Miscellaneous series identities with Cauchy and harmonic numbers, and their interpretation as Ramanujan summation

Marc-Antoine Coppo

Miscellaneous series identities with Cauchy and harmonic numbers, and their interpretation as Ramanujan summation

Marc-Antoine Coppo*

Université Côte d'Azur, CNRS, LJAD (UMR 7351), Nice, France

Integers, **23** (2023)

Abstract We provide an overview of several series identities involving the Cauchy numbers and various types of harmonic numbers, all of which are closely related to certain alternating series with zeta values (or harmonic zeta values). We then give, for each of these identities, an interpretation in terms of Ramanujan summation with the hope that this unusual but interesting interpretation of still little-known formulas could inspire further research on the topic.

Keywords Cauchy numbers; harmonic numbers; binomial identities; series with zeta values; Ramanujan summation of series.

Mathematics Subject Classification (2020) 05A19, 11B75, 11M06, 40G99.

1 Introduction

A decade ago, we showed how Ramanujan's method of summation of series could be useful to generate a number of identities linking together Cauchy numbers (also known as Bernoulli numbers of the second kind), harmonic numbers, and special values of the Riemann zeta function at positive integers [4]. This powerful method is based on a binomial transformation formula that relates the Cauchy numbers to the Ramanujan summation of series [3, Theorem 18]. More recently, new series identities were obtained by a refinement of the same method, thanks notably to the consideration of a rather natural generalization of the harmonic numbers and

*Corresponding author. *Email address:* `coppo@unice.fr`

new binomial identities (see [8] for details). Some special cases of these results are discussed in Section 3. On the other hand, an alternative efficient method for generating such identities, based on a similar binomial transformation formula, is presented in Section 4. Combining these two methods and reversing the procedure enables us to provide an interesting interpretation in terms of Ramanujan summation of almost all the identities mentioned above, as explained in Section 5.

2 Reminder of some basic definitions

We first recall some basic facts about the Cauchy numbers [1, 2, 4, 10]. We then introduce various types of harmonic numbers using the notations in [8].

If $s(n,k)$ denotes the (signed) Stirling numbers of the first kind, the non-alternating Cauchy numbers $\{\lambda_n\}_{n\geq 1}$ can be defined explicitly as follows:

$$\lambda_n := \left|\sum_{k=1}^{n} \frac{s(n,k)}{k+1}\right| \qquad (n \geq 1)\,.$$

Alternatively, they can also be defined recursively by means of the relation

$$\sum_{k=1}^{n-1} \frac{\lambda_k}{k!\,(n-k)} = \frac{1}{n} \qquad (n \geq 2)\,.$$

The first ones are

$$\lambda_1 = \frac{1}{2}\,,\ \lambda_2 = \frac{1}{6}\,,\ \lambda_3 = \frac{1}{4}\,,\ \lambda_4 = \frac{19}{30}\,,\ \lambda_5 = \frac{9}{4}\,,\ \lambda_6 = \frac{863}{84}\,,\ \text{etc.}$$

The non-alternating Cauchy numbers λ_n are closely linked to the Bernoulli numbers of the second kind b_n (first introduced by Jordan [9]) through the relation

$$\lambda_n = (-1)^{n-1} n!\, b_n = n!\, |b_n| \qquad (n \geq 1)\,.$$

Otherwise, the exponential generating function of the non-alternating Cauchy numbers is given by

$$\sum_{n=1}^{\infty} \lambda_n \frac{x^n}{n!} = 1 + \frac{x}{\ln(1-x)} \qquad (|x| < 1)\,.$$

In particular, the series $\sum_{n\geq 1} \frac{\lambda_n}{n!}$ converges to 1.

The classical harmonic numbers $\{H_n\}_{n\geq 1}$ are defined by

$$H_n = \sum_{j=1}^{n} \frac{1}{j} = \psi(n+1) + \gamma\,,$$

where $\psi = \Gamma'/\Gamma$ denotes the digamma function and $\gamma = -\psi(1)$ is the Euler constant [5].

For any integer $k \geq 1$, the generalized harmonic numbers $\{H_n^{(k)}\}_{n\geq 1}$ are defined by $H_n^{(1)} = H_n$, and

$$H_n^{(k)} = \sum_{j=1}^{n} \frac{1}{j^k} = \frac{(-1)^{k-1}}{(k-1)!}\partial^{k-1}\psi(n+1) + \zeta(k) \qquad (k \geq 2)\,.$$

For any integer $k \geq 0$, the Roman harmonic numbers $\{H_{n,k}\}_{n\geq 1}$ are defined by $H_{n,0} = 1$, and

$$H_{n,k} = \sum_{n\geq j_1 \geq \cdots \geq j_k \geq 1} \frac{1}{j_1\, j_2 \cdots j_k} \qquad (k \geq 1).$$

The Roman harmonic numbers[1] can be expressed as polynomials in the generalized harmonic numbers $H_n, H_n^{(2)}, \cdots, H_n^{(k)}$ [4, Equation (18)], [11, Equation (29)]. More precisely, $H_{n,1} = H_n$, and

$$H_{n,k} = \frac{1}{k!}(H_n)^k + \cdots + \frac{1}{k}H_n^{(k)} = P_k(H_n, \cdots, H_n^{(k)}) \qquad (k \geq 2)\,,$$

with

$$P_k(x_1, \cdots, x_k) = \sum_{n_1+2n_2+\cdots+kn_k=k} \frac{1}{n_1!n_2!\cdots n_k!}\left(\frac{x_1}{1}\right)^{n_1}\left(\frac{x_2}{2}\right)^{n_2}\cdots\left(\frac{x_k}{k}\right)^{n_k}.$$

These polynomials are a (slight) modification of the Bell polynomials [4, 6]. A natural generalization of the ordinary Roman harmonic numbers, noted $H_{n,k}^{(r)}$, such that $H_{n,1}^{(r)} = H_n^{(r)}$ was also introduced in [8]. It is given by the following expression [8, Definition 2]:

$$H_{n,k}^{(r)} = \sum_{n\geq j_1 \geq \cdots \geq j_k \geq 1} \frac{1}{j_1\, j_2 \cdots j_k^r} \qquad (k \geq 1, r \geq 1)\,.$$

3 Overview of some known formulas

In this section, we enumerate a number of more or less known identities, starting with the most classic examples and ending with the lesser-known, and make some comments about them.

The formula

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\,n} = \sum_{n=2}^{\infty} \frac{(-1)^n}{n}\,\zeta(n) = \gamma \tag{1}$$

1. Introduced three decades ago by S. Roman, G-C. Rota and D. Loeb (see [11] for historical details).

is a classical representation of γ dating back to Mascheroni and Euler which can be slightly modified as follows:

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{(n+1)!\, n} = \sum_{n=2}^{\infty} \frac{(-1)^n}{n} \{\zeta(n) - 1\} = \gamma + \ln 2 - 1\,. \tag{2}$$

A non-trivial generalization of Equation (1) is given by

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\, n^2} = \frac{1}{2}\gamma^2 + \frac{1}{2}\zeta(2) + \gamma_1 - \sum_{n=2}^{\infty} \frac{(-1)^n}{n} \zeta(n+1)\,, \tag{3}$$

where $\gamma_1 = \lim_{n\to\infty} \left\{ \sum_{j=1}^{n} \frac{\ln j}{j} - \frac{1}{2}\ln^2 n \right\}$ is the first Stieltjes constant. This nice identity is already known and appears in [7, 8].

The formula

$$\sum_{n=1}^{\infty} \frac{\lambda_n\, H_n}{n!\, n} = \zeta(2) - 1 \tag{4}$$

is a fairly known representation of $\zeta(2) = \frac{\pi^2}{6}$. It is in fact a special case of the more general formula

$$\sum_{n=1}^{\infty} \frac{\lambda_n\, H_{n,k}}{n!\, n} = \zeta(k+1) - \frac{1}{k} \qquad (k \geq 1)\,,$$

which is called *Hermite's formula* in [4, 6]. A non-trivial generalization of Equation (4) is given by

$$\sum_{n=1}^{\infty} \frac{\lambda_n\, H_n^{(2)}}{n!\, n} = \zeta(3) + \{\gamma + \ln(2\pi) - 12 \ln A\}\, \zeta(2) + \sum_{n=2}^{\infty} \frac{(-1)^n}{n} \zeta(n+2)\,, \tag{5}$$

where $A = \lim_{n\to\infty} \left\{ \frac{\prod_{k=1}^{n} k^k}{n^{\frac{n^2}{2}+\frac{n}{2}+\frac{1}{12}}\, e^{-\frac{n^2}{4}}} \right\} = \exp\left\{ \frac{1}{12} - \zeta'(-1) \right\}$ is the Glaisher-Kinkelin constant[2]. This identity is a direct consequence of [8, Equation (19)] and of the well-known relation:

$$\zeta'(2) = \{\gamma + \ln(2\pi) - 12 \ln A\}\, \zeta(2)\,.$$

2. This constant plays a role similar to the Stirling constant $\sqrt{2\pi}$ in the celebrated formula

$$\sqrt{2\pi} = \lim_{n\to\infty} \left\{ \frac{n!}{n^{n+\frac{1}{2}}\, e^{-n}} \right\} = \exp\{-\zeta'(0)\}\,.$$

Furthermore, Equation (5) admits a kind of reciprocal which is given by [8, Equation (18)]. More precisely, this is the following identity:

$$\sum_{n=1}^{\infty} \frac{\lambda_n\, H_n}{n!\, n^2} = \{12 \ln A - \ln(2\pi)\}\, \zeta(2) - 1 - \sum_{n=2}^{\infty} \frac{(-1)^n}{n}\, \zeta(n+2)\,, \tag{6}$$

which is another non-trivial generalization of Equation (4).

Remark 1. When k is greater than 2, no explicit formula, even conjectural, appears to be known for the sum $\sum_{n=1}^{\infty} \frac{\lambda_n\, H_n^{(k)}}{n!\, n}$, nor for the reciprocal sum $\sum_{n=1}^{\infty} \frac{\lambda_n\, H_n}{n!\, n^k}$ (see however Remark 4 below where an interpretation of these sums in terms of Ramanujan summations is given).

4 New supplementary formulas

The following binomial formula is nothing else than a variant of [2, Proposition 1] which is an elementary but efficient tool to generate several series identities with Cauchy and harmonic numbers. For appropriate analytic functions f with moderate growth, we have the relation:

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!} \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} f(k) = \int_0^1 f(x)\, dx\,. \tag{7}$$

For instance, using adequate functions f, we can easily derive from Equation (7) the identities given by Equations (1) and (4) above (see [2, Example 4]). We now present a number of interesting new identities that can also be deduced from this method.

Applying Equation (7) with $f(x) = \dfrac{\psi(x+1)+\gamma}{x+1}$, we can use the binomial identity [1, Equation (9.32)]

$$\frac{H_n}{n+1} = \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} \frac{H_k}{k+1}\,,$$

to derive the following identity:

$$\sum_{n=1}^{\infty} \frac{\lambda_n H_n}{(n+1)!} = \frac{1}{2}\zeta(2) + \ln 2 - 1 + \sum_{n=3}^{\infty} \frac{(-1)^n}{n} \left\{ \sum_{k=2}^{n} (\zeta(k) - 1) \right\}\,, \tag{8}$$

which is a refinement of a formula previously given by Boyadzhiev [2, Example 5]. Moreover, subtracting Equation (8) from Equation (4) leads to the new identity:

$$\sum_{n=1}^{\infty} \frac{\lambda_n\, H_n}{(n+1)!\, n} = \frac{1}{2}\zeta(2) - \ln 2 - \sum_{n=3}^{\infty} \frac{(-1)^n}{n} \left\{ \sum_{k=2}^{n} (\zeta(k) - 1) \right\}\,, \tag{9}$$

which can be seen as a modification of Equation (4) quite similar to Equation (2).

Applying Equation (7) with $f(x) = \dfrac{\psi(x+1)+\gamma}{x}$, we can use the binomial identity [1, Equation (5.22)]

$$H_n^{(2)} = \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} \frac{H_k}{k}\,,$$

to obtain the following new identity:

$$\sum_{n=1}^{\infty} \frac{\lambda_n\, H_n^{(2)}}{n!} = \int_0^1 \frac{\psi(x+1)+\gamma}{x}\, dx = \zeta(2) - \sum_{n=2}^{\infty} \frac{(-1)^n}{n} \zeta(n+1)\,. \tag{10}$$

Moreover, writing $H_n^{(2)} = H_{n-1}^{(2)} + \dfrac{1}{n^2}$ in the left member of Equation (10), we can deduce from Equation (3) yet another interesting new identity:

$$\sum_{n=1}^{\infty} \frac{\lambda_{n+1}\, H_n^{(2)}}{(n+1)!} = \frac{1}{2}\zeta(2) - \frac{1}{2}\gamma^2 - \gamma_1\,. \tag{11}$$

Remark 2. If ζ_H denotes the harmonic zeta function defined by

$$\zeta_H(s) = \sum_{n=1}^{\infty} \frac{H_n}{n^s} \qquad (\mathrm{Re}(s) > 1)\,,$$

the following nice identity [7, Equation (14)]:

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{n}\, \zeta_H(n) = \frac{1}{2}\zeta(2) + \frac{1}{2}\gamma^2 + \gamma_1\,,$$

allows us to rewrite Equation (11) above as follows:

$$\sum_{n=1}^{\infty} \frac{\lambda_{n+1}\, H_n^{(2)}}{(n+1)!} = \zeta(2) - \sum_{n=2}^{\infty} \frac{(-1)^n}{n}\, \zeta_H(n)\,. \tag{12}$$

Remark 3. Very recently, using a more powerful method, Young [12] has established the identity below which significantly generalizes Equation (10):

$$\sum_{n=1}^{\infty} \frac{\lambda_n\, H_n^{(k)}}{n!} = \zeta(k) - \sum_{n=2}^{\infty} \frac{(-1)^n}{n}\, \zeta(n+1, \underbrace{1, \ldots, 1}_{k-2}) \qquad (k \geq 2)\,.$$

However, it should be noted that this formula cannot be extended to the case $k = 1$. Indeed, the series $\sum_{n \geq 1} \dfrac{\lambda_n\, H_n}{n!}$ is divergent since $\dfrac{\lambda_n\, H_n}{n!} \sim \dfrac{1}{n \ln n}$.

5 Interpretation as Ramanujan summation

If $\sum_{n\geq 1}^{\mathcal{R}} f(n)$ denotes the $\mathcal{R}$-sum of the series $\sum_{n\geq 1} f(n)$ (i.e. the sum of the series in the sense of Ramanujan's summation method, following the masterful exposition in [3]), then, under certain appropriate conditions of growth and analyticity, we can make use of the following binomial transformation formula [8, Equation (10)]:

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\, n} \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} k f(k) = \sum_{n\geq 1}^{\mathcal{R}} f(n)\,. \tag{13}$$

This formula allows us to give an interesting interpretation in terms of Ramanujan summation of almost all the series identities mentioned above.

From the binomial identities

$$\sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} = 1 \quad \text{and} \quad \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} \frac{k}{k+1} = \frac{1}{n+1}\,,$$

we derive respectively the formulas

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\, n} = \sum_{n\geq 1}^{\mathcal{R}} \frac{1}{n}\,, \tag{14}$$

and

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{(n+1)!\, n} = \sum_{n\geq 1}^{\mathcal{R}} \frac{1}{n+1}\,, \tag{15}$$

whose expressions are given by Equations (1) and (2).

From the reciprocal binomial identities

$$\sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} H_k = \frac{1}{n} \quad \text{and} \quad \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} \frac{1}{k} = H_n\,,$$

we derive respectively the reciprocal formulas

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\, n^2} = \sum_{n\geq 1}^{\mathcal{R}} \frac{H_n}{n}\,, \tag{16}$$

and

$$\sum_{n=1}^{\infty} \frac{\lambda_n H_n}{n!\, n} = \sum_{n\geq 1}^{\mathcal{R}} \frac{1}{n^2}\,, \tag{17}$$

whose expressions are given by Equations (3) and (4).

From the reciprocal binomial identities

$$\sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}\frac{H_k}{k} = H_n^{(2)} \quad \text{and} \quad \sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}H_k^{(2)} = \frac{H_n}{n}\,,$$

we derive respectively the reciprocal formulas

$$\sum_{n=1}^{\infty}\frac{\lambda_n\,H_n^{(2)}}{n!\,n} = \sum_{n\geq 1}^{\mathcal{R}}\frac{H_n}{n^2}\,, \tag{18}$$

and

$$\sum_{n=1}^{\infty}\frac{\lambda_n\,H_n}{n!\,n^2} = \sum_{n\geq 1}^{\mathcal{R}}\frac{H_n^{(2)}}{n}\,, \tag{19}$$

whose expressions are given by Equations (5) and (6).

Remark 4. 1) Formulas (14) and (16) are two particular cases of the more general relation [8, Equation (12)]:

$$\sum_{n=1}^{\infty}\frac{\lambda_n}{n!\,n^k} = \sum_{n\geq 1}^{\mathcal{R}}\frac{H_{n,k-1}}{n} \qquad (k\geq 1)\,.$$

2) Formulas (17) and (18) are two particular cases of the more general relation [8, Equation (11)]:

$$\sum_{n=1}^{\infty}\frac{\lambda_n\,H_n^{(k)}}{n!\,n} = \sum_{n\geq 1}^{\mathcal{R}}\frac{H_{n,k-1}}{n^2} \qquad (k\geq 1)\,.$$

3) Formula (19) is also a special case of the more general formula:

$$\sum_{n=1}^{\infty}\frac{\lambda_n\,H_n}{n!\,n^k} = \sum_{n\geq 1}^{\mathcal{R}}\frac{H_{n,k-1}^{(2)}}{n} \qquad (k\geq 2)\,,$$

where $H_{n,k}^{(r)}$ are the generalized Roman harmonic numbers introduced in [8].

From the the self-reciprocal binomial identity

$$\sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}\frac{H_k}{k+1} = \frac{H_n}{n+1}\,,$$

we derive the formula

$$\sum_{n=1}^{\infty}\frac{\lambda_n\,H_n}{(n+1)!\,n} = \sum_{n\geq 1}^{\mathcal{R}}\frac{H_n}{n(n+1)}\,, \tag{20}$$

whose expression is given by Equation (9).

Applying Equation (13) to the function

$$f(x) = \frac{\psi(x+1)+\gamma-1}{x(x-1)}\,,$$

whose values at positive integers are

$$f(n) = \frac{H_n - 1}{n(n-1)} \qquad (n \geq 2)\,,$$

and

$$f(1) = \zeta(2) - 1 = \lim_{x\to 1} \frac{\psi(x+1)+\gamma-1}{x(x-1)}\,,$$

then, using the binomial identity

$$\sum_{k=2}^{n} (-1)^{k-1} \binom{n}{k} \frac{1-H_k}{k-1} = nH_n^{(2)} - n\,,$$

which is the reciprocal of the identity [1, Equation (5.24)]:

$$\sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} kH_k^{(2)} = \frac{1-H_n}{n-1} \qquad (n \geq 2)\,,$$

allows us to deduce the nice formula

$$\sum_{n=1}^{\infty} \frac{\lambda_n\, H_n^{(2)}}{n!} = \zeta(2) - \sum_{n\geq 1}^{\mathcal{R}} \frac{H_n - 1}{n(n-1)}\,, \tag{21}$$

where the $\mathcal{R}$-summed sequence in the right member of Equation (21) has the value $\zeta(2) - 1$ when $n = 1$; this is the interpretation of Equation (10) in terms of Ramanujan summation. Furthermore, subtracting Equation (16) from Equation (21) allows us to derive yet another formula:

$$\sum_{n=1}^{\infty} \frac{\lambda_{n+1}\, H_n^{(2)}}{(n+1)!} = \zeta(2) - \sum_{n\geq 1}^{\mathcal{R}} \frac{H_{n-1}}{n-1}\,, \tag{22}$$

where the $\mathcal{R}$-summed sequence in the right member of Equation (22) has the value $\zeta(2)$ when $n = 1$; this provides an interpretation of Equation (12) in terms of Ramanujan summation.

Remark 5. Very recently, using a more sophisticated method, Young [12] has established the relation below which is a substantial improvement of Equation (21):

$$\sum_{n=1}^{\infty} \frac{\lambda_n\, H_n^{(k)}}{n!} = \zeta(k) - \sum_{n\geq 1}^{\mathcal{R}} \frac{H_{n,k-1} - H_{n,k-2}}{n(n-1)} \qquad (k \geq 2)\,.$$

References

[1] K. N. Boyadzhiev, *Notes on the Binomial Transform, Theory and Table*, World Scientific, 2018.

[2] K. N. Boyadzhiev, New series identities with Cauchy, Stirling, and harmonic numbers, and Laguerre polynomials, *J. Integer Sequences* **23** (2020), article 20.11.7.

[3] B. Candelpergher, *Ramanujan Summation of Divergent Series*, Lecture Notes in Math. 2185, Springer, 2017.

[4] B. Candelpergher and M-A. Coppo, A new class of identities involving Cauchy numbers, harmonic numbers and zeta values, *Ramanujan J.* **27** (2012), 305–328.

[5] H. Cohen, *Number Theory, Volume II: Analytic and Modern Tools*, Graduate Texts in Math., vol. 240, Springer, 2007.

[6] M-A. Coppo, Nouvelles expressions des formules de Hasse et de Hermite pour la fonction zêta d'Hurwitz, *Expo. Math.* **27** (2009), 79–86.

[7] M-A. Coppo, A note on some alternating series involving zeta and multiple zeta values, *J. Math. Anal. App.* **475** (2019), 1831–1841.

[8] M-A. Coppo, New identities involving Cauchy numbers, harmonic numbers and zeta values, *Results in Math.* **76** (2021), article 189.

[9] C. Jordan, *Calculus of Finite Differences*, Chelsea, New York, 1965.

[10] D. Merlini, R. Sprugnoli, and M. C. Verri, The Cauchy numbers, *Discrete Math.*, **306** (2006), 1906–1920.

[11] J. Sesma, The Roman harmonic numbers revisited, *J. Number Theory* **180** (2017), 544–565.

[12] P. T. Young, Global series for height 1 multiple zeta functions, preprint, to appear in *Eur. J. Math.*