A Complement to Laurent expansion of harmonic zeta functions

Marc-Antoine Coppo, Bernard Candelpergher

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Abstract We complement an earlier article dedicated to harmonic zeta functions by outlining a method for obtaining closed-form expressions of the Laurent series coefficients of the harmonic zeta function ζ_H about its pole at $s = 1$. These coefficients are named harmonic Stieltjes constants by analogy with the classical case.

1 Two representations of the harmonic zeta function

We recall that the harmonic zeta function ζ_H (noted h in [2]) is defined by

$$\zeta_H(s) := \sum_{n=1}^{\infty} \frac{H_n}{n^s} \quad \text{for } \mathrm{Re}(s) > 1\,,$$

where H_n are the classical harmonic numbers

$$H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}\,.$$

We consider the function F defined by the integral representation

$$F(s) := \frac{\Gamma(1-s)}{2i\pi} \int_L \frac{z^{s-1}e^z}{e^z - 1} \mathrm{Log}\left(\frac{e^z - 1}{z}\right) dz\,,$$

where Log denotes the principal logarithm and L is the Hankel contour defined by the parametrization

$$z = \mathrm{Log}(1 + e^{ix}) \text{ with } x \in]-\pi, \pi[\quad \text{for all } z \in L\,.$$

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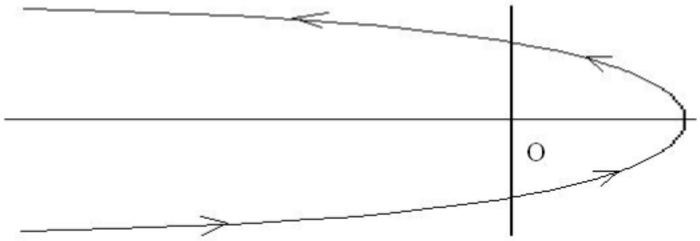

This function appears in [2] and plays a central role in the study of the function ζ_H around its poles at the negative integers. The following representation:

$$\zeta_H(s) = \pi\cot(\pi s)\,\zeta(s) + \zeta(s+1) - \psi(1-s)\zeta(s) - \zeta'(s) + F(s) \quad \text{for } s \text{ in } \mathbb{C}\setminus\mathbb{Z}\,, \tag{1}$$

where $\psi(s) = \Gamma'(s)/\Gamma(s)$ is the digamma function, is a direct consequence of [2, Theorem 1] (see [2, Eqs. (8), (12) and (25)]).

On the other hand, differentiating the Hankel integral representation of ζ:

$$\zeta(s) = \frac{\Gamma(1-s)}{2i\pi}\int_L \frac{z^{s-1}e^z}{1-e^z}\,dz$$

leads to the following identity:

$$\zeta'(s) + \psi(1-s)\zeta(s) = \frac{\Gamma(1-s)}{2i\pi}\int_L \frac{z^{s-1}e^z\mathrm{Log}(z)}{1-e^z}\,dz \quad \text{for } s \neq 1, 2, 3, \ldots \tag{2}$$

(see [1, Eqs. (2.4)–(2.7)]). Furthermore, the above integral splits into two parts:

$$\int_L \frac{z^{s-1}e^z\mathrm{Log}(z)}{1-e^z}\,dz = \int_L \frac{z^{s-1}e^z}{e^z-1}\mathrm{Log}\left(\frac{e^z-1}{z}\right)dz - \int_L \frac{z^{s-1}e^z}{e^z-1}\mathrm{Log}(e^z-1)\,dz\,,$$

allowing a rewriting of formula (2) as follows:

$$\zeta'(s) + \psi(1-s)\zeta(s) = F(s) - G(s)\,, \tag{3}$$

with

$$G(s) := \frac{\Gamma(1-s)}{2i\pi}\int_L \frac{z^{s-1}e^z}{e^z-1}\mathrm{Log}(e^z-1)\,dz\,.$$

Finally, a substitution of (3) in formula (1) leads to another simpler representation of ζ_H involving the function G. We have

$$\zeta_H(s) = \pi\cot(\pi s)\,\zeta(s) + \zeta(s+1) + G(s) \quad \text{for } s \text{ in } \mathbb{C}\setminus\mathbb{Z}\,. \tag{4}$$

2 Laurent series expansions at $s = 1$

The splitting of $\zeta_H(s)$ into three parts given by formula (4) above is the key formula for obtaining the Laurent series expansion of the harmonic zeta function about its (double) pole at $s = 1$. To show this, we will make use of the following expansions:

a) The Laurent expansions of $\pi\cot(\pi s)$ and $\zeta(s)$ at $s = 1$ are known. They are respectively

$$\pi\cot(\pi s) = \frac{1}{s-1} - 2\zeta(2)(s-1) - 2\zeta(4)(s-1)^3 - \cdots\,,$$

and

$$\zeta(s) = \frac{1}{s-1} + \gamma - \gamma_1(s-1) + \frac{1}{2}\gamma_2(s-1)^2 + \cdots\,,$$

where γ_n are the classical Stieltjes constants. The expansion of $\pi\cot(\pi s)\,\zeta(s)$ is then deduced by Cauchy's product as follows:

$$\pi\cot(\pi s)\,\zeta(s) = \frac{1}{(s-1)^2} + \frac{\gamma}{s-1} - 2\zeta(2) - \gamma_1 + \left(\frac{1}{2}\gamma_2 - 2\gamma\zeta(2)\right)(s-1) + \cdots \tag{5}$$

b) The Taylor series expansion of $\zeta(s+1)$ at $s = 1$ is given by

$$\zeta(s+1) = \zeta(2) + \zeta'(2)(s-1) + \frac{1}{2}\zeta''(2)(s-1)^2 + \cdots \tag{6}$$

Moreover, we will use afterwards a well-known expression of $\zeta'(2)$:

$$\zeta'(2) = \zeta(2)\gamma + \zeta(2)\log(2\pi) - 2\pi^2\log(A)\,,$$

where A is the Glaisher-Kinkelin constant defined by

$$\log A = \lim_{n\to\infty}\left\{\sum_{k=1}^{n} k\log k - \left(\frac{n^2}{2} + \frac{n}{2} + \frac{1}{12}\right)\log n + \frac{n^2}{4}\right\}.$$

c) The Laurent expansion of $\Gamma(z)$ at $z = 0$ is given by

$$\Gamma(z) = \sum_{k\geq 0} P_k(-\gamma,\ldots,(-1)^k\zeta(k))z^{k-1}\,,$$

where the polynomials P_k are the modified Bell polynomials defined by

$$\begin{aligned}\exp\Big(\sum_{k\geq 1} x_k\frac{z^k}{k}\Big) &= \sum_{k\geq 0} P_k(x_1,\ldots,x_k)z^k \\ &= 1 + x_1 z + \left(\frac{1}{2}x_2 + \frac{1}{2}x_1^2\right)z^2 + \left(\frac{1}{3}x_3 + \frac{1}{2}x_1x_2 + \frac{1}{6}x_1^3\right)z^3 + \cdots\end{aligned}$$

The Laurent expansion of $\Gamma(1-s)$ at $s=1$ is easily deduced from this expression by setting $z=1-s$. We have

$$\Gamma(1-s) = -\frac{1}{s-1} - \gamma - \left(\frac{1}{2}\zeta(2) + \frac{1}{2}\gamma^2\right)(s-1) - \left(\frac{1}{3}\zeta(3) + \frac{1}{2}\gamma\zeta(2) + \frac{1}{6}\gamma^3\right)(s-1)^2 + \cdots \quad (7)$$

d) Let us consider the function

$$g(s) := G(s)/\Gamma(1-s) = \frac{1}{2i\pi}\int_L \frac{z^{s-1}e^z}{e^z-1}\mathrm{Log}(e^z-1)\,dz\,.$$

This is an entire function of s, and the change of variables

$$z = \mathrm{Log}(1+e^{ix}) \text{ with } x \in]-\pi,\pi[$$

leads to the integral representation

$$g(s) = -\frac{1}{2i\pi}\int_{-\pi}^{\pi} x(\mathrm{Log}(1+e^{ix}))^{s-1}\,dx\,.$$

If we now define K_n by

$$K_n := g^{(n)}(1) = \frac{i\pi}{2}\int_{-1}^{1} x\mathrm{Log}^n(\mathrm{Log}(1+e^{i\pi x}))\,dx\,,$$

then $K_0 = g(1) = 0$, and the Taylor series expansion of $g(s)$ at $s=1$ is given by

$$g(s) = \sum_{n\geq 1}\frac{K_n}{n!}(s-1)^n\,. \quad (8)$$

e) The Laurent expansion of $G(s)$ at $s=1$ is deduced from (7) and (8) by Cauchy's product. The result is as follows:

$$G(s) = -K_1 - \left(\frac{K_2}{2} + \gamma K_1\right)(s-1) - \left(\frac{K_3}{6} + \gamma\frac{K_2}{2} + (\gamma^2+\zeta(2))\frac{K_1}{2}\right)(s-1)^2 + \cdots \quad (9)$$

In particular, it results from the decomposition (4) and the expansions (5), (6), and (9) above that the constant term in the Laurent expansion of $\zeta_H(s)$ at $s=1$ is $-\zeta(2)-\gamma_1-K_1$. However, by using yet another representation of ζ_H, we have shown (see [3, Corollary 1 and Lemma 2]) that this constant term is $\frac{1}{2}\gamma^2 + \frac{1}{2}\zeta(2)$. By comparing these two expressions, we derive the relation

$$-\zeta(2) - \gamma_1 - K_1 = \frac{1}{2}\gamma^2 + \frac{1}{2}\zeta(2)$$

which gives an expression of the integral K_1 in terms of $\zeta(2)$, γ and γ_1. We have

$$-K_1 = -\frac{i\pi}{2}\int_{-1}^{1} x\mathrm{Log}(\mathrm{Log}(1+e^{i\pi x}))\,dx = \frac{3}{2}\zeta(2) + \frac{\gamma^2}{2} + \gamma_1 = 2.561174\ldots \quad (10)$$

Remark 1. Unfortunately, no such formula is known for the integral K_2 whose decimal approximation is $K_2 = -1.924491$. More generally, it is not known whether the integrals K_n for $n \geq 2$ may be expressed using classical constants.

3 Evaluation of the harmonic Stieltjes constants

The expansions (5), (6), and (9) above and the decomposition (4) enable us to write the Laurent series expansion of the harmonic zeta function ζ_H about its double pole $s = 1$. We have

$$\zeta_H(s) = \frac{1}{(s-1)^2} + \frac{\gamma}{(s-1)} + \tilde{\gamma}_0 - \tilde{\gamma}_1(s-1) + \sum_{n=2}^{\infty}\frac{(-1)^n}{n!}\tilde{\gamma}_n(s-1)^n \quad (0 < |s-1| < 1)\,,$$

where the coefficents $\tilde{\gamma}_n$ are the *harmonic Stieltjes constants* which are so called by analogy with the classical Stieltjes constants (see [3, Remark 3]). Moreover, this provides a method to evaluate these constants. Thus, for the first two ones, we obtain the following closed-form expressions:

$$\tilde{\gamma}_0 = -\zeta(2) - \gamma_1 - K_1 = \frac{\gamma^2}{2} + \frac{\pi^2}{12} = 0,98905599\ldots$$

and

$$\tilde{\gamma}_1 = \frac{1}{2}K_2 + 2\pi^2\log A - \frac{1}{2}\gamma_2 - \frac{\pi^2}{12}\gamma - \frac{\pi^2}{6}\log 2\pi - \frac{1}{2}\gamma^3 - \gamma\gamma_1 = 0,40076\ldots$$

Remark 2. The expressions for $\tilde{\gamma}_0$ and $\tilde{\gamma}_1$ given above coincide with the asymptotic formula for $\tilde{\gamma}_n$ mentioned in [3] (see [3, Remark 3 c)]). In this order, we have

$$\tilde{\gamma}_0 = \lim_{n\to\infty}\left\{\sum_{k=1}^{n}\frac{H_k}{k} - \frac{1}{2}\log^2(n) - \gamma\log(n)\right\}$$

and

$$\tilde{\gamma}_1 = \lim_{n\to\infty}\left\{\sum_{k=1}^{n}\frac{H_k}{k}\log k - \frac{1}{3}\log^3(n) - \frac{1}{2}\gamma\log^2(n)\right\}.$$

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