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Marc-Antoine Coppo[*] and Bernard Candelpergher
Université Côte d'Azur, CNRS, LJAD (UMR 7351), Nice, France

Abstract We study a range of constants that are closely linked to the harmonic zeta function ζ_H. In addition, by means of a new representation of ζ_H, we derive possibly new integral formulas for $\zeta(n)$ and $\zeta'(n)$ and new evaluations for a variety of integrals of the same type as those previously considered by Glasser and Manna.

Keywords Harmonic zeta function; Stieltjes constants; zeta values; digamma function; Glasser-Manna integrals, Ramanujan summation of series.

1 Introduction

Let ζ_H be the harmonic zeta function defined for $\mathrm{Re}(s) > 1$ by

$$\zeta_H(s) := \sum_{n=1}^{\infty} \frac{H_n}{n^s} \,,$$

where, for all $n \geq 1$,

$$H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$$

are the classical harmonic numbers. Four decades ago, Apostol and Vu [2] showed that this function could be continued as a meromorphic function with a double pole at $s = 1$, and an infinite number of simple poles at $s = 0$ and $s = 1 - 2k$ for each integer $k \geq 1$. The Laurent expansion of the harmonic zeta function ζ_H around its double pole $s = 1$ can be written as

$$\zeta_H(s) = \frac{1}{(s-1)^2} + \frac{\gamma}{s-1} + \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \gamma_H^{(k)} (s-1)^k \quad (0 < |s-1| < 1) \,,$$

[*]Corresponding author. *Email address:* coppo@unice.fr

where $\gamma = -\Gamma'(1)$ is Euler's constant, and the coefficents $\gamma_H^{(k)}$ are called *harmonic Stieltjes constants* by analogy with the classical Stieltjes constants γ_k. A common definition of these constants is

$$\gamma_k = \lim_{s\to 1}\left\{(-1)^k\zeta^{(k)}(s) - \frac{k!}{(s-1)^{k+1}}\right\} \qquad (k \geq 0)\,,$$

where $\zeta^{(k)}(s)$ is the kth derivative of the Riemann zeta function. From now on and throughout the text, we will use the lighter notation $\gamma^{(k)}$ in place of $\gamma_H^{(k)}$. An explicit expression of $\gamma^{(0)}$ is given by the following nice formula [10, Equation (6)]:

$$\gamma^{(0)} = \frac{1}{2}\gamma^2 + \frac{1}{2}\zeta(2) = \frac{1}{2}\,\Gamma''(1)\,. \tag{1}$$

Incidentally, this formula can also be generalized to the case of height 1 multiple zeta functions $\zeta(s,1,\ldots,1)$ [17, Corollary 4.1]. Regarding the classical Stieltjes constants, we recall the well-known asymptotic formula

$$\gamma_k = \lim_{N\to\infty}\left\{\sum_{n=1}^{N}\frac{(\log n)^k}{n} - \frac{(\log N)^{k+1}}{k+1}\right\} \qquad (k \geq 0)\,.$$

From our point of view, it should be noted that γ_k is nothing else than the $\mathcal{R}$-sum (i.e. the sum of the series in the sense of Ramanujan's summation method) of the divergent series $\sum_{n\geqslant 1}\frac{(\log n)^k}{n}$ [9, p. 67]. The existence of a similar asymptotic formula for $\gamma^{(k)}$:

$$\gamma^{(k)} = \lim_{N\to\infty}\left\{\sum_{n=1}^{N}\frac{H_n(\log n)^k}{n} - \frac{(\log N)^{k+2}}{k+2} - \gamma\frac{(\log N)^{k+1}}{k+1}\right\} \qquad (k \geq 0)\,,$$

which results from an extension of a formula of Briggs and Buschman [4, 8], strongly suggests that the constant $\gamma^{(k)}$ is closely linked to the $\mathcal{R}$-sum of the divergent series $\sum_{n\geqslant 1}\frac{H_n\,(\log n)^k}{n}$. This is indeed the case: such a relation exists although it is more intricate than in the classical case. We have already encountered in earlier studies [9, Equation (3.23)], [10, Equation (4)] the intriguing relation:

$$r_0 = \gamma_1 + \tau_1 + \gamma^{(0)} - \zeta(2) = \gamma_1 + \tau_1 + \frac{1}{2}\gamma^2 - \frac{1}{2}\zeta(2)\,, \tag{2}$$

where

$$r_0 := \sum_{n\geq 1}^{\mathcal{R}}\frac{H_n}{n} \quad \text{and} \quad \tau_1 = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{\zeta(n+1)}{n} = -\sum_{n=2}^{\infty}\zeta'(n)\,.$$

In the next section, we will see to what extent our formula (2) can be generalized (see Proposition 1). We also mention that another natural generalization of this formula with Roman's harmonic numbers $H_{n,k}$ instead of H_n was given by Young [17, Theorem 6.1 (d)]. In the subsequent section, we will use the new representation of $\zeta_H(s)$ provided by our Theorem 1 (which is a refinement of a formula due to Boyadzhiev, Gadiyar and Padma [7]) to obtain new expressions of the harmonic Stieltjes constants $\gamma^{(k)}$ (see Corollary 1). Moreover, we derive from this representation new integral formulas for $\zeta(n)$ and $\zeta'(n)$ (see Corollary 2) and new evaluations for a variety of integrals of the same type as those introduced earlier by Glasser and Manna in relation with the Laplace transform of the digamma function [1, 15] (see Corollaries 4–6).

2 How the constants $\sum_{n\geqslant 1}^{\mathcal{R}} \frac{H_n\,(\log n)^k}{n}$ and $\gamma^{(k)}$ are linked together

The forthcoming Proposition 1 successfully generalizes the previous relation (2).

Proposition 1. Let k be a natural number. If r_k and τ_k are respectively defined by

$$r_k := \sum_{n\geqslant 1}^{\mathcal{R}} \frac{H_n(\log n)^k}{n} \qquad (k \geq 0)\,,$$

and

$$\tau_k := \sum_{n=1}^{\infty} (-1)^{n+k}\,\frac{\zeta(n+1)}{n^k} \qquad (k \geq 1)\,,$$

then

$$r_0 = \gamma_1 + \tau_1 + \gamma^{(0)} - \zeta(2)\,,$$

and the following relations hold true for all positive values of k:

$$r_{2k-1} = \frac{\gamma_{2k}}{2k} + (2k-1)!\,\tau_{2k} + \gamma^{(2k-1)} + \sum_{n=1}^{k} \frac{(2k-1)!}{(2k-2n)!} \times \frac{(2^{2n-1}-1)}{2^{2n-2}}\,\gamma_{2k-2n}\,\zeta(2n)\,, \tag{3}$$

and

$$r_{2k} = \frac{\gamma_{2k+1}}{2k+1} + (2k)!\,\tau_{2k+1} + \gamma^{(2k)} + \sum_{n=1}^{k} \frac{(2k)!}{(2k-2n+1)!} \times \frac{(2^{2n-1}-1)}{2^{2n-2}}\,\gamma_{2k-2n+1}\zeta(2n) - (2k)!\frac{(2^{2k+1}-1)}{2^{2k}}\zeta(2k+2)\,. \tag{4}$$

In particular,

$$\begin{aligned} r_1 &= \frac{1}{2}\gamma_2 + \tau_2 + \gamma^{(1)} + \gamma\zeta(2)\,, \\ r_2 &= \frac{1}{3}\gamma_3 + 2\tau_3 + \gamma^{(2)} + 2\gamma_1\zeta(2) - \frac{7}{2}\zeta(4)\,. \end{aligned}$$

Example 1. Numerical evaluations of r_k, τ_k and $\gamma^{(k)}$ for small values of k are given below. Note that the constant τ_0 is undefined.

$$\begin{array}{lll} r_0 = 0.529052\ldots & \tau_1 = 1.257746\ldots & \gamma^{(0)} = 0.989055\ldots \\ r_1 = -0.078850\ldots & \tau_2 = -1.424248\ldots & \gamma^{(1)} = 0.400761\ldots \\ r_2 = -0.008095\ldots & \tau_3 = 1.523800\ldots & \gamma^{(2)} = 0.971304\ldots \end{array}$$

Remark 1. Although not used afterwards, we also mention another expression of the constant τ_k which can be easily deduced from its definition:

$$\tau_k = (-1)^k \sum_{n=1}^{\infty} \frac{1}{n} \operatorname{Li}_k\left(-\frac{1}{n}\right) \qquad (k \geq 1)\,,$$

where Li_k denotes the kth polylogarithm. In particular, for $k = 1$, the formula

$$\tau_1 = \sum_{n=1}^{\infty} \frac{1}{n} \log\left(1 + \frac{1}{n}\right)$$

is regained. Note that the definition of τ_k given here differs from that given in [10] when $k > 1$.

Proof of Proposition 1. The key formula to derive the general relations (3) and (4) is the decomposition of $\sum_{n\geqslant 1}^{\mathcal{R}} \frac{H_n}{n^s}$ given by [10, Theorem 1] which is reproduced below:

$$\sum_{n\geqslant 1}^{\mathcal{R}} \frac{H_n}{n^s} = \frac{\pi}{\sin(\pi s)}\,\zeta(s) + \int_0^1 \frac{\psi(x+1)+\gamma}{x^s}\,dx + \zeta_H(s)\,, \tag{5}$$

where $\psi(x) = \frac{d}{dx}\log\Gamma(x)$ is the digamma function. This formula applies to all complex numbers s such that $\operatorname{Re}(s) < 2$ and $s \neq 1, 0, 1-2k$ for each integer $k > 0$. Fortunately, the Laurent expansion of each component in (5) can be written explicitly.

a) The expansion of $\frac{\pi}{\sin(\pi s)}\,\zeta(s)$ at $s = 1$ can be obtained as follows: first, we write the successive equations:

$$\begin{aligned}\frac{-\pi}{\sin(\pi s)} &= \frac{\pi}{\sin(\pi(s-1))} = \frac{\exp(i\pi(s-1))}{s-1}\,\frac{2i\pi(s-1)}{\exp(2i\pi(s-1))-1}\\ &= \frac{1}{s-1}\left(\sum_{k\geq 0} i^k\pi^k\frac{1}{k!}(s-1)^k\right)\left(\sum_{k\geq 0} i^k(2\pi)^k\frac{B_k}{k!}(s-1)^k\right)\\ &= \frac{1}{s-1}+\sum_{k\geq 1}(-1)^k\left(\sum_{j=0}^{2k}\binom{2k}{j}2^jB_j\right)\frac{\pi^{2k}}{(2k)!}(s-1)^{2k-1},\end{aligned}$$

where B_j are the Bernoulli numbers. Euler's identity:

$$\zeta(2k) = (-1)^{k+1}2^{2k-1}B_{2k}\frac{\pi^{2k}}{(2k)!} \qquad (k\geq 1),$$

then allows us to rewrite this expansion as follows:

$$\frac{-\pi}{\sin(\pi s)} = \frac{1}{s-1} - \sum_{k\geq 1}\frac{2^{1-2k}}{B_{2k}}\sum_{j=0}^{2k}\binom{2k}{j}2^jB_j\,\zeta(2k)(s-1)^{2k-1}.$$

Morover, the latter expression can be simplified thanks to the identity

$$\sum_{j=0}^{k}\binom{k}{j}2^jB_j = 2^kB_k(\frac{1}{2}) = 2(1-2^{k-1})B_k \qquad (k\geq 2).$$

Hence, the expansion of $\dfrac{\pi}{\sin(\pi s)}$ at $s=1$ is given by

$$\frac{\pi}{\sin(\pi s)} = -\frac{1}{s-1} - \sum_{k=1}^{\infty}\frac{2^{2k-1}-1}{2^{2k-2}}\,\zeta(2k)(s-1)^{2k-1}.$$

On the other hand, the expansion of $\zeta(s)$ at $s=1$ is

$$\zeta(s) = \frac{1}{s-1}+\gamma+\sum_{k=1}^{\infty}\frac{(-1)^k}{k!}\gamma_k(s-1)^k,$$

where γ_k are the Stieltjes constants. The expansion of $\frac{\pi}{\sin(\pi s)}\,\zeta(s)$ is then obtained by Cauchy product. We have

$$\begin{aligned}\frac{\pi}{\sin(\pi s)}\,\zeta(s) &= -\frac{1}{(s-1)^2} - \frac{\gamma}{(s-1)} + \gamma_1 - \zeta(2)\\ &\quad -\left(\frac{1}{2}\gamma_2+\gamma\zeta(2)\right)(s-1) + \left(\frac{1}{6}\gamma_3+\gamma_1\zeta(2)-\frac{7}{4}\zeta(4)\right)(s-1)^2\\ &\quad -\left(\frac{1}{24}\gamma_4+\frac{1}{2}\gamma_2\zeta(2)+\frac{7}{4}\gamma\zeta(4)\right)(s-1)^3\\ &\quad +\left(\frac{1}{120}\gamma_5+\frac{1}{6}\gamma_3\zeta(2)+\frac{7}{4}\gamma_1\zeta(4)-\frac{31}{16}\zeta(6)\right)(s-1)^4 - \cdots \qquad (6)\end{aligned}$$

In (6), the coefficient of $(s-1)^{2k}$ and $(s-1)^{2k-1}$ are respectively

$$\frac{\gamma_{2k+1}}{(2k+1)!}+\sum_{j=1}^{k}\frac{\gamma_{2k-2j+1}}{(2k-2j+1)!}\times\frac{(2^{2j-1}-1)}{2^{2j-2}}\,\zeta(2j)-\frac{(2^{2k+1}-1)}{2^{2k}}\zeta(2k+2)\,,$$

and

$$-\frac{\gamma_{2k}}{(2k)!}-\sum_{j=1}^{k}\frac{\gamma_{2k-2j}}{(2k-2j)!}\times\frac{(2^{2j-1}-1)}{2^{2j-2}}\,\zeta(2j)\,.$$

b) The well-known Taylor series expansion of the digamma function:

$$\psi(x+1)+\gamma=\sum_{n=1}^{\infty}(-1)^{n+1}\zeta(n+1)x^n\qquad(|x|<1),$$

allows us write

$$\int_0^1\frac{\psi(x+1)+\gamma}{x^s}\,dx=\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\zeta(n+1)}{n-(s-1)}.$$

This leads to the following expansion:

$$\begin{aligned}\int_0^1\frac{\psi(x+1)+\gamma}{x^s}\,dx&=\sum_{k=0}^{\infty}\left(\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\zeta(n+1)}{n^{k+1}}\right)(s-1)^k\\&=\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\times k!\,\tau_{k+1}\,(s-1)^k\qquad(|s-1|<1).\end{aligned}\tag{7}$$

c) To obtain the expansion of $\sum_{n\geqslant1}^{\mathcal{R}}\frac{H_n}{n^s}$ at $s=1$, we proceed as follows: first we write the identities

$$\frac{H_n}{n^s}=\frac{H_n}{n}e^{-(s-1)\log n}=\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\times\frac{H_n(\log n)^k}{n}\,(s-1)^k\,,$$

so that

$$\sum_{n\geqslant1}^{\mathcal{R}}\frac{H_n}{n^s}=\sum_{n\geq1}^{\mathcal{R}}\left(\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\times\frac{H_n(\log n)^k}{n}\,(s-1)^k\right)\,.$$

Moreover, [9, Theorem 9] allows us interchange $\sum_{n\geqslant1}^{\mathcal{R}}$ and $\sum_{k=0}^{\infty}$. This leads to the expansion:

$$\sum_{n\geqslant1}^{\mathcal{R}}\frac{H_n}{n^s}=\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\left(\sum_{n\geq1}^{\mathcal{R}}\frac{H_n(\log n)^k}{n}\right)(s-1)^k=\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\,r_k\,(s-1)^k\,.\tag{8}$$

The desired formulas (3) and (4) can then be easily obtained by combining the previous expansions (6), (7), and (8). □

3 A new expresssion of $\zeta_H(s)$ and its implications

In this section, we give a new expression of the harmonic zeta function (Theorem 1) from which follows a new evaluation of the harmonic Stieltjes constants (Corollary 1). By means of this new representation of ζ_H, we can also derive new integral formulas for $\zeta(n)$ and $\zeta'(n)$ (Corollary 2), and new evaluations for Glasser-Manna type integrals (Corollaries 4–6).

Theorem 1. *For all complex numbers s in $\mathbb{C} \setminus \mathbb{Z}$, we have*

$$\zeta_H(s) = \pi \cot(\pi s)\,\zeta(s) + \zeta(s+1) + \Gamma(1-s)\Phi(s) \tag{9}$$

with

$$\Phi(s) := -\frac{1}{2i\pi}\int_{-\pi}^{\pi} x \left(\mathrm{Log}(1+e^{ix})\right)^{s-1}\,dx\,, \tag{10}$$

where Log *denotes the principal value of the complex logarithm.*

Proof. Let $z = \mathrm{Log}(1+e^{ix}) = \log\left(2\cos(\frac{x}{2})\right) + \frac{ix}{2}$, $-\pi < x < \pi$. When x varies from $-\pi$ to π, the variable z traverses the path L defined by the parametric equations $\mathrm{Re}\,z = \log(2\cos(x/2))$ and $\mathrm{Im}\,z = x/2$. This path extends from $-\infty$ below the line $\mathrm{Im}\,z = 0$, passes through the point $(\log 2, 0)$, then extends back to $-\infty$ above the line $\mathrm{Im}\,z = 0$. Since L is homotopic to the Hankel contour C defined in [6, 7], then by means of [7, Equation (25)], we have, for $s \in \mathbb{C} \setminus \mathbb{Z}$, the representation

$$\zeta_H(s) = \pi\cot(\pi s)\,\zeta(s) + \zeta(s+1) + \Gamma(1-s)\phi(s) - \psi(1-s)\zeta(s) - \zeta'(s)\,, \tag{11}$$

where

$$\phi(s) := \frac{1}{2i\pi}\int_L \frac{z^{s-1}e^z}{e^z-1}\mathrm{Log}\left(\frac{e^z-1}{z}\right)dz\,.$$

We now show that the equations (9) and (11) are equivalent. To do this, we proceed as follows. Differentiation of the integral representation

$$\zeta(s) = \frac{\Gamma(1-s)}{2i\pi}\int_L \frac{z^{s-1}e^z}{1-e^z}\,dz$$

[6, Equation (2.4)], leads to the following identity:

$$\psi(1-s)\zeta(s) + \zeta'(s) = \frac{\Gamma(1-s)}{2i\pi}\int_L \frac{z^{s-1}e^z\mathrm{Log}(z)}{1-e^z}\,dz \qquad (s \neq 1,2,3,\ldots) \tag{12}$$

[6, Equation (2.7)]. Splitting the right-hand member of (12), we can write

$$\frac{1}{2i\pi}\int_L \frac{z^{s-1}e^z\mathrm{Log}(z)}{1-e^z}\,dz = \phi(s) - \Phi(s)\,, \tag{13}$$

with

$$\Phi(s) = \frac{1}{2i\pi} \int_L \frac{z^{s-1} e^z}{e^z - 1} \mathrm{Log}(e^z - 1)\, dz\,.$$

From (12) and (13), we deduce the identity

$$\Gamma(1-s)\phi(s) - \psi(1-s)\zeta(s) - \zeta'(s) = \Gamma(1-s)\Phi(s)\,.$$

Moreover, the substitution

$$z = \mathrm{Log}(1 + e^{ix}), \quad -\pi < x < \pi\,,$$

in the contour integral defining Φ gives this function the equivalent form (10), proving that (9) and (11) are equivalent equations. □

3.1 New formulas for $\gamma^{(k)}$

We deduce from (9) an expression of $\gamma^{(k)}$ which seems to us more meaningful than the one given by Kargin et al. [16, Equation (21)].

Corollary 1. For any non-negative integer n, let $\mathcal{L}_n$ and ξ_n be defined as follows:

$$\mathcal{L}_n := \Phi^{(n)}(1) = -\frac{1}{2i\pi} \int_{-\pi}^{\pi} x\, \mathrm{Log}^n(\mathrm{Log}(1 + e^{ix}))\, dx\,,$$

and

$$\xi_n := (-1)^n\, \Gamma^{(n)}(1) = (-1)^n \int_0^{+\infty} e^{-x} \log^n(x)\, dx\,.$$

Then we have

$$\gamma^{(0)} = -\mathcal{L}_1 - \gamma_1 \ - \zeta(2)\,, \tag{14}$$

and, for all positive integers k, the following general relations hold true:

$$\begin{aligned}\gamma^{(2k-1)} = -\zeta^{(2k-1)}(2) - \frac{\gamma_{2k}}{2k} + \frac{1}{2k} \sum_{n=0}^{2k-1} \binom{2k}{n} \xi_n\, \mathcal{L}_{2k-n} \\ + \sum_{n=1}^{k} \frac{2(2k-1)!}{(2k-2n)!} \gamma_{2k-2n}\, \zeta(2n)\,,\end{aligned} \tag{15}$$

and

$$\begin{aligned}\gamma^{(2k)} = \zeta^{(2k)}(2) - \frac{\gamma_{2k+1}}{2k+1} - \frac{1}{2k+1} \sum_{n=0}^{2k} \binom{2k+1}{n} \xi_n\, \mathcal{L}_{2k+1-n} \\ + \sum_{n=1}^{k} \frac{2(2k)!}{(2k+1-2n)!} \gamma_{2k+1-2n}\, \zeta(2n) - 2(2k)!\, \zeta(2k+2)\,.\end{aligned} \tag{16}$$

Moreover, it follows from (14) and (1) the explicit evaluation:

$$\mathcal{L}_1 = -\frac{1}{2i\pi}\int_{-\pi}^{\pi} x\,\mathrm{Log}(\mathrm{Log}(1+e^{ix}))\,dx = -\frac{3}{2}\zeta(2) - \frac{1}{2}\gamma^2 - \gamma_1\,. \tag{17}$$

Example 2. Applying formulas (15), (16) to the case $k = 1$ and using (17) above, we then obtain

$$\gamma^{(1)} = -\zeta'(2) - \frac{1}{2}\gamma^3 - \frac{1}{2}\gamma_2 + \frac{1}{2}\mathcal{L}_2 + \frac{1}{2}\gamma\zeta(2) - \gamma\gamma_1\,,$$

and

$$\gamma^{(2)} = \zeta''(2) + \frac{1}{2}\gamma^4 - \frac{1}{3}\gamma_3 - \frac{1}{3}\mathcal{L}_3 - \gamma\mathcal{L}_2 + (2\gamma^2 + 5\gamma_1)\zeta(2) + \gamma^2\gamma_1 - \frac{1}{10}(\zeta(2))^2\,.$$

Numerical evaluations of the constants $\mathcal{L}_2$ and $\mathcal{L}_3$ are

$$\mathcal{L}_2 = -1.924491\ldots \quad \text{and} \quad \mathcal{L}_3 = 7.158075\ldots$$

Proof of Corollary 1. The key formula to derive the relations (14), (15) and (16) is the representation of $\zeta_H(s)$ given by (9). Fortunately, the Laurent series expansion of each component in (9) can be written explicitly.

a) The Laurent expansions of $\pi\cot(\pi s)$ and $\zeta(s)$ at $s = 1$ are respectively

$$\pi\cot(\pi s) = \frac{1}{s-1} - \sum_{k\geq 1} 2\zeta(2k)(s-1)^{2k-1}\,,$$

and

$$\zeta(s) = \frac{1}{s-1} + \gamma + \sum_{k\geq 1}\frac{(-1)^k}{k!}\gamma_k(s-1)^k\,,$$

where the coefficents γ_k are the Stieltjes constants. The expansion of $\pi\cot(\pi s)\,\zeta(s)$ is then obtained by Cauchy product:

$$\begin{aligned}\pi\cot(\pi s)\,\zeta(s) = {}& \frac{1}{(s-1)^2} + \frac{\gamma}{s-1} - \gamma_1 - 2\zeta(2) - \left(-\frac{1}{2}\gamma_2 + 2\gamma\zeta(2)\right)(s-1)\\ &+ \frac{1}{2}\left(-\frac{1}{3}\gamma_3 + 4\zeta(2)\gamma_1 - 4\zeta(4)\right)(s-1)^2\\ &- \frac{1}{6}\left(-\frac{1}{4}\gamma_4 + 6\zeta(2)\gamma_2 + 12\gamma\zeta(4)\right)(s-1)^3 + \cdots\end{aligned} \tag{18}$$

b) It follows from the definition of ξ_k as $(-1)^k\,\Gamma^{(k)}(1)$ that the Laurent expansion of $\Gamma(1-s)$ at $s = 1$ is given by

$$\Gamma(1-s) = -\frac{1}{s-1} - \gamma - \sum_{k=2}^{\infty}\frac{\xi_k}{k!}(s-1)^{k-1} \qquad (|s-1| < 1)\,,$$

On the other hand, the function Φ defined by (10) is an entire function of s with $\Phi(1) = \mathcal{L}_0 = 0$, and the definition of $\mathcal{L}_k$ as $\Phi^{(k)}(1)$ implies that the Taylor expansion of $\Phi(s)$ at $s = 1$ is given by

$$\Phi(s) = \sum_{k=1}^{\infty} \frac{\mathcal{L}_k}{k!} (s-1)^k .$$

The Laurent expansion of $\Gamma(1-s)\Phi(s)$ then follows by Cauchy product:

$$\begin{aligned}\Gamma(1-s)\Phi(s) = -\mathcal{L}_1 - \left(\frac{1}{2}\mathcal{L}_2 + \gamma\mathcal{L}_1\right)(s-1) \\ + \frac{1}{2}\left(-\frac{1}{3}\mathcal{L}_3 - \gamma\mathcal{L}_2 - 2\gamma^{(0)}\mathcal{L}_1\right)(s-1)^2 \\ - \frac{1}{6}\left(\frac{1}{4}\mathcal{L}_4 + \gamma\mathcal{L}_3 + 3\gamma^{(0)}\mathcal{L}_2 + (2\zeta(3) + 3\gamma\zeta(2) + \gamma^3)\mathcal{L}_1\right)(s-1)^3 + \cdots \quad (19)\end{aligned}$$

c) The Taylor expansion of $\zeta(s+1)$ at $s = 1$ can be written as follows:

$$\zeta(s+1) = \zeta(2) + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} (-1)^k \zeta^{(k)}(2) (s-1)^k . \quad (20)$$

By assembling equations (18), (19) and (20) above, we obtain, by identifying the constant term, the relation (14). In the same way, the general formulae (15) and (16) are derived by identifying the coefficients of higher degree. □

Remark 2. a) The constants ξ_n involved in formulae (15)–(16) have a closed form expression in terms of Euler's constant γ and $\zeta(2), \ldots, \zeta(n)$ [3, 11]. More precisely, we have $\xi_0 = 1$, $\xi_1 = \gamma$, $\xi_2 = \gamma^2 + \zeta(2) = 2\gamma^{(0)}$, and by the recursion formula [11, Equation (2.2)]

$$\xi_{n+1} = \gamma\xi_n + \sum_{k=1}^{n} \frac{n!}{(n-k)!} \zeta(k+1)\, \xi_{n-k} \qquad (n \geq 1),$$

we also have

$$\xi_3 = \gamma^3 + 3\gamma\zeta(2) + 2\zeta(3)\, , \; \xi_4 = \gamma^4 + 6\gamma^2\zeta(2) + 8\gamma\zeta(3) + 3\left(\zeta(2)\right)^2 + 6\zeta(4)\, , \text{ etc.}$$

A clever reformulation of this result is

$$\xi_n = P_n(\gamma, \zeta(2), \ldots, \zeta(n)) \qquad (n \geq 2)\, ,$$

where the polynomials P_n are defined by the exponential generating function

$$\exp\left(\sum_{k=1}^{\infty} x_k \frac{t^k}{k}\right) = \sum_{n=0}^{\infty} P_k(x_1, x_2, \ldots, x_n) \frac{t^n}{n!} .$$

b) By means of a theorem of Ramanujan [12, Theorem 4] whose proof is given in [5, p. 224], we can write the following identity:

$$(-1)^n \zeta^{(n)}(2) = n! + \gamma_n + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \gamma_{k+n} \qquad (n \geq 1)\,.$$

3.2 New formulas for $\zeta(n)$ and $\zeta'(n)$

The relation between $\Phi(s)$ and $\zeta_H(s)$ resulting from formula (9) when written in a neighborhood of $s = n$ (for an integer $n \geq 2$) enables to derive new formulas for $\zeta(n)$ and $\zeta'(n)$.

Corollary 2. For all integers $n \geq 2$, we have

$$\zeta(n) = \frac{(-1)^n}{(n-1)!} \times \frac{1}{2i\pi} \int_{-\pi}^{\pi} x \left(\operatorname{Log}(1 + e^{ix}) \right)^{n-1} dx\,, \tag{21}$$

and

$$\zeta'(n) = \frac{(-1)^n}{(n-1)!} \times \frac{1}{2i\pi} \int_{-\pi}^{\pi} x \operatorname{Log}\left(\operatorname{Log}(1 + e^{ix}) \right) \left(\operatorname{Log}(1 + e^{ix}) \right)^{n-1} dx \\ + \zeta_H(n) - \zeta(n)\, \psi(n) - \zeta(n+1)\,. \tag{22}$$

Proof of Corollary 2. For $n \geq 2$, we can write

$$\begin{aligned} \pi \cot(\pi s)\, \zeta(s) &= \frac{\zeta(n)}{s-n} + \zeta'(n) + \mathrm{O}(s-n)\,, \\ \Gamma(1-s) &= \frac{(-1)^n}{(n-1)!} \left(\frac{1}{s-n} + (\gamma - H_{n-1}) \right) + \mathrm{O}(s-n)\,, \end{aligned}$$

and

$$\Gamma(1-s)\, \Phi(s) = \frac{(-1)^n}{(n-1)!} \frac{\Phi(n)}{s-n} + \frac{(-1)^n}{(n-1)!} \left(\Phi'(n) + \Phi(n)(\gamma - H_{n-1}) \right) + \mathrm{O}(s-n)\,.$$

By applying formula (9) around $s = n$, we deduce the equation

$$\zeta_H(n) + \mathrm{O}(s-n) = \left(\zeta(n) + \frac{(-1)^n}{(n-1)!} \Phi(n) \right) \frac{1}{s-n} \\ + \frac{(-1)^n}{(n-1)!} \left(\Phi'(n) + \Phi(n)(\gamma - H_{n-1}) \right) + \zeta'(n) + \zeta(n+1) + \mathrm{O}(s-n)\,.$$

This leads to the identities

$$\zeta(n) + \frac{(-1)^n}{(n-1)!} \Phi(n) = 0\,,$$

and

$$\zeta_H(n) = \frac{(-1)^n}{(n-1)!}\left(\Phi'(n) + \Phi(n)(\gamma - H_{n-1})\right) + \zeta'(n) + \zeta(n+1)\,,$$

thus proving formulas (21) and (22). $\square$

Remark 3. It should be noted that our formula (21) for $\zeta(n)$ is similar to but different from [6, Equation (2.6)] which, after substitution $z = \mathrm{Log}(1+e^{ix})$, translates into

$$\zeta(n) = \frac{(-1)^{n-1}}{(n-1)!} \times \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{Log}\left(\mathrm{Log}(1+e^{ix})\right)\left(\mathrm{Log}(1+e^{ix})\right)^{n-1}\,dx\,.$$

Corollary 3. For all positive integers n, we have

$$\Phi(2n+1) = (2n)!\,\zeta(2n+1)\,, \tag{23}$$

$$(-1)^n\,\Phi(2n) = (-1)^{n-1}\,(2n-1)!\,\zeta(2n) = \frac{B_{2n}}{4n}\,(2\pi)^{2n}\,, \tag{24}$$

and

$$\Phi'(2n) = (2n-1)!\,[n\zeta(2n+1) - \zeta(2n)\psi(2n) - \sum_{k=1}^{n-1}\zeta(2n-k)\,\zeta(k+1) - \zeta'(2n)]. \tag{25}$$

Proof of Corollary 3. By Euler's formula [5, p. 252], we have

$$\zeta_H(2n) = (n+1)\,\zeta(2n+1) - \sum_{k=1}^{n-1}\zeta(2n-k)\,\zeta(k+1)\,,$$

and thus formulas (23)–(25) derive from (21) and (22). $\square$

Example 3. Applying (23), (24) and (25) to the case $n=1$, we obtain

$$\begin{aligned}
\Phi(3) = 2\zeta(3) = \zeta_H(2) &= -\frac{1}{2i\pi}\int_{-\pi}^{\pi} x\,\mathrm{Log}^2(1+e^{ix})\,dx\,,\\
-\Phi(2) = \zeta(2) &= \frac{1}{2i\pi}\int_{-\pi}^{\pi} x\,\mathrm{Log}(1+e^{ix})\,dx = \frac{\pi^2}{6}\,,\\
\Phi'(2) &= \zeta(3) - \zeta(2)(1-\gamma) - \zeta'(2)\,.
\end{aligned}$$

3.3 New formulas for Glasser-Manna type integrals

The relation between $\Phi(s)$ and $\zeta_H(s)$ resulting from formula (9) when written in a neighborhood of $s = -n$ (for a non-negative integer n) enables to derive new formulas for Glasser-Manna type integrals.

Corollary 4. We have

$$\Phi(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{ix}{\operatorname{Log}(1+e^{ix})}\,dx = \frac{1}{2}\log(2\pi) - \frac{1}{2}\gamma + \frac{1}{2}\,, \tag{26}$$

and

$$\begin{aligned}\Phi'(0) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{ix\operatorname{Log}(\operatorname{Log}(1+e^{ix}))}{\operatorname{Log}(1+e^{ix})}\,dx \\ &= \frac{1}{2}\gamma_1 + \frac{1}{4}\log^2(2\pi) + \frac{1}{4}\gamma^2 - \frac{1}{2}\gamma - \frac{1}{2}\gamma\log(2\pi) - \frac{7}{8}\zeta(2) + \beta\,, \end{aligned}\tag{27}$$

where β is the linear coefficient in the Laurent expansion of ζ_H at $s=0$.

Proof of Corollary 4. Around 0, we have the decomposition given by (9):

$$\zeta_H(s) = \zeta(s+1) + \pi\cot(\pi s)\,\zeta(s) + \Gamma(1-s)\Phi(s)\,,$$

and the expansion [10, Equation (8)]

$$\zeta_H(s) = \frac{1}{2s} + \frac{1}{2}\gamma + \frac{1}{2} + \beta s + \mathrm{O}(s^2)\,.$$

On the other side, we have the expansions

$$\begin{aligned}\zeta(s+1) &= \frac{1}{s} + \gamma - \gamma_1\,s + \mathrm{O}(s^2)\,,\\ \pi\cot(\pi s)\,\zeta(s) &= -\frac{1}{2s} - \frac{1}{2}\log(2\pi) + \Big(\frac{1}{2}\gamma_1 - \frac{1}{4}\log^2(2\pi) + \frac{1}{4}\gamma^2 + \frac{7}{8}\zeta(2)\Big)s + \mathrm{O}(s^2)\,,\\ \Gamma(1-s)\Phi(s) &= \Phi(0) + (\Phi'(0) + \gamma\Phi(0)) + \mathrm{O}(s^2)\,.\end{aligned}$$

By identifying the constant coefficient in the expansion of the right-hand member of (9), we deduce the equation

$$\frac{1}{2}\gamma + \frac{1}{2} = \gamma - \frac{1}{2}\log(2\pi) + \Phi(0)\,,$$

which is equivalent to (26). In the same way, formula (27) is derived by identifying the linear coefficient in the expansion of the right-hand member of (9). □

Remark 4. We derive from [17, Corollary 4.2] the following evaluation of the constant β which appears in formula (27):

$$\beta = -\gamma_1^{[2]}(0) - \gamma_1 = 1 + \frac{1}{2}\gamma - \frac{1}{4}\gamma^2 - \gamma_1 + \frac{1}{4}\zeta(2) - \sum_{n=2}^{\infty}\frac{|b_n|}{(n-1)^2} = 1.589935\ldots\,,$$

where b_n are the Bernoulli numbers of the second kind defined by way of their generating function

$$\frac{x}{\log(x+1)} = \sum_{n=0}^{\infty} b_n\,x^n \qquad (|x|<1).$$

Remark 5. Formula (26) appears in a slightly different but equivalent form in [15, Proposition 3.4], due to the identities

$$\Phi(0) = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{ix\,dx}{2\log(2\cos(\frac{x}{2})) + ix} = \frac{2}{\pi}\int_0^{\pi} \frac{x^2}{x^2 + 4\log^2(2\cos(\frac{x}{2}))}\,dx\,.$$

Corollary 5. We have

$$\Phi(-1) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{ix}{\mathrm{Log}^2(1+e^{ix})}\,dx = \log(\mathcal{A}) - \frac{1}{12}\gamma + \frac{7}{24}\,, \tag{28}$$

where $\mathcal{A} = \exp(\frac{1}{12} - \zeta'(-1))$ is the Glaisher-Kinkelin constant, and for all integers $n \geq 2$,

$$(2n-1)!\,\Phi(1-2n) = \log(\mathcal{A}_n) - \frac{B_{2n}}{2n}\gamma + \sum_{k=0}^{2n-1}\binom{2n-1}{k}\frac{B_k\,B_{2n-k}}{(2n-k)^2}\,, \tag{29}$$

with $\log(\mathcal{A}_n) := \dfrac{H_{2n-1}B_{2n}}{2n} - \zeta'(1-2n)$.[1]

Example 4.

$$\Phi(-3) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{ix}{\mathrm{Log}^4(1+e^{ix})}\,dx = \frac{1}{6}\log(\mathcal{A}_2) + \frac{1}{720}\gamma + \frac{1}{320}\,.$$

Proof of Corollary 5. Around $s = -1$, we have the expansion [10, Equation (9)]

$$\zeta_H(s) = -\frac{1}{12(s+1)} - \frac{1}{12}\gamma - \frac{1}{8} + \mathrm{O}(s+1)\,.$$

On the other side, we have

$$\begin{aligned}
\zeta(s+1) &= -\frac{1}{2} + \mathrm{O}(s+1)\,,\\
\pi\cot(\pi s)\,\zeta(s) &= \zeta'(-1) + \mathrm{O}(s+1)\,,\\
\Gamma(1-s)\Phi(s) &= \Phi(-1) + \mathrm{O}(s+1)\,.
\end{aligned}$$

Formula (28) is then deduced by identifying the constant coefficient in the expansion of the right-hand member of (9). In the same way, from [10, Proposition 2], the Laurent expansion around $s = 1-2n$ for $n \geq 2$ is given by

$$\zeta_H(s) = \frac{\zeta(1-2n)}{s+2n-1} - \frac{B_{2n}}{2n}\gamma + \frac{H_{2n-1}B_{2n}}{2n} + \sum_{k=0}^{2n-1}\binom{2n-1}{k}\frac{B_k\,B_{2n-k}}{(2n-k)^2} + \mathrm{O}(s+2n-1)$$

which, by the same method, allows us to deduce formula (29). $\square$

[1] The numbers $\mathcal{A}_n$ are called *generalized Glaisher-Kinkelin constants* (cf. [13]).

Corollary 6. For all positive integers n, we have

$$(2n)!\,\Phi(-2n) = -\zeta'(-2n) + (2n+1)\,\frac{B_{2n}}{4n}\,, \tag{30}$$

from which follows the reflection formula:

$$2(2n)!\,\Phi(-2n) = (-1)^{n+1}(2\pi)^{-2n}\,\Phi(2n+1) + (2n+1)\,\frac{B_{2n}}{2n}\,. \tag{31}$$

Example 5.

$$2\Phi(-2) = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{ix}{\mathrm{Log}^3(1+e^{ix})}\,dx = -\zeta'(-2n) + \frac{1}{8} = \frac{1}{4\pi^2}\,\zeta(3) + \frac{1}{8}\,. \tag{32}$$

Proof of Corollary 6. Around $s = -2n$, we have

$$\zeta_H(s) = -\frac{B_{2n}}{4n} + \frac{B_{2n}}{2} + \mathrm{O}(s+2n)\,.$$

On the other side, we have the expansions

$$\begin{aligned}
\zeta(s+1) &= -\frac{B_{2n}}{2n} + \mathrm{O}(s+2n)\,,\\
\pi\cot(\pi s)\,\zeta(s) &= \zeta'(-2n) + \mathrm{O}(s+2n)\,,\\
\Gamma(1-s)\Phi(s) &= (2n)!\Phi(-2n) + \mathrm{O}(s+2n)\,.
\end{aligned}$$

By identifying the constant coefficient in the expansion of the right-hand member of (9), we obtain formula (30). Moreover, the well-known identity

$$-2\zeta'(-2n) = (-1)^{n+1}\,(2\pi)^{-2n}\,(2n)!\,\zeta(2n+1)$$

and the relation

$$\Phi(2n+1) = (2n)!\,\zeta(2n+1)$$

given by (23) enable to deduce formula (31) from (30). □

Appendix: Evaluation of Glasser-Manna integrals with shifted Mascheroni series

For any natural number k, let us consider the series

$$\sigma_k := \sum_{n=k+1}^{\infty} \frac{|b_n|}{n-k}\,,$$

where b_n are the Bernoulli numbers of the second kind defined by means of their generating function

$$\frac{x}{\log(x+1)} = \sum_{n=0}^{\infty} b_n\, x^n = 1 + \frac{x}{2} - \frac{x^2}{12} + \frac{x^3}{24} - \frac{19x^4}{720} + \cdots$$

For positive values of k, the series σ_k are called *shifted Mascheroni series* [14]. The nice identities $\sigma_0 = \gamma$ and $\sigma_1 = \frac{1}{2}\log(2\pi) - \frac{1}{2}\gamma - \frac{1}{2}$ are well-known [14, Proposition 2]. The first one is a classical result due to Mascheroni, and the second allows us to deduce from formula (26) the following evaluation:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{ix}{\mathrm{Log}(1+e^{ix})}\,dx = \frac{4}{\pi}\int_0^{\pi/2} \frac{x^2}{x^2+\log^2(2\cos(x))}\,dx = \sigma_1 + 1 = 1.13033\ldots \tag{A1}$$

This formula was first given by Glasser and Manna [15, Proposition 3.3] in a slightly different but equivalent form. Moreover, thanks to the simplest special cases of the following formula arising from [14, Proposition 3]:

$$\sum_{k=1}^{n}(-1)^{n+k}k!\,S(n,k)\,\sigma_{k+1} = -\zeta'(-n) - \frac{B_{n+1}}{n+1}\,(\gamma + H_{n+1} - H_n) \qquad (n \geq 1)\,, \tag{A2}$$

with $S(n,k)$ denoting the Stirling numbers of the second kind, we can easily derive from formulas (28) and (32) above further evaluations of the same kind, such as

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{ix}{\mathrm{Log}^2(1+e^{ix})}\,dx = \sigma_2 + \frac{5}{12} = 0.49232\ldots \tag{A3}$$

and

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \frac{ix}{\mathrm{Log}^3(1+e^{ix})}\,dx = -\sigma_2 + 2\sigma_3 + \frac{1}{8} = 0.15544\ldots \tag{A4}$$

More generally, it is possible to deduce from (A2) the following identities that are nothing more than reinterpretations of formulas (29) and (30) in terms of shifted

Mascheroni series σ_k. These are

$$\frac{(2n-1)!}{2\pi}\int_{-\pi}^{\pi}\frac{ix}{\mathrm{Log}^{2n}(1+e^{ix})}\,dx=\sum_{k=1}^{2n-1}(-1)^{k+1}k!\,S(2n-1,k)\,\sigma_{k+1}$$
$$+\frac{B_{2n}\,H_{2n}}{2n}+\sum_{k=0}^{2n-1}\binom{2n-1}{k}\frac{B_k\,B_{2n-k}}{(2n-k)^2}\qquad(n\geq 2),\quad \text{(A5)}$$

and

$$\frac{(2n)!}{2\pi}\int_{-\pi}^{\pi}\frac{ix}{\mathrm{Log}^{2n+1}(1+e^{ix})}\,dx=\sum_{k=1}^{2n}(-1)^{k}k!\,S(2n,k)\,\sigma_{k+1}+(2n+1)\,\frac{B_{2n}}{4n}$$
$$(n\geq 1).\quad \text{(A6)}$$

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