq 1)\,,$$

sometimes called *Hermite's formula* [4].

d) A non-trivial generalization of (4) consists of the following formula:

$$\sum_{n=1}^{\infty} \frac{\lambda_n\, H_n^{(2)}}{n!\, n} = \zeta(3) + \{\gamma + \log(2\pi) - 12\log(A)\}\,\zeta(2) + \sum_{n=2}^{\infty} \frac{(-1)^n}{n}\zeta(n+2) \tag{5}$$

with A denoting the Glaisher-Kinkelin constant

$$A = \lim_{n\to\infty} \left\{ n^{-\frac{n^2}{2} - \frac{n}{2} - \frac{1}{12}}\, e^{\frac{n^2}{4}} \prod_{k=1}^{n} k^k \right\}.$$

This identity results directly from [6, Eq. (19)] and the well-known relation:

$$\zeta'(2) = (\gamma + \log(2\pi) - 12\log(A))\,\zeta(2)\,.$$

Furthermore, formula (5) admits a kind of reciprocal which is given by [6, Eq. (16)] in the case $p = 2$. More precisely, we have the following formula:

$$\sum_{n=1}^{\infty} \frac{\lambda_n\, H_n}{n!\, n^2} = \zeta(2)\,(12\log(A) - \log(2\pi)) - 1 - \sum_{n=2}^{\infty} \frac{(-1)^n}{n}\zeta(n+2)\,. \tag{6}$$

Remark 1. When k is greater than 2, no explicit formula, even conjectural, appears to be known for the sum $\sum_{n=1}^{\infty} \frac{\lambda_n\, H_n^{(k)}}{n!\, n}$, nor for the reciprocal sum $\sum_{n=1}^{\infty} \frac{\lambda_n\, H_n}{n!\, n^k}$ (see however Remark 3 below for an interpretation of these sums in terms of Ramanujan summation).

4 Additional formulas

The following binomial formula is nothing else than a variant of [1, Prop. 1] which is an elementary but efficient tool for obtaining several series identities with Cauchy numbers. For appropriate analytic functions f with moderate growth, we have the relation:

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!} \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} f(k) = \int_0^1 f(x)\, dx\,. \tag{7}$$

For instance, formulas (1) and (4) above can be easily derived from formula (7) using adequate functions f (see [1, Ex. 4]). We now give another new interesting identities that can also be deduced by this method.

e) Applying (7) with $f(x) = \dfrac{\psi(x+1)+\gamma}{x+1}$, and using the binomial identity

$$\frac{H_n}{n+1} = \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} \frac{H_k}{k+1}\,,$$

allows us to deduce the following identity:

$$\sum_{n=1}^{\infty} \frac{\lambda_n H_n}{(n+1)!} = \int_0^1 f(x)\, dx = \frac{1}{2}\zeta(2) + \log 2 - 1 + \sum_{n=3}^{\infty} \frac{(-1)^n}{n} \left\{ \sum_{k=2}^{n} (\zeta(k) - 1) \right\} \tag{8}$$

which is a refinement of a formula previously given by Boyadzhiev [1, Ex. 5]. Moreover, substracting (8) from (4) leads to the following new formula:

$$\sum_{n=1}^{\infty} \frac{\lambda_n H_n}{(n+1)!\, n} = \frac{1}{2}\zeta(2) - \log 2 - \sum_{n=3}^{\infty} \frac{(-1)^n}{n} \left\{ \sum_{k=2}^{n} (\zeta(k) - 1) \right\}. \tag{9}$$

This last formula is a modification of (4) quite similar to (2).

f) Applying (7) with $f(x) = \dfrac{\psi(x+1)+\gamma}{x}$, and using the binomial identity

$$H_n^{(2)} = \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} \frac{H_k}{k}\,,$$

allows us to deduce the following identity:

$$\sum_{n=1}^{\infty} \frac{\lambda_n H_n^{(2)}}{n!} = \int_0^1 f(x)\, dx = \zeta(2) - \sum_{n=2}^{\infty} \frac{(-1)^n}{n} \zeta(n+1)\,. \tag{10}$$

Moreover, writing $H_n^{(2)} = H_{n-1}^{(2)} + \dfrac{1}{n^2}$ and using (3) leads to another interesting identity:

$$\sum_{n=1}^{\infty} \frac{\lambda_{n+1}\, H_n^{(2)}}{(n+1)!} = \frac{1}{2}\zeta(2) - \frac{1}{2}\gamma^2 - \gamma_1\,. \tag{11}$$

If ζ_H denotes the harmonic zeta function defined by

$$\zeta_H(s) = \sum_{n=1}^{\infty} \frac{H_n}{n^s} \qquad (\mathrm{Re}(s) > 1)\,,$$

then, thanks to [5, Eq. (14)], this last identity may also be rewritten as follows:

$$\sum_{n=1}^{\infty} \frac{\lambda_{n+1}\, H_n^{(2)}}{(n+1)!} = \zeta(2) - \sum_{n=2}^{\infty} \frac{(-1)^n}{n} \zeta_H(n)\,. \tag{11 bis}$$

Remark 2. Very recently, Young [9] has shown the following formula:

$$\sum_{n=1}^{\infty} \frac{\lambda_n\, H_n^{(k)}}{n!} = \zeta(k) - \sum_{n=2}^{\infty} \frac{(-1)^n}{n} \zeta(n+1, \{1\}^{k-2}) \qquad (k \geq 2)$$

which considerably generalizes (10). It should be noted that, in contrast to the series $\sum_{n\geq1} \dfrac{\lambda_n\, H_n^{(k)}}{n!}$, the series $\sum_{n\geq1} \dfrac{\lambda_n\, H_n}{n!}$ is divergent since $\dfrac{\lambda_n\, H_n}{n!} \sim \dfrac{1}{n\log(n)}$.

5 Interpretation as Ramanujan summation

If $\sum_{n\geq1}^{\mathcal{R}}$ denotes the $\mathcal{R}$-sum of the series (i.e. the sum of the series in the sense of Ramanujan's summation method [3]), then, under certain appropriate conditions of growth and analyticity, we can make use of the following binomial formula [6, Eq. (10)]:

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\, n} \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} f(k) = \sum_{n\geq1}^{\mathcal{R}} \frac{f(n)}{n}\,. \tag{12}$$

This formula allows us to give an interesting interpretation in terms of Ramanujan summation of almost all the series identities mentioned above.

a) From the identities

$$\sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} = 1 \quad \text{and} \quad \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} \frac{k}{k+1} = \frac{1}{n+1}\,,$$

we derive respectively

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\, n} = \sum_{n\geq 1}^{\mathcal{R}} \frac{1}{n}\,, \tag{A}$$

and the shifted formula

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{(n+1)!\, n} = \sum_{n\geq 1}^{\mathcal{R}} \frac{1}{n+1}\,. \tag{B}$$

b) From the reciprocal binomial identities

$$\sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} H_k = \frac{1}{n} \quad \text{and} \quad \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} \frac{1}{k} = H_n\,,$$

we derive respectively the formulas

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\, n^2} = \sum_{n\geq 1}^{\mathcal{R}} \frac{H_n}{n}\,. \tag{C}$$

and

$$\sum_{n=1}^{\infty} \frac{\lambda_n H_n}{n!\, n} = \sum_{n\geq 1}^{\mathcal{R}} \frac{1}{n^2}\,, \tag{D}$$

c) From the reciprocal binomial identities

$$\sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} H_k^{(2)} = \frac{H_n}{n} \quad \text{and} \quad \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} \frac{H_k}{k} = H_n^{(2)}\,,$$

we derive respectively the formulas

$$\sum_{n=1}^{\infty} \frac{\lambda_n H_n}{n!\, n^2} = \sum_{n\geq 1}^{\mathcal{R}} \frac{H_n^{(2)}}{n}\,. \tag{E}$$

and

$$\sum_{n=1}^{\infty} \frac{\lambda_n H_n^{(2)}}{n!\, n} = \sum_{n\geq 1}^{\mathcal{R}} \frac{H_n}{n^2}\,, \tag{F}$$

Remark 3. a) Formulas (A) and (C) are two particular cases of the more general relation [6, Eq. (12)]:

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\, n^k} = \sum_{n\geq 1}^{\mathcal{R}} \frac{H_{n,k-1}}{n} \qquad (k \geq 1)\,.$$

b) Formulas (D) and (F) are two particular cases of the more general relation [6, Eq. (11)]:

$$\sum_{n=1}^{\infty} \frac{\lambda_n\, H_n^{(k)}}{n!\, n} = \sum_{n\geq 1}^{\mathcal{R}} \frac{H_{n,k-1}}{n^2} \qquad (k \geq 1)\,.$$

c) Formula (E) can be generalized by means of the generalized Roman harmonic numbers $H_{n,k}^{(r)}$ introduced in [6]. Indeed, by [6, Eq. (12)], we have

$$\sum_{n=1}^{\infty} \frac{\lambda_n\, H_n}{n!\, n^k} = \sum_{n\geq 1}^{\mathcal{R}} \frac{H_{n,k-1}^{(2)}}{n} \qquad (k \geq 2)\,.$$

d) From the the self-reciprocal binomial identity

$$\sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} \frac{H_k}{k+1} = \frac{H_n}{n+1}\,,$$

we deduce the formula

$$\sum_{n=1}^{\infty} \frac{\lambda_n\, H_n}{(n+1)!\, n} = \sum_{n\geq 1}^{\mathcal{R}} \frac{H_n}{n(n+1)}\,. \tag{G}$$

e) Let us now apply the formula (12) to the function

$$f(x) = \frac{\psi(x+1)+\gamma-1}{x(x-1)}\,,$$

which verifies

$$f(1) = \lim_{x\to 1} \frac{\psi(x+1)+\gamma-1}{x(x-1)} = \zeta(2)-1\,,$$

and

$$f(n) = \frac{H_n - 1}{n(n-1)} \qquad (n \geq 2)\,.$$

Then, by means of the binomial identity

$$\sum_{k=2}^{n} (-1)^{k-1} \binom{n}{k} \frac{1-H_k}{k-1} = nH_n^{(2)} - n$$

which is the reciprocal of the identity [2, Eq. (5.24)]:

$$\sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} kH_k^{(2)} = \frac{1-H_n}{n-1} \qquad (n \geq 2)\,,$$

we deduce the formula

$$\sum_{n=1}^{\infty} \frac{\lambda_n\, H_n^{(2)}}{n!} = \zeta(2) - \sum_{n\geq 1}^{\mathcal{R}} \frac{H_n - 1}{n(n-1)}\,. \tag{H}$$

Furthermore, subtracting (C) from (H), leads to the formula

$$\sum_{n=1}^{\infty} \frac{\lambda_{n+1}\, H_n^{(2)}}{(n+1)!} = \zeta(2) - \sum_{n\geq 1}^{\mathcal{R}} \frac{H_{n-1}}{n-1}\,. \tag{I}$$

Remark 4. Very recently, Young [9] has established the following relation:

$$\sum_{n=1}^{\infty} \frac{\lambda_n\, H_n^{(k+1)}}{n!} = \zeta(k+1) - \sum_{n\geq 1}^{\mathcal{R}} \frac{H_{n,k} - H_{n,k-1}}{n(n-1)} \qquad (k \geq 1)$$

which is a deep generalization of our formula (H).

References

[1] K. N. Boyadzhiev, New series identities with Cauchy, Stirling, and harmonic numbers, and Laguerre polynomials, *J. Integer Sequences* **23** (2020).

[2] K. N. Boyadzhiev, *Notes on the Binomial Transform, Theory and Table*, World Scientific, 2018.

[3] B. Candelpergher, *Ramanujan Summation of Divergent Series*, Lecture Notes in Math. 2185, Springer, 2017.

[4] B. Candelpergher, M-A. Coppo, A new class of identities involving Cauchy numbers, harmonic numbers and zeta values, *Ramanujan J.* **27** (2012), 305–328.

[5] M-A. Coppo, A note on some alternating series involving zeta and multiple zeta values, *J. Math. Anal. App.* **475** (2019), 1831–1841.

[6] M-A. Coppo, New identities involving Cauchy numbers, harmonic numbers and zeta values, *Results in Math.* **76** (2021).

[7] C. Jordan, Calculus of finite differencies, Chelsea, New York, 1965.

[8] J. Sesma, The Roman harmonic numbers revisited, *J. Number Theory* **180** (2017), 544–565.

[9] P. T. Young, Global series for height 1 multiple zeta functions, preprint, August 2022.