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Abstract We provide an overview of several series identities involving Cauchy numbers and harmonic numbers, all of them closely related to certain alternating series with zeta (or harmonic zeta) values, and then give, for each of them, their interpretation in terms of Ramanujan summation. We believe that this unusual interpretation of still little known formulas should be useful for further research on the topic.

1 Reminder of some basic definitions

We first recall some basic facts about the Cauchy numbers (also known as Bernoulli numbers of the second kind) and introduce various types of harmonic numbers.

a) The non-alternating Cauchy numbers $\{\lambda_n\}_{n\geq 1}$ are defined explicitely by the formula

$$\lambda_n := \int_0^1 x(1-x)\cdots(n-1-x)\,dx\,.$$

Alternatively, they can be defined recursively by means of the relation

$$\sum_{k=1}^{n-1} \frac{\lambda_k}{k!\,(n-k)} = \frac{1}{n} \qquad (n\geq 2)\,.$$

The first ones are the following:

$$\lambda_1 = \frac{1}{2}\,,\ \lambda_2 = \frac{1}{6}\,,\ \lambda_3 = \frac{1}{4}\,,\ \lambda_4 = \frac{19}{30}\,,\ \lambda_5 = \frac{9}{4}\,,\ \lambda_6 = \frac{863}{84}\,,\ \text{etc.}$$

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The non-alternating Cauchy numbers λ_n are closely linked to the Bernoulli numbers of the second kind b_n through the relation

$$\lambda_n = n!\,|b_n| \quad (n \geq 1)\,.$$

Otherwise, with the current notation c_n used in [1], we simply have

$$\lambda_n = (-1)^{n-1}\,c_n \quad (n \geq 1)\,.$$

b) The classical harmonic numbers $\{H_n\}_{n\geq 1}$ are defined by

$$H_n = \sum_{j=1}^{n} \frac{1}{j} = \psi(n+1) + \gamma\,,$$

where ψ denotes the digamma function and $\gamma = -\psi(1)$ is the Euler constant.

c) For any integer $k \geq 2$, the generalized harmonic numbers $\{H_n^{(k)}\}_{n\geq 1}$ are defined by

$$H_n^{(k)} = \sum_{j=1}^{n} \frac{1}{j^k} = \frac{(-1)^{k-1}}{(k-1)!}\partial^{k-1}\psi(n+1) + \zeta(k)\,.$$

d) For any integer $k \geq 0$, the (ordinary) Roman harmonic numbers $\{H_{n,k}\}_{n\geq 1}$ are defined by

$$H_{n,0} = 1, \quad \text{and} \quad H_{n,k} = \sum_{n\geq j_1\geq\cdots\geq j_k\geq 1} \frac{1}{j_1\, j_2 \cdots j_k} \quad \text{for } k \geq 1.$$

The Roman harmonic numbers can be expressed as polynomials in the generalized harmonic numbers $H_n, H_n^{(2)}, \cdots, H_n^{(k)}$. More precisely, $H_{n,1} = H_n$, and

$$H_{n,k} = \frac{1}{k!}(H_n)^k + \cdots + \frac{1}{k}H_n^{(k)} = P_k(H_n, \cdots, H_n^{(k)}) \qquad (k \geq 2)\,,$$

where P_k are the modifed Bell polynomials (cf. [4, Eq. (18)]). In particular, we have

$$\begin{aligned} H_{n,2} &= \frac{1}{2}(H_n)^2 + \frac{1}{2}H_n^{(2)}\,, \\ H_{n,3} &= \frac{1}{6}(H_n)^3 + \frac{1}{2}H_n H_n^{(2)} + \frac{1}{3}H_n^{(3)}\,, \end{aligned}$$

etc.

2 Overview of some known formulas

In this section, we review a number of more or less well-known identities with some comments.

a) The formula

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\,n} = \gamma = \sum_{n=2}^{\infty} \frac{(-1)^n}{n}\zeta(n) \tag{1}$$

is a classical representation of γ due to Mascheroni (for the first equality) and Euler (for the second) which can be slightly modified as follows:

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{(n+1)!\,n} = \gamma + \log 2 - 1 = \sum_{n=2}^{\infty} \frac{(-1)^n}{n} \{\zeta(n) - 1\} \ . \tag{2}$$

b) The formula

$$\sum_{n=1}^{\infty} \frac{\lambda_n\, H_n}{n!\,n} = \zeta(2) - 1 \tag{3}$$

is a fairly known representation of $\zeta(2) = \dfrac{\pi^2}{6}$ which is in fact a particular case of the more general formula

$$\sum_{n=1}^{\infty} \frac{\lambda_n\, H_{n,k}}{n!\,n} = \zeta(k+1) - \frac{1}{k} \qquad (k \geq 1)\,,$$

sometimes called *Hermite's formula* (cf. [4]).

c) A non-trivial generalization of (1) consists of the following formula:

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\,n^2} = \frac{1}{2}\gamma^2 + \frac{1}{2}\zeta(2) + \gamma_1 - \sum_{n=2}^{\infty} \frac{(-1)^n}{n}\zeta(n+1)\,, \tag{4}$$

where γ_1 denotes the first Stieltjes constant. This notable identity is already known (cf. [6]).

d) A non-trivial generalization of (3) consists of the following formula:

$$\sum_{n=1}^{\infty} \frac{\lambda_n\, H_n^{(2)}}{n!\,n} = \zeta(3) + \{\gamma + \log(2\pi) - 12\log(A)\}\,\zeta(2) + \sum_{n=2}^{\infty} \frac{(-1)^n}{n}\zeta(n+2) \tag{5}$$

where A is the Glaisher-Kinkelin constant. This identity results directly from [6, Eq. (19)] and the well-known relation:

$$\zeta'(2) = (\gamma + \log(2\pi) - 12\log(A))\,\zeta(2)\,.$$

Remark 1. No explicit formula, even conjectural, appears to be known for the sum $\displaystyle\sum_{n=1}^{\infty} \frac{\lambda_n\, H_n^{(k)}}{n!\,n}$ for $k > 2$ (see however Remark 4 below).

3 Two additional formulas

We now give two new formulas.

a) Applying [1, Proposition 1] with $f(x) = \dfrac{\psi(x+1)+\gamma}{x}$, and using the binomial identity

$$H_n^{(2)} = \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} \frac{H_k}{k},$$

leads to this new formula:

$$\sum_{n=1}^{\infty} \frac{\lambda_n \, H_n^{(2)}}{n!} = \zeta(2) - \sum_{n=2}^{\infty} \frac{(-1)^n}{n} \zeta(n+1) \,. \tag{6}$$

Moreover, writing $H_n^{(2)} = H_{n-1}^{(2)} + \dfrac{1}{n^2}$ and using (4) allows us to deduce another interesting identity:

$$\sum_{n=1}^{\infty} \frac{\lambda_{n+1} \, H_n^{(2)}}{(n+1)!} = \frac{1}{2}\zeta(2) - \frac{1}{2}\gamma^2 - \gamma_1 \,. \tag{7}$$

As a consequence of [5, Eq. (14)]), this last identity may be rewritten as follows:

$$\sum_{n=1}^{\infty} \frac{\lambda_{n+1} \, H_n^{(2)}}{(n+1)!} = \zeta(2) - \sum_{n=2}^{\infty} \frac{(-1)^n}{n} \zeta_H(n) \,, \tag{7 bis}$$

where ζ_H denotes the harmonic zeta function defined by

$$\zeta_H(s) = \sum_{n=1}^{\infty} \frac{H_n}{n^s} \qquad (\mathrm{Re}(s) > 1) \,.$$

Remark 2. It should be noted that, in contrast to series $\sum_{n\geq 1} \dfrac{\lambda_n \, H_n^{(k)}}{n!}$ for $k \geq 2$, $\sum_{n\geq 1} \dfrac{\lambda_n \, H_n}{n!}$ is a divergent series since $\dfrac{\lambda_n \, H_n}{n!} \sim \dfrac{1}{n \log(n)}$, $n \to +\infty$.

b) Applying [1, Proposition 1] with $f(x) = \dfrac{\psi(x+1)+\gamma}{x+1}$ leads to the identity

$$\sum_{n=1}^{\infty} \frac{\lambda_n H_n}{(n+1)!} = \frac{1}{2}\zeta(2) + \log 2 - 1 + \sum_{n=3}^{\infty} \frac{(-1)^n}{n} \left\{ \sum_{k=2}^{n} (\zeta(k) - 1) \right\} \tag{8}$$

which is a refinement of a formula previously given by Boyadzhiev ([1, Example 5]). Substracting (8) from (3) allows us to write the following formula:

$$\sum_{n=1}^{\infty} \frac{\lambda_n \, H_n}{(n+1)! \, n} = \frac{1}{2}\zeta(2) - \log 2 - \sum_{n=3}^{\infty} \frac{(-1)^n}{n} \left\{ \sum_{k=2}^{n} (\zeta(k) - 1) \right\} \tag{9}$$

which is a modification of (3) quite similar to (2).

4 Interpretation as Ramanujan summation

If $\sum_{n\geq 1}^{\mathcal{R}}$ denotes the $\mathcal{R}$-sum of the series (i.e. the sum of the series in the sense of Ramanujan's summation method), then, under certain appropriate conditions of growth and analyticity, we can make use of the transformation formula (cf. [3, Theorem 18]) to write the formula

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\,n} \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} k\, f(k) = \sum_{n\geq 1}^{\mathcal{R}} f(n)\,, \tag{10}$$

(cf. [6, Eq. (10)]). This enables us to give an interesting interpretation of each of the previous series identities in terms of Ramanujan summation.

a) Thus, by means of the identities

$$1 = \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k}$$

and

$$\frac{1}{n+1} = \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} \frac{k}{k+1}\,,$$

we obtain respectively

$$\boxed{\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\,n} = \sum_{n\geq 1}^{\mathcal{R}} \frac{1}{n}} \tag{A}$$

and the shifted formula

$$\boxed{\sum_{n=1}^{\infty} \frac{\lambda_n}{(n+1)!\,n} = \sum_{n\geq 1}^{\mathcal{R}} \frac{1}{n+1}} \tag{B}$$

b) By means of the binomial identity

$$H_n = \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} \frac{1}{k}\,,$$

we obtain

$$\boxed{\sum_{n=1}^{\infty} \frac{\lambda_n H_n}{n!\,n} = \sum_{n\geq 1}^{\mathcal{R}} \frac{1}{n^2}} \tag{C}$$

and, by inversion of this identity,

$$\boxed{\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\,n^2} = \sum_{n\geq 1}^{\mathcal{R}} \frac{H_n}{n}} \tag{D}$$

Remark 3. Formula (D) is a particular case of the more general formula (cf. [6, Eq. (12)]):

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\, n^k} = \sum_{n\geq 1}^{\mathcal{R}} \frac{H_{n,k-1}}{n} \qquad (k \geq 1)\,.$$

c) By means of the identity

$$H_n^{(2)} = \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} \frac{H_k}{k}\,,$$

we obtain

$$\boxed{\sum_{n=1}^{\infty} \frac{\lambda_n\, H_n^{(2)}}{n!\, n} = \sum_{n\geq 1}^{\mathcal{R}} \frac{H_n}{n^2}} \tag{E}$$

and, by inversion of this identity,

$$\boxed{\sum_{n=1}^{\infty} \frac{\lambda_n\, H_n}{n!\, n^2} = \sum_{n\geq 1}^{\mathcal{R}} \frac{H_n^{(2)}}{n}} \tag{F}$$

Remark 4. Formulas (C) and (E) are two particular cases of the more general formula (cf. [6, Eq. (11)]):

$$\sum_{n=1}^{\infty} \frac{\lambda_n\, H_n^{(k)}}{n!\, n} = \sum_{n\geq 1}^{\mathcal{R}} \frac{H_{n,k-1}}{n^2} \qquad (k \geq 1)\,.$$

d) By means of the binomial identity

$$\frac{H_n}{n+1} = \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} \frac{H_k}{k+1}\,,$$

we obtain the self-reciprocal identity

$$\boxed{\sum_{n=1}^{\infty} \frac{\lambda_n\, H_n}{(n+1)!\, n} = \sum_{n\geq 1}^{\mathcal{R}} \frac{H_n}{n(n+1)}} \tag{G}$$

e) In order to give an interpretation of the series $\displaystyle\sum_{n=1}^{\infty} \frac{\lambda_n\, H_n^{(2)}}{n!}$ in terms of Ramananujan summation, we can make use of the binomial identity

$$nH_n^{(2)} = n + \sum_{k=2}^{n} (-1)^{k-1} \binom{n}{k} \frac{1-H_k}{k-1} \tag{11}$$

which is obtained by inversion of the identity ([2, Eq. (5.24)]):

$$\sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}kH_k^{(2)}=\frac{1-H_n}{n-1}\qquad\text{for } n\geq 2\,.$$

Applying (10) to the function

$$f(x)=\frac{\psi(x+1)+\gamma-1}{x(x-1)}\,,$$

which verifies

$$f(1)=\lim_{x\to 1}\frac{\psi(x+1)+\gamma-1}{x(x-1)}=\zeta(2)-1\,,$$

and

$$f(n)=\frac{H_n-1}{n(n-1)}\qquad\text{for } n\geq 2\,,$$

we deduce, by means of the binomial identity (11), the formula

$$\boxed{\sum_{n=1}^{\infty}\frac{\lambda_n\,H_n^{(2)}}{n!}=\zeta(2)-\sum_{n\geq 1}^{\mathcal{R}}\frac{H_n-1}{n(n-1)}}\tag{H}$$

Remark 5. Very recently, Young [7] has established the following identity:

$$\sum_{n=1}^{\infty}\frac{\lambda_n\,H_n^{(k+1)}}{n!}=\zeta(k+1)-\sum_{n\geq 1}^{\mathcal{R}}\frac{H_{n,k}-H_{n,k-1}}{n(n-1)}\qquad(k\geq 1)$$

which is a deep generalization of our formula (H).

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