Miscellaneous series with Cauchy and harmonic numbers and their interpretation as Ramanujan summation

Marc-Antoine Coppo

Miscellaneous series with Cauchy and harmonic numbers and their interpretation as Ramanujan summation

Marc-Antoine Coppo*

Université Côte d'Azur, CNRS, LJAD (UMR 7351), Nice, France

Abstract We provide an overview of several series identities involving Cauchy numbers and harmonic numbers, all of which are closely linked to certain alternating series with zeta (or harmonic zeta) values; we then give, for each of them, their interpretation in terms of Ramanujan summation. We believe that this unusual interpretation of still little-known formulas should be useful for further research on the topic.

Keywords Cauchy numbers; harmonic numbers; binomial transform; series with zeta values; Ramanujan summation of series.

Mathematics Subject Classification (2020) 05A19, 11B75, 11M06, 40G99.

1 Introduction

A decade ago, we used Ramanujan's method of summation of series to generate a number of identities linking together Cauchy numbers (also known as Bernoulli numbers of the second kind), harmonic numbers, and values of the Riemann zeta function at positive integers [4]. Recently, we have refined this method by making a systematic use of a binomial transformation formula that relates the Cauchy numbers to the Ramanujan summation of series, and by the consideration of a natural generalization of the classical Roman harmonic numbers (see [6] for details). An alternative method to generate similar identities based on another binomial

*Corresponding author. *Email address:* `coppo@unice.fr`

transformation is presented in [1]. In the present article, we offer a comprehensive overview of these methods with the hope that the unusual but interesting interpretation of most of the formulas mentioned here in terms of Ramanujan summation (see Section 5) will inspire further research on the subject.

2 Reminder of some basic definitions

We first recall some basic facts about the Cauchy numbers (also known as Bernoulli numbers of the second kind) and introduce various types of harmonic numbers.

a) The non-alternating Cauchy numbers $\{\lambda_n\}_{n\geq 1}$ are defined explicitely by the formula

$$\lambda_n := \int_0^1 x(1-x)\cdots(n-1-x)\,dx\,.$$

Alternatively, they can be defined recursively by means of the relation

$$\sum_{k=1}^{n-1} \frac{\lambda_k}{k!\,(n-k)} = \frac{1}{n} \qquad (n\geq 2)\,.$$

The first ones are the following:

$$\lambda_1 = \frac{1}{2}\,,\ \lambda_2 = \frac{1}{6}\,,\ \lambda_3 = \frac{1}{4}\,,\ \lambda_4 = \frac{19}{30}\,,\ \lambda_5 = \frac{9}{4}\,,\ \lambda_6 = \frac{863}{84}\,,\ \text{etc.}$$

The non-alternating Cauchy numbers λ_n are closely linked to the Bernoulli numbers of the second kind b_n first introduced by Jordan [7] through the relation

$$\lambda_n = n!\,|b_n| \qquad (n\geq 1)\,.$$

Otherwise, with the current notation c_n used in [1], we have the simple relation

$$\lambda_n = (-1)^{n-1}\,c_n \qquad (n\geq 1)\,.$$

b) The classical harmonic numbers $\{H_n\}_{n\geq 1}$ are defined by

$$H_n = \sum_{j=1}^{n} \frac{1}{j} = \psi(n+1)+\gamma\,,$$

where ψ denotes the digamma function and $\gamma = -\psi(1)$ is the Euler constant.

c) For any integer $k\geq 2$, the generalized harmonic numbers $\{H_n^{(k)}\}_{n\geq 1}$ are defined by

$$H_n^{(k)} = \sum_{j=1}^{n} \frac{1}{j^k} = \frac{(-1)^{k-1}}{(k-1)!}\partial^{k-1}\psi(n+1)+\zeta(k)\,.$$

d) For any integer $k \geq 0$, the Roman harmonic numbers $\{H_{n,k}\}_{n\geq 1}$ are defined by

$$H_{n,0} = 1, \quad \text{and} \quad H_{n,k} = \sum_{n\geq j_1\geq\cdots\geq j_k\geq 1} \frac{1}{j_1\, j_2 \cdots j_k} \qquad (k \geq 1).$$

The Roman harmonic numbers can be expressed as polynomials in the generalized harmonic numbers $H_n, H_n^{(2)}, \cdots, H_n^{(k)}$ [4, Eq. (18)], [8, Eq. (29)]. More precisely, we have $H_{n,1} = H_n$, and

$$H_{n,k} = \frac{1}{k!}(H_n)^k + \cdots + \frac{1}{k}H_n^{(k)} = P_k(H_n, \cdots, H_n^{(k)}) \qquad (k \geq 2)\,,$$

where P_k are the modifed Bell polynomials [4, Definition 2]. In particular,

$$H_{n,2} = \sum_{k=1}^{n} \frac{H_k}{k} = \frac{1}{2}(H_n)^2 + \frac{1}{2}H_n^{(2)}\,.$$

3 Overview of some known formulas

In this section, we mention a number of more or less known identities and make some comments about them.

a) The formula

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\,n} = \sum_{n=2}^{\infty} \frac{(-1)^n}{n}\zeta(n) = \gamma \tag{1}$$

is a classical representation of γ due to Mascheroni and Euler which can be slightly modified as follows:

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{(n+1)!\,n} = \sum_{n=2}^{\infty} \frac{(-1)^n}{n}\,\{\zeta(n) - 1\} = \gamma + \log 2 - 1\,. \tag{2}$$

b) The formula

$$\sum_{n=1}^{\infty} \frac{\lambda_n\, H_n}{n!\,n} = \zeta(2) - 1 \tag{3}$$

is a fairly known representation of $\zeta(2) = \dfrac{\pi^2}{6}$ which is in fact a particular case of the more general formula

$$\sum_{n=1}^{\infty} \frac{\lambda_n\, H_{n,k}}{n!\,n} = \zeta(k+1) - \frac{1}{k} \qquad (k \geq 1)\,,$$

which is sometimes called *Hermite's formula* [4].

c) A non-trivial generalization of (1) is the following formula:

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\, n^2} = \sum_{n=2}^{\infty} \frac{(-1)^n}{n} \zeta(n,1) = \frac{1}{2}\gamma^2 + \frac{1}{2}\zeta(2) + \gamma_1 - \sum_{n=2}^{\infty} \frac{(-1)^n}{n} \zeta(n+1)\,, \tag{4}$$

where γ_1 denotes the first Stieltjes constant. This intriguing identity is already known and appears in [3], [5] and [6].

d) A non-trivial generalization of (3) consists of the following formula:

$$\sum_{n=1}^{\infty} \frac{\lambda_n\, H_n^{(2)}}{n!\, n} = \zeta(3) + \{\gamma + \log(2\pi) - 12\log(A)\}\,\zeta(2) + \sum_{n=2}^{\infty} \frac{(-1)^n}{n} \zeta(n+2) \tag{5}$$

with A denoting the Glaisher-Kinkelin constant

$$A = \lim_{n\to\infty} \left\{ n^{-\frac{n^2}{2} - \frac{n}{2} - \frac{1}{12}}\, e^{\frac{n^2}{4}} \prod_{k=1}^{n} k^k \right\} .$$

This identity results directly from [6, Eq. (19)] and the well-known relation:

$$\zeta'(2) = (\gamma + \log(2\pi) - 12\log(A))\,\zeta(2)\,.$$

Furthermore, formula (5) admits a kind of reciprocal which is given by [6, Eq. (16)] in the case $p = 2$. This is

$$\sum_{n=1}^{\infty} \frac{\lambda_n\, H_n}{n!\, n^2} = \zeta(2)\,(12\log(A) - \log(2\pi)) - 1 - \sum_{n=2}^{\infty} \frac{(-1)^n}{n} \zeta(n+2)\,. \tag{6}$$

Remark 1. When k is greater than 2, no explicit formula, even conjectural, appears to be known for the sum $\sum_{n=1}^{\infty} \frac{\lambda_n\, H_n}{n!\, n^k}$, nor for the sum $\sum_{n=1}^{\infty} \frac{\lambda_n\, H_n^{(k)}}{n!\, n}$ (see however Remark 4 below).

4 Additional formulas

The following binomial formula is nothing else than a variant of [1, Proposition 1] which is an elementary but powerful tool for obtaining several series identities with Cauchy numbers. For appropriate analytic functions f with moderate growth, we have the relation:

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!} \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} f(k) = \int_0^1 f(x)\, dx\,. \tag{7}$$

For instance, formulas (1) and (3) above can be easily derived from this formula using adequate functions f (see [1, Example 4]). We now give some new interesting formulas that can also be deduced by means of (7).

e) Applying (7) with $f(x) = \dfrac{\psi(x+1)+\gamma}{x+1}$, and using the binomial identity

$$\frac{H_n}{n+1} = \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} \frac{H_k}{k+1},$$

allows us to deduce the following identity:

$$\sum_{n=1}^{\infty} \frac{\lambda_n H_n}{(n+1)!} = \int_0^1 f(x)\,dx = \frac{1}{2}\zeta(2) + \log 2 - 1 + \sum_{n=3}^{\infty} \frac{(-1)^n}{n} \left\{ \sum_{k=2}^{n} (\zeta(k)-1) \right\} \tag{8}$$

which is a refinement of a formula previously given by Boyadzhiev [1, Example 5]. Moreover, substracting (8) from (3) leads to the following new formula:

$$\sum_{n=1}^{\infty} \frac{\lambda_n H_n}{(n+1)!\,n} = \frac{1}{2}\zeta(2) - \log 2 - \sum_{n=3}^{\infty} \frac{(-1)^n}{n} \left\{ \sum_{k=2}^{n} (\zeta(k)-1) \right\}. \tag{9}$$

We can notice that this last formula is a modification of (3) quite similar to (2).

f) Applying (7) with $f(x) = \dfrac{\psi(x+1)+\gamma}{x}$, and using the binomial identity

$$H_n^{(2)} = \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} \frac{H_k}{k},$$

leads to the formula:

$$\sum_{n=1}^{\infty} \frac{\lambda_n H_n^{(2)}}{n!} = \int_0^1 f(x)\,dx = \zeta(2) - \sum_{n=2}^{\infty} \frac{(-1)^n}{n} \zeta(n+1). \tag{10}$$

Moreover, writing $H_n^{(2)} = H_{n-1}^{(2)} + \dfrac{1}{n^2}$ and using (4) gives another interesting identity:

$$\sum_{n=1}^{\infty} \frac{\lambda_{n+1} H_n^{(2)}}{(n+1)!} = \frac{1}{2}\zeta(2) - \frac{1}{2}\gamma^2 - \gamma_1. \tag{11}$$

If ζ_H denotes the harmonic zeta function defined by

$$\zeta_H(s) = \sum_{n=1}^{\infty} \frac{H_n}{n^s} \qquad (\mathrm{Re}(s) > 1),$$

then, thanks to [5, Eq. (14)], this last identity may also be rewritten as follows:

$$\sum_{n=1}^{\infty} \frac{\lambda_{n+1} H_n^{(2)}}{(n+1)!} = \zeta(2) - \sum_{n=2}^{\infty} \frac{(-1)^n}{n} \zeta_H(n). \tag{11 bis}$$

Remark 2. Very recently, Young [9] has shown the following formula:

$$\sum_{n=1}^{\infty} \frac{\lambda_n H_n^{(k)}}{n!} = \zeta(k) - \sum_{n=2}^{\infty} \frac{(-1)^n}{n} \zeta(n+1, \{1\}^{k-2}) \qquad (k \geq 2)$$

that generalizes (10). It should be noted that, in contrast to the series $\sum_{n\geq 1} \frac{\lambda_n H_n^{(k)}}{n!}$, the series $\sum_{n\geq 1} \frac{\lambda_n H_n}{n!}$ is divergent since $\frac{\lambda_n H_n}{n!} \sim \frac{1}{n \log(n)}$, $n \to +\infty$.

5 Interpretation as Ramanujan summation

If $\sum_{n\geq 1}^{\mathcal{R}}$ denotes the $\mathcal{R}$-sum of the series (i.e. the sum of the series in the sense of Ramanujan's summation method [3]), then, under certain appropriate conditions of growth and analyticity, we can make use of the following binomial formula [6, Eq. (10)]:

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\, n} \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} k\, f(k) = \sum_{n\geq 1}^{\mathcal{R}} f(n)\,. \tag{12}$$

This formula allows us to give, for most of the series identities mentioned above, an interesting interpretation in terms of Ramanujan summation.

a) Thus, by means of the identities

$$1 = \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k}$$

and

$$\frac{1}{n+1} = \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} \frac{k}{k+1}\,,$$

we obtain respectively

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\, n} = \sum_{n\geq 1}^{\mathcal{R}} \frac{1}{n}\,, \tag{A}$$

and the shifted formula

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{(n+1)!\, n} = \sum_{n\geq 1}^{\mathcal{R}} \frac{1}{n+1}\,. \tag{B}$$

b) By means of the binomial identity

$$H_n = \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} \frac{1}{k} ,$$

we obtain

$$\sum_{n=1}^{\infty} \frac{\lambda_n H_n}{n!\, n} = \sum_{n \geq 1}^{\mathcal{R}} \frac{1}{n^2} , \tag{C}$$

and, by inversion of this identity, the formula

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\, n^2} = \sum_{n \geq 1}^{\mathcal{R}} \frac{H_n}{n} . \tag{D}$$

Remark 3. Formula (D) is a particular case of the more general relation [6, Eq. (12)]:

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\, n^k} = \sum_{n \geq 1}^{\mathcal{R}} \frac{H_{n,k-1}}{n} \qquad (k \geq 1) .$$

c) By means of the identity

$$H_n^{(2)} = \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} \frac{H_k}{k} ,$$

we obtain

$$\sum_{n=1}^{\infty} \frac{\lambda_n H_n^{(2)}}{n!\, n} = \sum_{n \geq 1}^{\mathcal{R}} \frac{H_n}{n^2} , \tag{E}$$

and, by inversion of this identity, the formula

$$\sum_{n=1}^{\infty} \frac{\lambda_n H_n}{n!\, n^2} = \sum_{n \geq 1}^{\mathcal{R}} \frac{H_n^{(2)}}{n} . \tag{F}$$

Remark 4. Formulas (C) and (E) are two particular cases of the more general relation [6, Eq. (11)]:

$$\sum_{n=1}^{\infty} \frac{\lambda_n H_n^{(k)}}{n!\, n} = \sum_{n \geq 1}^{\mathcal{R}} \frac{H_{n,k-1}}{n^2} \qquad (k \geq 1) .$$

d) By means of the binomial identity

$$\frac{H_n}{n+1} = \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} \frac{H_k}{k+1} ,$$

we obtain the self-reciprocal identity

$$\sum_{n=1}^{\infty} \frac{\lambda_n H_n}{(n+1)!\, n} = \sum_{n\geq 1}^{\mathcal{R}} \frac{H_n}{n(n+1)} \,. \tag{G}$$

e) In order to give an interpretation of the series $\sum_{n=1}^{\infty} \frac{\lambda_n H_n^{(2)}}{n!}$ in terms of Ramananujan summation, we apply formula (12) to the function

$$f(x) = \frac{\psi(x+1) + \gamma - 1}{x(x-1)} \,,$$

which verifies

$$f(1) = \lim_{x\to 1} \frac{\psi(x+1) + \gamma - 1}{x(x-1)} = \zeta(2) - 1 \,,$$

and

$$f(n) = \frac{H_n - 1}{n(n-1)} \qquad (n \geq 2) \,.$$

Then, by means of the binomial identity

$$nH_n^{(2)} = n + \sum_{k=2}^{n} (-1)^{k-1} \binom{n}{k} \frac{1 - H_k}{k-1}$$

obtained by inversion of the identity [2, Eq. (5.24)]:

$$\sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} k H_k^{(2)} = \frac{1 - H_n}{n-1} \qquad (n \geq 2) \,,$$

we deduce the formula

$$\sum_{n=1}^{\infty} \frac{\lambda_n H_n^{(2)}}{n!} = \zeta(2) - \sum_{n\geq 1}^{\mathcal{R}} \frac{H_n - 1}{n(n-1)} \,. \tag{H}$$

Remark 5. Very recently, Young [9] has established the following relation:

$$\sum_{n=1}^{\infty} \frac{\lambda_n H_n^{(k+1)}}{n!} = \zeta(k+1) - \sum_{n\geq 1}^{\mathcal{R}} \frac{H_{n,k} - H_{n,k-1}}{n(n-1)} \qquad (k \geq 1)$$

which is a deep generalization of our formula (H).

References

[1] K. N. Boyadzhiev, New series identities with Cauchy, Stirling, and harmonic numbers, and Laguerre polynomials, *J. Integer Sequences* **23** (2020).

[2] K. N. Boyadzhiev, *Notes on the Binomial Transform, Theory and Table*, World Scientific, 2018.

[3] B. Candelpergher, *Ramanujan Summation of Divergent Series*, Lecture Notes in Math. 2185, Springer, 2017.

[4] B. Candelpergher, M-A. Coppo, A new class of identities involving Cauchy numbers, harmonic numbers and zeta values, *Ramanujan J.* **27** (2012), 305–328.

[5] M-A. Coppo, A note on some alternating series involving zeta and multiple zeta values, *J. Math. Anal. App.* **475** (2019), 1831–1841.

[6] M-A. Coppo, New identities involving Cauchy numbers, harmonic numbers and zeta values, *Results in Math.* **76** (2021).

[7] C. Jordan, Calculus of finite differencies, Chelsea, New York, 1965.

[8] J. Sesma, The Roman harmonic numbers revisited, *J. Number Theory* **180** (2017), 544–565.

[9] P. T. Young, Global series for height 1 multiple zeta functions, preprint, August 2022.