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Miscellaneous series with Cauchy and harmonic numbers and their interpretation as Ramanujan summation

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Abstract We provide an overview of several series identities involving Cauchy numbers and harmonic numbers, all of which are closely linked to certain alternating series with zeta (or harmonic zeta) values; we then give, for each of them, their interpretation in terms of Ramanujan summation. We believe that this unusual interpretation of still little-known formulas should be useful for further research on the topic.

Keywords Cauchy numbers; harmonic numbers; binomial identities; series with zeta values; Ramanujan summation of series.

Mathematics Subject Classification (2020) 05A19, 11B75, 11M06, 40G99.

1 Introduction

A decade ago, we used Ramanujan's method of summation of series to generate a number of identities linking together Cauchy numbers (also known as Bernoulli numbers of the second kind), harmonic numbers, and values of the Riemann zeta function at positive integers [4]. Recently, we were able to improve this method by making a systematic use of a general transformation formula that relates the Cauchy numbers to the Ramanujan summation of series, in combination with the consideration of a natural generalization of the Roman harmonic numbers (see [6] for details). In the present article, we offer a comprehensive overview of our methods with the hope that the unusual but interesting interpretation in terms of Ramanujan summation of most of the basic formulas given here will inspire further research on the subject.

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2 Reminder of some basic definitions

We first recall some basic facts about the Cauchy numbers (also known as Bernoulli numbers of the second kind) and introduce various types of harmonic numbers.

a) The non-alternating Cauchy numbers $\{\lambda_n\}_{n\geq 1}$ are defined explicitely by the formula

$$\lambda_n := \int_0^1 x(1-x)\cdots(n-1-x)\,dx\,.$$

Alternatively, they can be defined recursively by means of the relation

$$\sum_{k=1}^{n-1} \frac{\lambda_k}{k!\,(n-k)} = \frac{1}{n} \qquad (n\geq 2)\,.$$

The first ones are the following:

$$\lambda_1 = \frac{1}{2}\,,\ \lambda_2 = \frac{1}{6}\,,\ \lambda_3 = \frac{1}{4}\,,\ \lambda_4 = \frac{19}{30}\,,\ \lambda_5 = \frac{9}{4}\,,\ \lambda_6 = \frac{863}{84}\,,\ \text{etc.}$$

The non-alternating Cauchy numbers λ_n are closely linked to the Bernoulli numbers of the second kind b_n first introduced by Jordan [7] through the relation

$$\lambda_n = n!\,|b_n| \quad (n\geq 1)\,.$$

Otherwise, with the current notation c_n used in [1], we have the simple relation

$$\lambda_n = (-1)^{n-1}\,c_n \quad (n\geq 1)\,.$$

b) The classical harmonic numbers $\{H_n\}_{n\geq 1}$ are defined by

$$H_n = \sum_{j=1}^{n} \frac{1}{j} = \psi(n+1)+\gamma\,,$$

where ψ denotes the digamma function and $\gamma = -\psi(1)$ is the Euler constant.

c) For any integer $k\geq 2$, the generalized harmonic numbers $\{H_n^{(k)}\}_{n\geq 1}$ are defined by

$$H_n^{(k)} = \sum_{j=1}^{n} \frac{1}{j^k} = \frac{(-1)^{k-1}}{(k-1)!}\partial^{k-1}\psi(n+1)+\zeta(k)\,.$$

d) For any integer $k\geq 0$, the (ordinary) Roman harmonic numbers $\{H_{n,k}\}_{n\geq 1}$ are defined by

$$H_{n,0} = 1, \quad \text{and} \quad H_{n,k} = \sum_{n\geq j_1\geq\cdots\geq j_k\geq 1} \frac{1}{j_1\,j_2\cdots j_k} \quad \text{for } k\geq 1.$$

The Roman harmonic numbers can be expressed as polynomials in the generalized harmonic numbers $H_n, H_n^{(2)}, \cdots, H_n^{(k)}$ [4, Eq. (18)]. More precisely, we have

$$H_{n,1} = H_n \quad \text{and} \quad H_{n,k} = \frac{1}{k!}(H_n)^k + \cdots + \frac{1}{k}H_n^{(k)} = P_k(H_n, \cdots, H_n^{(k)}) \qquad (k \geq 2)\,,$$

where P_k are the modifed Bell polynomials [4, Definition 2]. In particular,

$$H_{n,2} = \frac{1}{2}(H_n)^2 + \frac{1}{2}H_n^{(2)}\,,$$

and

$$H_{n,3} = \frac{1}{6}(H_n)^3 + \frac{1}{2}H_n H_n^{(2)} + \frac{1}{3}H_n^{(3)}\,.$$

3 Overview of some known formulas

In this section, we mention a number of more or less known identities with some comments about them.

a) The formula

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\,n} = \sum_{n=2}^{\infty} \frac{(-1)^n}{n}\zeta(n) = \gamma \tag{1}$$

is a classical representation of γ due to Mascheroni and Euler which can be slightly modified as follows:

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{(n+1)!\,n} = \sum_{n=2}^{\infty} \frac{(-1)^n}{n}\,\{\zeta(n) - 1\} = \gamma + \log 2 - 1\,. \tag{2}$$

b) The formula

$$\sum_{n=1}^{\infty} \frac{\lambda_n\, H_n}{n!\,n} = \zeta(2) - 1 \tag{3}$$

is a fairly known representation of $\zeta(2) = \frac{\pi^2}{6}$ which is in fact a particular case of the more general formula

$$\sum_{n=1}^{\infty} \frac{\lambda_n\, H_{n,k}}{n!\,n} = \zeta(k+1) - \frac{1}{k} \qquad (k \geq 1)\,,$$

which is sometimes called *Hermite's formula* [4].

c) A non-trivial generalization of (1) is the following formula:

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\, n^2} = \frac{1}{2}\gamma^2 + \frac{1}{2}\zeta(2) + \gamma_1 - \sum_{n=2}^{\infty} \frac{(-1)^n}{n}\zeta(n+1)\,, \tag{4}$$

where γ_1 denotes the first Stieltjes constant. This interesting identity is already known and appears in [3], [5] and [6].

d) A non-trivial generalization of (3) consists of the following formula:

$$\sum_{n=1}^{\infty} \frac{\lambda_n\, H_n^{(2)}}{n!\, n} = \zeta(3) + \{\gamma + \log(2\pi) - 12\log(A)\}\,\zeta(2) + \sum_{n=2}^{\infty} \frac{(-1)^n}{n}\zeta(n+2) \tag{5}$$

where A is the Glaisher-Kinkelin constant. This identity results directly from [6, Eq. (19)] and the well-known relation:

$$\zeta'(2) = (\gamma + \log(2\pi) - 12\log(A))\,\zeta(2)\,.$$

Furthermore, formula (5) admits a kind of reciprocal given by [6, Eq. (16)] in the case $p = 2$:

$$\sum_{n=1}^{\infty} \frac{\lambda_n\, H_n}{n!\, n^2} = \zeta(2)\,(12\log(A) - \log(2\pi)) - 1 - \sum_{n=2}^{\infty} \frac{(-1)^n}{n}\zeta(n+2)\,. \tag{6}$$

Remark 1. No explicit formula, even conjectural, appears to be known for the sum $\sum_{n=1}^{\infty} \frac{\lambda_n\, H_n^{(k)}}{n!\, n}$ for $k > 2$ (see however Remark 4 below), nor for the sum $\sum_{n=1}^{\infty} \frac{\lambda_n\, H_n}{n!\, n^k}$ for $k > 2$.

4 Additional formulas

We now give some new formulas deduced from a variant of [1, Proposition 1] which is an elementary but powerful tool for obtaining several series identities with Cauchy numbers. For appropriate analytic functions f with moderate growth, we have the following relation:

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!} \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} f(k) = \int_0^1 f(x)\, dx\,. \tag{7}$$

e) Applying (7) with $f(x) = \dfrac{\psi(x+1) + \gamma}{x}$, and using the binomial identity

$$H_n^{(2)} = \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} \frac{H_k}{k}\,,$$

we obtain the formula:

$$\sum_{n=1}^{\infty} \frac{\lambda_n H_n^{(2)}}{n!} = \int_0^1 f(x)\, dx = \zeta(2) - \sum_{n=2}^{\infty} \frac{(-1)^n}{n} \zeta(n+1)\,. \tag{8}$$

Moreover, writing $H_n^{(2)} = H_{n-1}^{(2)} + \dfrac{1}{n^2}$ and using (4) leads us to another new identity:

$$\sum_{n=1}^{\infty} \frac{\lambda_{n+1} H_n^{(2)}}{(n+1)!} = \frac{1}{2}\zeta(2) - \frac{1}{2}\gamma^2 - \gamma_1\,. \tag{9}$$

As an interesting consequence of [5, Eq. (14)], this last identity may also be rewritten as follows:

$$\sum_{n=1}^{\infty} \frac{\lambda_{n+1} H_n^{(2)}}{(n+1)!} = \zeta(2) - \sum_{n=2}^{\infty} \frac{(-1)^n}{n} \zeta_H(n)\,, \tag{8 bis}$$

where ζ_H denotes the harmonic zeta function defined by

$$\zeta_H(s) = \sum_{n=1}^{\infty} \frac{H_n}{n^s} \qquad (\mathrm{Re}(s) > 1)\,.$$

Remark 2. It should be noted that, in contrast to the series $\sum_{n\geq 1} \dfrac{\lambda_n H_n^{(k)}}{n!}$ for $k \geq 2$, the series $\sum_{n\geq 1} \dfrac{\lambda_n H_n}{n!}$ is divergent since $\dfrac{\lambda_n H_n}{n!} \sim \dfrac{1}{n \log(n)}$, $n \to +\infty$.

f) Applying (7) with $f(x) = \dfrac{\psi(x+1)+\gamma}{x+1}$, and using the binomial identity

$$\frac{H_n}{n+1} = \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} \frac{H_k}{k+1}\,,$$

allows us to deduce the following identity:

$$\sum_{n=1}^{\infty} \frac{\lambda_n H_n}{(n+1)!} = \int_0^1 f(x)\, dx = \frac{1}{2}\zeta(2) + \log 2 - 1 + \sum_{n=3}^{\infty} \frac{(-1)^n}{n} \left\{ \sum_{k=2}^{n} (\zeta(k) - 1) \right\} \tag{10}$$

which is a refinement of a formula previously given by Boyadzhiev [1, Example 5]. Moreover, substracting (10) from (3) leads to the following new formula:

$$\sum_{n=1}^{\infty} \frac{\lambda_n H_n}{(n+1)!\, n} = \frac{1}{2}\zeta(2) - \log 2 - \sum_{n=3}^{\infty} \frac{(-1)^n}{n} \left\{ \sum_{k=2}^{n} (\zeta(k) - 1) \right\}\,. \tag{11}$$

It should be noted that this last formula is a modification of (3) quite similar to (2).

5 Interpretation as Ramanujan summation

If $\sum_{n\geq 1}^{\mathcal{R}}$ denotes the $\mathcal{R}$-sum of the series (i.e. the sum of the series in the sense of Ramanujan's summation method [3]), then, under certain appropriate conditions of growth and analyticity, we can make use of the following formula [6, Eq. (10)]:

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\, n} \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} k\, f(k) = \sum_{n\geq 1}^{\mathcal{R}} f(n)\,. \tag{12}$$

This formula allows us to give, for each of the series identities mentioned above, an interesting interpretation in terms of Ramanujan summation.

a) Thus, by means of the identities

$$1 = \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k}$$

and

$$\frac{1}{n+1} = \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} \frac{k}{k+1}\,,$$

we obtain respectively

$$\boxed{\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\, n} = \sum_{n\geq 1}^{\mathcal{R}} \frac{1}{n}} \tag{A}$$

and the shifted formula

$$\boxed{\sum_{n=1}^{\infty} \frac{\lambda_n}{(n+1)!\, n} = \sum_{n\geq 1}^{\mathcal{R}} \frac{1}{n+1}} \tag{B}$$

b) By means of the binomial identity

$$H_n = \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} \frac{1}{k}\,,$$

we obtain

$$\boxed{\sum_{n=1}^{\infty} \frac{\lambda_n H_n}{n!\, n} = \sum_{n\geq 1}^{\mathcal{R}} \frac{1}{n^2}} \tag{C}$$

and, by inversion of this identity,

$$\boxed{\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\, n^2} = \sum_{n\geq 1}^{\mathcal{R}} \frac{H_n}{n}} \tag{D}$$

Remark 3. Formula (D) is a particular case of the more general formula [6, Eq. (12)]:

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\, n^k} = \sum_{n\geq 1}^{\mathcal{R}} \frac{H_{n,k-1}}{n} \qquad (k \geq 1)\,.$$

c) By means of the identity

$$H_n^{(2)} = \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} \frac{H_k}{k}\,,$$

we obtain

$$\boxed{\sum_{n=1}^{\infty} \frac{\lambda_n\, H_n^{(2)}}{n!\, n} = \sum_{n\geq 1}^{\mathcal{R}} \frac{H_n}{n^2}} \tag{E}$$

and, by inversion of this identity,

$$\boxed{\sum_{n=1}^{\infty} \frac{\lambda_n\, H_n}{n!\, n^2} = \sum_{n\geq 1}^{\mathcal{R}} \frac{H_n^{(2)}}{n}} \tag{F}$$

Remark 4. Formulas (C) and (E) are two particular cases of the more general formula [6, Eq. (11)]:

$$\sum_{n=1}^{\infty} \frac{\lambda_n\, H_n^{(k)}}{n!\, n} = \sum_{n\geq 1}^{\mathcal{R}} \frac{H_{n,k-1}}{n^2} \qquad (k \geq 1)\,.$$

d) By means of the binomial identity

$$\frac{H_n}{n+1} = \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} \frac{H_k}{k+1}\,,$$

we obtain the self-reciprocal identity

$$\boxed{\sum_{n=1}^{\infty} \frac{\lambda_n\, H_n}{(n+1)!\, n} = \sum_{n\geq 1}^{\mathcal{R}} \frac{H_n}{n(n+1)}} \tag{G}$$

e) In order to give an interpretation of the series $\sum_{n=1}^{\infty} \frac{\lambda_n\, H_n^{(2)}}{n!}$ in terms of Ramananujan summation, we can make use of the binomial identity

$$nH_n^{(2)} = n + \sum_{k=2}^{n} (-1)^{k-1} \binom{n}{k} \frac{1-H_k}{k-1} \tag{13}$$

which is obtained by inversion of the identity [2, Eq. (5.24)]:

$$\sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}kH_k^{(2)} = \frac{1-H_n}{n-1} \qquad \text{for } n \geq 2\,.$$

Applying (12) to the function

$$f(x) = \frac{\psi(x+1)+\gamma-1}{x(x-1)}\,,$$

which verifies

$$f(1) = \lim_{x\to 1}\frac{\psi(x+1)+\gamma-1}{x(x-1)} = \zeta(2)-1\,,$$

and

$$f(n) = \frac{H_n-1}{n(n-1)} \qquad \text{for } n \geq 2\,,$$

by means of the binomial identity (13), we then deduce the formula

$$\boxed{\sum_{n=1}^{\infty}\frac{\lambda_n\, H_n^{(2)}}{n!} = \zeta(2) - \sum_{n\geq 1}^{\mathcal{R}}\frac{H_n-1}{n(n-1)}} \tag{H}$$

Remark 5. Very recently, Young [8] has established the following relation:

$$\sum_{n=1}^{\infty}\frac{\lambda_n\, H_n^{(k+1)}}{n!} = \zeta(k+1) - \sum_{n\geq 1}^{\mathcal{R}}\frac{H_{n,k}-H_{n,k-1}}{n(n-1)} \qquad (k\geq 1)$$

which is a deep generalization of our formula (H).

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