Descartes' rule of signs

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DESCARTES' RULE OF SIGNS

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ABSTRACT. A sequence of $d+1$ signs $+$ and $-$ beginning with a $+$ is termed as a *sign pattern (SP)*. We say that the real polynomial $P := x^d + \sum_{j=0}^{d-1} a_j x^j$, $a_j \neq 0$, defines the SP $\sigma := (+, \operatorname{sgn} a_{d-1}, \ldots, \operatorname{sgn} a_0)$. By Descartes' rule of signs, for the quantity of positive (resp. negative) roots of P, one has $pos \leq c$ (resp. $neg \leq p = d - c$), where c and p are the numbers of sign changes and sign preservations in σ; the numbers $c - pos$ and $p - neg$ are even. We say that P realizes the SP σ with the pair (pos, neg). For SPs with $c = 2$, we give some sufficient conditions for the realizability of pairs (pos, neg) of the form $(0, d-2k)$, $k = 1, \ldots, [(d-2)/2]$.

Key words: real polynomial in one variable; Descartes' rule of signs; sign pattern

AMS classification: 26C10, 30C15

1. Introduction

In the present paper we consider a problem which is a natural continuation of Descartes' rule of signs. The latter states that the number of positive roots of a real univariate polynomial (counted with multiplicity) is majorized by the number of sign changes in the sequence of its coefficients. We focus on polynomials without zero coefficients. Such a polynomial (say, of degree d) is representable in the form $P := x^d + a_{d-1}x^{d-1} + \cdots + a_1 x + a_0$, $a_j \in \mathbb{R}^*$. Denoting by c and p the numbers of sign changes and sign preservations in the sequence 1, a_{d-1}, $\ldots$, a_1, a_0 and by pos and neg the number of positive and negative roots of P (hence $c+p = pos+neg = d$) one obtains the conditions

$$\begin{array}{llll} pos \leq c \quad , & neg \leq p \quad , & c + p = d \quad , \\ c - pos \in 2\mathbb{N} \cup 0 \quad , & p - neg \in 2\mathbb{N} \cup 0 \quad , & (-1)^{pos} = \operatorname{sgn}\,(a_0) \end{array} \tag{1.1}$$

(the condition $neg \leq p$ results from Descartes' rule applied to the polynomial $P(-x)$).

We call *sign pattern (SP)* a sequence of $+$ or $-$ signs of length $d+1$ beginning with a $+$. We say that the polynomial P defines the SP $(+, \operatorname{sgn}(a_{d-1}), \ldots, \operatorname{sgn}(a_1), \operatorname{sgn}(a_0))$. A pair (pos, neg) satisfying conditions (1.1) is called *admissible*. An admissible pair (AP) is called *realizable* if there exists a polynomial P with exactly pos positive distinct and exactly neg negative distinct roots.

Example 1. For $c = 0$ the all-pluses SP is realizable with any AP (which is of the form $(0, d-2k)$, $k = 0, 1, \ldots, [d/2]$, where $[\alpha]$ denotes the integer part of $\alpha \in \mathbb{R}$). Indeed, one can construct a polynomial P with d distinct negative roots and $d-1$ distinct critical levels. Then in the family of polynomials $P+t$, $t > 0$, one encounters polynomials with exactly $d-2$, $d-4$, …, $d-2[d/2]$ negative distinct roots and with no positive roots.

We show below (see Remark 1) that any SP with $c=1$ is realizable with any AP $(1, d-1-2k)$, $k \leq [(d-1)/2]$. To this end we remind the formulation of a *concatenation lemma* (see [2]):

Lemma 1. *Suppose that the monic polynomials P_1 and P_2 of degrees d_1 and d_2 with SPs $(+, \sigma_1)$ and $(+, \sigma_2)$ respectively realize the pairs (pos_1, neg_1) and (pos_2, neg_2). Here σ_j denote what remains of the SPs when the initial sign + is deleted. Then*

(1) if the last position of σ_1 is +, then for any $\varepsilon > 0$ small enough, the polynomial $\varepsilon^{d_2} P_1(x) P_2(x/\varepsilon)$ realizes the SP $(1, \sigma_1, \sigma_2)$ and the pair $(pos_1 + pos_2, neg_1 + neg_2)$;

(2) if the last position of σ_1 is −, then for any $\varepsilon > 0$ small enough, the polynomial $\varepsilon^{d_2} P_1(x) P_2(x/\varepsilon)$ realizes the SP $(1, \sigma_1, -\sigma_2)$ and the pair $(pos_1 + pos_2, neg_1 + neg_2)$. Here $-\sigma_2$ is obtained from σ_2 by changing each + by − and vice versa.

Remark 1. For $d = 1$, the SP $(+, +)$ (resp. $(+, -)$) is realizable with the AP $(0, 1)$ (resp. $(1, 0)$) by the polynomial $x + 1$ (resp. $x - 1$). Applying Lemma 1 with P_1 and P_2 of the form $x \pm 1$ one realizes for $d = 2$ all the three SPs with $c = 0$ or $c = 1$ with the APs of the form $(0, 2)$ or $(1, 1)$. Suppose that for $d = d_0 \geq 2$ all SPs with $c = 0$ or $c = 1$ are realizable by monic polynomials (denoted by P). Then to realize for $d = d_0 + 1$ a SP with $c = 0$ or $c = 1$ with the pair $(0, d_0 + 1)$ or $(1, d_0)$ it suffices to apply Lemma 1 with $P_1 = P$ and with $P_2 = x - 1$ (resp. $P_2 = x + 1$) if $c = 0$ and the last two signs of the SP defined by P are $(+, -)$ (resp. if $c = 1$ and these last two signs are $(-, -)$).

To realize for $c = 1$ a SP with any AP $(1, d-1-2k)$, $k \leq [(d-1)/2]$, it suffices to perturb a polynomial P realizing this SP with the pair $(1, d-1)$ so that all critical levels become distinct and then choose suitable values of $t > 0$ in the family of polynomials $P - t$.

In the present paper we consider the case $c = 2$. This is the first truly nontrivial case. We should point out that due to the possibility to consider instead of the polynomial $P(x)$ the polynomial $P(-x)$ (this change exchanges the quantities c and p and the quantities *pos* and *neg*), it suffices to consider (for a given degree d) the cases $c \leq [d/2]$.

Notation 1. We denote by $\Sigma_{m,n,q}$ the SP consisting of $m \geq 1$ pluses followed by $n \geq 1$ minuses followed by $q \geq 1$ pluses, where $m + n + q = d + 1$. By P^R we denote the *reverted* of the polynomial P, i.e. $P^R := x^d P(1/x)$. If P defines the SP $\Sigma_{m,n,q}$, then P^R defines the SP $\Sigma_{q,n,m}$.

For small values of n, we have the following result:

Theorem 1. *(1) For $n = 1$, $d \geq 2$, and $n = 2$, $d \geq 3$, any SP $\Sigma_{m,n,q}$ is realizable with the AP $(0, d-2)$.*

(2) For $n = 3$ and $d \geq 5$, any SP $\Sigma_{m,3,q}$ is realizable with the AP $(0, d-2)$. For $d = 4$, the SP $(+, -, -, -, +)$ is not realizable with the AP $(0, 2)$.

(3) For $n = 4$, the SP $\Sigma_{m,4,q}$ is realizable with the AP $(0, d-2)$ if $q \geq 3$ and $m \geq 3$ and $d \geq 10$, or if $q > m = 2$ and $d \geq 11$.

(4) For $m = 1$ and $n \geq 4$, the SP $\Sigma_{1,n,q}$ is not realizable with the AP $(0, d-2)$.

The theorem is proved in Section 2.

Remarks 1. (1) If a SP $\Sigma_{m,n,q}$ is realizable with the AP $(0, d-2)$, then it is realizable with any AP of the form $(0, d-2k)$, $k = 1, \ldots, [(d-2)/2]$. Indeed, if a polynomial P with distinct nonzero roots realizes the SP $\Sigma_{m,n,q}$, then one can perturb P to make all its critical levels distinct. Then in the family $P + t$ one encounters (for suitable positive values of t) polynomials with exactly $(0, d-2k)$ distinct negative roots and no positive ones, for $k = 1, \ldots, [(d-2)/2]$. As $t \geq 0$, the constant term of the polynomial P is positive hence $P + t$ defines the SP $\Sigma_{m,n,q}$.

(2) The exhaustive answer to the question which couples (SP, AP) are realizable for $d \leq 8$ is given in [2] and [4]. Thus for $d \geq 9$, $n \leq 4$ and $c = 2$, the only cases when the AP is $(0, d-2)$ and which are not covered by Theorem 1 are the ones of $\Sigma_{3,4,3}$ and $\Sigma_{2,4,4}$ for $d = 9$ and of $\Sigma_{2,4,5}$ for $d = 10$.

Proposition 1. *(1) For $d = 9$, the SPs $\Sigma_{3,4,3}$ and $\Sigma_{2,4,4}$ are not realizable with the AP $(0,7)$.*

(2) For $d = 10$, the SP $\Sigma_{2,4,5}$ is not realizable with the AP $(0,8)$.

Proposition 1 is proved in Section 3. Our next result contains sufficient conditions for realizability of a SP $\Sigma_{m,n,q}$ with the AP $(0, d-2)$:

Theorem 2. *The SP $\Sigma_{m,n,q}$ is realizable with the AP $(0, d-2)$ if one of the two following conditions holds true :*

$$A(d,m,n) := -dn^2 + 2dm + 3dn - 2m^2 - 2mn - 2d + 2m > 0$$

(1.2)

and

$$dn - dn^2/2 < A(d,m,n) < 0$$

(1.3)

Theorem 2 is proved in Section 4.

Remark 2. If a SP $\Sigma_{m,n,q}$ is realizable with the AP $(0, d-2)$, then it is realizable with any AP of the form $(0, d-2k)$, $k = 1, \ldots, [(d-2)/2]$. Indeed, if a polynomial P with distinct nonzero roots realizes the SP $\Sigma_{m,n,q}$, then one can perturb P to make all its critical levels distinct. Then in the family $P + t$ one encounters (for suitable positive values of t) polynomials with exactly $(0, d-2k)$ distinct negative roots and no positive ones, for $k = 1, \ldots, [(d-2)/2]$. As $t \geq 0$, the constant term of the polynomial P is positive hence $P + t$ defines the SP $\Sigma_{m,n,q}$.

2. Proof of Theorem 1

Part (1). For $d = 2$ and $d = 3$, the polynomials

$$(x-1)^2 + 1 = x^2 - 2x + 2 \ , \ (x+2)((x-2)^2 + 2) = x^3 - 2x^2 - 2x + 12$$

realize the APs $(0,0)$ and $(0,1)$ with the SPs $(+,-,+)$ and $(+,-,-,+)$ respectively. If a polynomial P realizes a SP $\Sigma_{m,n,q}$ (with $n = 1$ or 2) with the AP $(0, d-2)$, then the concatenation Q of P with $x + 1$ realizes the SP $\Sigma_{m,n,q+1}$ with the AP $(0, d-1)$, and the polynomial $Q^R := x^d Q(1/x)$ (the *reverted* of Q) realizes the SP $\Sigma_{q+1,n,m}$ with the AP $(0, d-1)$. Thus by means of concatenation and reversion one can realize all SPs $\Sigma_{m,1,q}$ and $\Sigma_{m,2,q}$ with the AP $(0, d-2)$.

Part (2). For $d = 4$, the nonrealizability of the SP $(+,-,-,-,+)$ with the AP $(0,2)$ is proved in [3]. For $d = 5$, the SP $(+,-,-,-,+,+)$ is realizable with the AP

$(0, 3)$, see [1]. To prove the first claim of part (2) one has to combine concatenation and reversion as in the proof of part (1).

Part (3). For $d = 10$, the polynomial

$$\begin{aligned}&(x+1)^8(x^2-2.49x+1.56) \;=\\ &x^{10}+5.51x^9+9.64x^8-1.24x^7-25.76x^6-30.94x^5\\ &-2.24x^4+25.64x^3+24.76x^2+9.99x+1.56\end{aligned}$$

defines the SP $\Sigma_{3,4,4}$. (The quadratic factor is without real roots.) One can perturb its 8-fold root at -1 so that the latter splits into 8 negative simple roots. Thus the perturbation will realize this SP with the AP $(0, 8)$. Similarly, for $d = 11$, a suitable perturbation of the polynomial

$$\begin{aligned}&(x+1)^9(x^2-4.69x+5.5) \;=\\ &x^{11}+4.31x^{10}-0.71x^9-35.34x^8-69.96x^7-2.94x^6\\ &+186.06x^5+335.04x^4+302.16x^3+156.79x^2+44.81x+5.5\end{aligned}$$

realizes the SP $\Sigma_{2,4,6}$ with the AP $(0, 9)$. As in the proof of parts (1) and (2), one deduces the realizability of all SPs as claimed by part (3) by applying concatenation and reversion.

Part (4). Suppose that the SP $\Sigma_{1,n,q}$ with $n \geq 4$ is realizable by the polynomial

$$P(x) = (x^{d-2} + e_1x^{d-3} + \ldots + e_{d-2})(x^2 - zx + y)$$

where $e_j > 0$ is the j^{th} elementary symmetric function of the moduli of the negative roots of P and $z^2 < 4y$ (As $y > 0$, the coefficient of x of the quadratic factor must be negative, otherwise all coefficients of P will be positive, so $z > 0$). Thus one obtains the conditions

$$\begin{aligned} e_1 - z &< 0\\ e_2 - e_1z + y &< 0\\ e_3 - e_2z + e_1y &< 0\\ e_4 - e_3z + e_2y &< 0 \end{aligned}$$

This means that

$$\begin{aligned} z^2 &< 4y\\ z &> e_1 > 0\\ e_1z &> y + e_2\\ e_2z &> e_3 + e_1y\\ e_3z &> e_4 + e_2y > 0 \end{aligned} \tag{2.4}$$

Keeping in mind that $e_j > 0$, $y > 0$ and $z > 0$, one gets $z > \dfrac{e_4 + e_2y}{e_3}$ and $z < 2\sqrt{y}$. Hence $2e_3\sqrt{y} > e_4 + e_2y$, i.e. $e_2y - 2e_3\sqrt{y} + e_4 < 0$. The last inequality implies

$$\sqrt{y} < \frac{e_3 + \sqrt{e_3^2 - e_2e_4}}{e_2}$$

and as $\sqrt{y} > \frac{z}{2} > \frac{e_1}{2}$ one deduces the condition

$$e_1e_2 - 2e_3 < \sqrt{4e_3^2 - 4e_2e_4} \quad (*)$$

From Newton's inequality $e_1e_2 \geq \frac{3d}{d-2}e_3$ one deduces that $e_1e_2 - 2e_3 > 0$. Hence inequality $(*)$ can be simplified by factoring squares of both hand-sides and then dividing by e_2 as follows :

$$e_1^2e_2 + 4e_4 < 4e_1e_3 \quad (**)$$

We are going to show that for $d \geq 3$,

$$e_1^2e_2 + 4e_4 > 4e_1e_3 \quad (***)$$

which contradiction proves part (4). For $d = 3$, one has $e_1e_2 \geq 9e_3$ (see Proposition 2 on page 2 of [6]). Suppose that $(***)$ holds true up to degree $d \geq 3$. We proceed by induction on d. Denote by $(-a_j)$ the negative roots of P. For degree $d+1$, we have to show that

$$(a_{d-1} + e_1)^2(a_{d-1}e_1 + e_2) + 4(a_{d-1}e_3 + e_4) > 4(a_{d-1} + e_1)(a_{d-1}e_2 + e_3), \quad (E1)$$

where e_j are the elementary symmetric functions of the quantities $a_1, \ldots, a_{d-2}$, which is simplified to

$$a_{d-1}^3e_1 + 2a_{d-1}^2e_1^2 + a_{d-1}e_1^3 > 3a_{d-1}^2e_2 + 2a_{d-1}e_1e_2, \quad (A1)$$

Newton's inequality $e_1^2 \geq \frac{2d}{d-1}e_2$ implies the following ones:

$$2a_{d-1}^2e_1^2 \geq 4a_{d-1}^2\frac{d}{d-1}e_2 \quad \textit{and}$$

$$a_{d-1}e_1^3 \geq \frac{2d}{d-1}a_{d-1}e_1e_2 \quad (B)$$

From inequalities (B) we conclude that

$$a_{d-1}^3e_1 + 2a_{d-1}^2e_1^2 + a_{d-1}e_1^3 > 4a_{d-1}^2\frac{d}{d-1}e_2 + \frac{2d}{d-1}a_{d-1}e_1e_2 > 3a_{d-1}^2e_2 + 2a_{d-1}e_1e_2$$

which proves (4.9) and hence $(***)$ as well.

3. Proof of Proposition 1

We give in detail the proof of part (1). For part (2), we point out only the differences w.r.t. the proof of part (1). These differences are only technical in character. In order to give easily references to the different parts of the proof, the latter are marked by 1^0, 2^0, $\ldots$, 6^0.

Proof of part (1) of Proposition 1. 1^0. Suppose that there exists a polynomial $P := RQ$, where

$$R := (x + u_1) \cdots (x + u_7) \ , \ u_j > 0 \ , \ \text{ and } \ Q := x^2 + rx + s \ ,$$

which realizes one of the two SPs $\Sigma_{3,4,3}$ or $\Sigma_{2,4,4}$ with the AP $(0, 7)$. We set $P := \sum_{j=0}^{9} p_jx^j$ and $Q := (x - a)^2 + b$, $a \in \mathbb{R}$, $b \geq 0$. We show that for $b = 0$, there exists no polynomial satisfying the conditions

$$p_3 < 0 \ , \ p_6 < 0 \ , \ \text{ resp. } \ p_4 < 0 \ , \ p_7 < 0 \ . \tag{3.5}$$

Hence this holds true also for $b > 0$ because $P = R \cdot Q|_{b=0} + bR$, and the polynomial R has all coefficients positive. This in turn implies that for $b \geq 0$, there exists no polynomial P realizing the SP $\Sigma_{3,4,3}$ or $\Sigma_{2,4,4}$. So from now on we concentrate on the case $b = 0$.

2^0. Suppose that a polynomial P with $b = 0$ and $u_1 \geq u_2 \geq \cdots \geq u_7 \geq 0$ satisfying the left or right couple of inequalities (3.5) exists. We make the change of variables $x \mapsto u_1 x$ and after this we multiply P by $(1/u_1)^9$ (these changes preserve the signs of the coefficients), so now we are in the case $u_1 = 1$. Denote by $\Delta \subset \mathbb{R}^7_+ = \{(u_2, u_3, \ldots, u_7, a)\}$ the set on which one has conditions (3.5). The closure $\overline{\Delta}$ of this set is compact. Indeed, one has $p_1 \geq 0$ hence

$$1 + u_2 + \cdots + u_7 - 2a \geq 0 \ \text{ and } \ u_j \leq 1 \ \text{ hence } \ a \in [0, 7/2] \ .$$

The set $\overline{\Delta}$ can be stratified according to the multiplicity vector of the variables $(u_2, \ldots, u_7)$ and the possible equalities $u_j = 0$, $u_i = 1$ and/or $a = 0$. Suppose that the set $\overline{\Delta}$ contains a polynomial satisfying the inequalities (3.5).

Remarks 2. (1) For this polynomial one has $a > 0$, otherwise all its coefficients are nonnegative. One has also $u_j > 0$, $j = 2, \ldots, 7$. Indeed, in the case of $\Sigma_{3,4,3}$ (resp. $\Sigma_{2,4,4}$), if three or more (resp. if four or more) of the variables u_j are 0, then the polynomial P has less than two sign changes in the sequence of its coefficients and by the Descartes rule of signs P cannot have two positive roots counted with multiplicity. For $\Sigma_{3,4,3}$, if exactly one or two of the variables u_j equal 0, then the polynomial P is the product of x with a polynomial defining the SP $(+,+,+,-,-,-,-,+,+)$ or of x^2 with a polynomial P^* defining the SP $(+,+,+,-,-,-,-,+)$. However these SPs are not realizable with the APs $(0,6)$ or $(0,5)$ respectively. For $\Sigma_{2,4,4}$, if exactly one, two or three of the variables u_j equal 0, then P is the product of x, x^2 or x^3 with a polynomial defining respectively the SP $(+,+,-,-,-,-,+,+,+)$, $(+,+,-,-,-,-,+,+)$ or $(+,+,-,-,-,-,+)$ which is not realizable with the AP $(0,6)$, $(0,5)$ or $(0,4)$.

(2) The set $\overline{\Delta}$ being compact the quantity $p_3 + p_6$, resp. $p_4 + p_7$, attains its minimum $-\delta$ on it ($\delta > 0$). Consider the set $\Delta^\bullet \subset \overline{\Delta}$ on which one has $p_3 + p_6 \leq -\delta/2$, resp. $p_4 + p_7 \leq -\delta/2$. On this set one has $a \geq 2^{-9}\delta$. Indeed, $P = x^2 R - 2axR + a^2 R$, so any coefficient of P is not less than $-2a\sigma$, where σ is the sum of all coefficients of R (they are all nonnegative); clearly $\sigma \leq 2^7$ (follows from $u_j \in [0,1]$).

(3) There exists $\delta_* > 0$ such that on the set $\Delta^\bullet$, one has also $u_j \geq \delta_*$. This follows from part (1) of the present remarks.

3^0. We need some technical lemmas:

Lemma 2. *The minimum of the quantity $p_3 + p_6$, resp. $p_4 + p_7$, is not attained at a point of the set $\Delta^\bullet$ with three or more distinct and distinct from 1 among the quantities u_j, $2 \leq j \leq 7$.*

The lemmas used in the proof of Proposition 1 are proved after the proposition.

Lemma 3. *Conditions (3.5) fail for $u_1 = u_2 = \cdots = u_7 = 1$ and any $a > 0$.*

Thus to prove Proposition 1 we have to consider only the case when exactly one or two of the quantities u_j are distinct from 1. We use the following result:

Lemma 4. *For $d \geq 4$, set $P := RQ$, where $R := \prod_{i=1}^{d-2}(x+u_i)$, $u_i > 0$, $Q := (x-a)^2$. Then the coefficients p_j of P, $j = 2, \ldots, d-2$, are quadratic polynomials in a with positive leading coefficients and with two distinct positive roots.*

4^0. Further we consider several different cases according to the multiplicity of u_{j_0}, the smallest of the variables u_j. In the proofs we use linear changes $x \mapsto \chi x$, $\chi > 0$, followed by $P \mapsto \chi^{-9}P$. These changes preserve the signs of the coefficients; the condition $u_1 = 1$ is lost and the condition $u_{j_0} = 1$, $j_0 \neq 1$, is obtained. The aim of this is to have more explicit computations. In all the cases the polynomial R is of the form $R = (x+1)^{s_1}(x+v)^{s_2}(x+w)^{s_3}$, $s_1+s_2+s_3 = 7$, and one has $v > 1$, $w > 1$, but v and w are not necessarily distinct and we do not suppose that $v > w$ or $v < w$. Allowing the equality $v = w$ means treating together cases of exactly two or exactly three distinct quantities u_j (counting also $u_1 = 1$). We list the triples (s_1, s_2, s_3) defining the cases:

$$(5,1,1) \quad , \quad (4,2,1) \quad , \quad (3,3,1) \quad , \quad (3,2,2) \quad , \quad (2,4,1) \quad ,$$

$$(2,3,2) \quad , \quad (1,5,1) \quad , \quad (1,4,2) \quad \text{and} \quad (1,3,3) \quad .$$

The cases when there are exactly two different quantities u_j one of which is $u_1 = 1$ can be coded in a similar way. E.g. $(5,2)$ means that $R = (x+1)^5(x+u)^2$, $u > 1$. The non-realizability of these cases follows automatically from the one of the above 9 ones (when v and w coalesce), with the only exception of $R = (x+1)^6(x+w)$ (the case $(6,1)$). In the latter case it suffices to observe that if the left or right two of conditions (3.5) fail for $R = (x+1)^5(x+v)(x+w)$, $1 < v < w$, then they also fail in the limit as $v \to 1$.

5^0. We consider the SP $\Sigma_{2,4,4}$ first. We compute using MAPLE the resultant $\mathrm{Res}\,(p_4, p_7, a)$ as a function of v and w. Then we set $v := 1+V$, $w := 1+W$, $V > 0$, $W > 0$. In all 9 cases this resultant is a polynomial in V and W with all coefficients positive. Hence for no value of $V > 0$ and $W > 0$ do the coefficients p_4 and p_7 vanish together.

In all 9 cases, the leading coefficients of p_4 and p_7 considered as quadratic polynomials in a are positive. In fact, they are polynomials in v and w with all coefficients positive. For $v = w = 2$, we compute the two roots $y_1 < y_2$ of p_4 and the two roots $y_3 < y_4$ of p_7. In all 9 cases, one has $y_1 < y_2 < y_3 < y_4$. By continuity, these inequalities hold true for all values of $v > 1$ and $w > 1$. Hence the intervals (y_1, y_2) and (y_3, y_4) on which p_4 and p_7 are negative do not intersect for any $v > 1$, $w > 1$. This proves the proposition in the case of $\Sigma_{2,4,4}$.

6^0. Consider now the SP $\Sigma_{3,4,3}$. Recall that the polynomials $P(x)$ and $x^9P(1/x)$ have one and the same numbers of positive and negative roots. Their roots are mutually reciprocal and they define the same SP. Hence the non-realizability of the case $(5,1,1)$ (resp. $(4,2,1)$, or $(3,3,1)$, or $(3,2,2)$) implies the one of $(1,5,1)$ (resp. $(2,4,1)$ and $(1,4,2)$, or $(1,3,3)$, or $(2,3,2)$).

As in the case of $\Sigma_{2,4,4}$, we express $\mathrm{Res}\,(p_3, p_6, a)$ as a polynomial of v and w, and then of V and W. In cases $(5,1,1)$, $(4,2,1)$ and $(3,2,2)$, this resultant has a single monomial with negative coefficient, this is UV. We give the monomials VW, V^2 and W^2 for these three cases:

$$(5,1,1) \quad -9408VW + 28224V^2 + 28224W^2 \quad ,$$

$$(4,2,1) \quad -18816VW + 47040V^2 + 28224W^2 \quad ,$$

$$(3,2,2) \quad -37632VW + 47040V^2 + 47040W^2 \quad .$$

The discriminants of these quadratic homogeneous polynomials are negative hence they are nonnegative (and positive for $V > 0$, $W > 0$). In the case of $(3,3,1)$, there are exactly two monomials with negative coefficients, namely VW and V^2W. The resultant equals

$$(-28224VW+56448V^2+28224W^2)+V(-42336VW+127008W^2+282240V^2)+\cdots$$

(we skip all other monomials; their coefficients are positive). The two quadratic homogeneous polynomials have negative discriminants, so they are positive for $V > 0$, $W > 0$.

The rest of the reasoning goes by exact analogy with the case of $\Sigma_{2,4,4}$.

□

Proof of Lemma 2. Denote by v_1, v_2 and v_3 three distinct and distinct from 1 of the variables u_j. We prove that one can choose v_1^*, v_2^*, v_3^*, $a^* \in \mathbb{R}$ such that the infinitesimal change $v_j \mapsto v_j + \varepsilon v_j^*$, $j = 1, 2, 3$, $a \mapsto a + \varepsilon a^*$, $\varepsilon > 0$, results in $p_\mu \mapsto p_\mu + \varepsilon p_\mu^* + o(\varepsilon)$, $p_\nu \mapsto p_\nu + \varepsilon p_\nu^* + o(\varepsilon)$, where $(\mu,\nu) = (3,6)$ or $(4,7)$ and $p_\mu^* < 0$, $p_\nu^* < 0$. Hence locally the quantity $p_\mu + p_\nu$ is not minimal.

Set $P := (x+v_1)^{\alpha_1}(x+v_2)^{\alpha_2}(x+v_3)^{\alpha_3}(x-a)^2P^\dagger$, where a, $-v_1$, $-v_2$ and $-v_3$ are not roots of $P^\dagger$ and α_j are the multiplicities of throots $-v_j$ of P. Set $P_{v_j} := P/(x+v_j)$, $P_a := P/(x-a)$, $P_{v_i,v_j} := P/((x+v_i)(x+v_j))$, $P_{a,v_j} := P/((x+a)(x+v_j))$ etc. Then the above infinitesimal change transforms P into

$$P + \varepsilon\tilde{P} + o(\varepsilon) \ , \ \text{ where } \ \tilde{P} := \sum_{j=1}^{3} \alpha_j v_j^* P_{v_j} - 2a^* P_a \ .$$

We show that one can choose v_j^* and a^* such that the coefficients of x^μ and x^ν of the polynomial $\tilde{P}$ (where $(\mu,\nu) = (3,6)$ or $(4,7)$) are both negative from which the lemma follows. To this end we observe that each of the polynomials P_{v_j} and P_a is a linear combination of $P^\diamond := P_{v_1,v_2,v_3,a} := x^5 + Ax^4 + Bx^3 + Cx^2 + Dx + E$, $xP^\diamond$, $x^2P^\diamond$ and $x^3P_{v,w,a}$.

We consider first the case of $\Sigma_{3,4,3}$, i.e $(\mu,\nu) = (3,6)$. The 2-vectors of coefficients of x^3 and x^6 of the polynomials $P^\diamond$, $xP^\diamond$, $x^2P^\diamond$ and $x^3P^\diamond$ equal $(B,0)$, $(C,1)$, (D,A) and (E,B) respectively. For $B \neq 0$, the first two of them are not collinear. As $E \neq 0$ (see parts (2) and (3) of Remarks 2), for $B = 0$, the second and fourth of these vectors are not collinear and the choice of v_j^* and a^* is possible.

If $(\mu,\nu) = (4,7)$, then the 2-vectors of coefficients of x^4 and x^7 equal $(A,0)$, $(B,0)$, $(C,1)$ and (D,A). One has either $A \neq 0$ or $B \neq 0$. Indeed, the polynomial $P^\diamond$ has all roots real and by the Rolle theorem this is the case of $(P^\diamond)'$ and $(P^\diamond)''$ as well. If $A = B = C = D = 0 \neq E$ (resp. $A = B = C = 0 \neq D$ or $A = B = 0 \neq C$), then $P^\diamond$ (resp. $(P^\diamond)'$ or $(P^\diamond)''$) has not all roots real. thus either $(A,0)$, $C,1)$ or $(B,0)$, $(C,1)$ are not collinear and the choice of v_j^* and a^* is possible.

□

Proof of Lemma 3. For the polynomial $(x+1)^7(x-a)^2$, we list its coefficients p_3, p_4, p_6 and p_7 and their roots:

$$\begin{array}{cccccccc} p_3 & = & 7-42a+35a^2 & , & p_4 & = & 21-70a+35a^2 & , \\ 0.2 & , & 1 & & 0.36\ldots & , & 1.63\ldots & \\ p_6 & = & 35-42a+7a^2 & , & p_7 & = & 21-14a+a^2 & . \\ 1 & , & 5 & & 1.70\ldots & , & 12.2\ldots & \end{array}$$

Hence for no value of $a \geq 0$ does one have the left or the right two of conditions (3.5) together.

□

Proof of Lemma 4. Set $R := r_{d-2}x^{d-2} + r_{d-3}x^{d-3} + \cdots + r_0$, $r_j > 0$, $r_{d-2} = 1$. The polynomial R has $d-2$ negative roots. Hence Newton's inequalities hold true:

$$(3.6) \quad \left(r_k / \binom{d-2}{k} \right)^2 \geq \left(r_{k-1} / \binom{d-2}{k-1} \right) \left(r_{k+1} / \binom{d-2}{k+1} \right) \ , \ k = 1, \ldots, d-3 \ .$$

The coefficient p_{k+1} equals $a^2 r_{k+1} - 2ar_k + r_{k-1}$, $k = 1, \ldots, d-3$, $r_{k+1} > 0$. This quadratic polynomial has two distinct positive roots if and only if $r_k^2 > r_{k-1}r_{k+1}$. These inequalities result from (3.6) because $\binom{d-2}{k}^2 > \binom{d-2}{k-1}\binom{d-2}{k+1}$ (the latter inequality is equivalent to $((k+1)/k)((d-1-k)/(d-2-k)) > 1$ which is true).

□

Proof of part (2) of Proposition 1. 1^0. In the analog of part 1^0 of the proof of part (1), we set $R := (x+u_1)\cdots(x+u_8)$, $u_j > 0$, and the analog of inequalities (3.5) reads $p_5 < 0$, $p_8 < 0$.

2^0. In the analog of part 2^0 we make the change of variables $x \mapsto u_1 x$ and then we multiply P by $(1/u_1)^{10}$. We denote by $\Delta \subset \mathbb{R}_+^8 = \{(u_2, u_3, \ldots, u_8, a)\}$ the set on which one has the conditions $p_5 < 0$, $p_8 < 0$. On the closure $\overline{\Delta}$ of this set one has $p_1 \geq 0$ hence

$$1 + u_2 + \cdots + u_8 - 2a \geq 0 \ \text{ and } \ u_j \leq 1 \ \text{ hence } \ a \in [0, 4] \ .$$

The analog of Remarks 2 reads:

Remarks 3. (1) One has $u_j > 0$, $j = 2, \ldots, 8$. Indeed, if exactly one of the quantities u_j is 0, then $P = xY$, where the polynomial Y defines the SP $\Sigma_{2,4,4}$ which by part (1) of Proposition 1 is impossible. If more than one of the quantities u_j is 0, then see part (1) of Remarks 2 about $\Sigma_{2,4,4}$.

(2) In the proof of part (2) of Proposition 1 we define $\Delta^\bullet \subset \overline{\Delta}$ as the one on which one has $p_5 + p_8 \leq -\delta/2$. On this set one has $a \geq 2^{-10}\delta$. Indeed, as $P = x^2R - 2axR + a^2R$, any coefficient of P is not less than $-2a\sigma$, where σ is the sum of all coefficients of R (they are all nonnegative); clearly $\sigma \leq 2^8$ (follows from $u_j \in [0,1]$).

3^0. The analog of Lemma 2 reads: *The minimum of the quantity $p_5 + p_8$ is not attained at a point of the set $\Delta^\bullet$ with three or more distinct and distinct from 1 among the quantities u_j, $2 \leq j \leq 8$.*

The proof is much the same as the one of Lemma 2. One sets $(\mu, \nu) = (5, 8)$. Each of the polynomials P_{v_j} and P_a is a linear combination of $P^\diamond := P_{v_1,v_2,v_3,a} := x^6 + Ax^5 + Bx^4 + Cx^3 + Dx^2 + Ex + F$, $xP^\diamond$, $x^2P^\diamond$ and $x^3P_{v,w,a}$. The 2-vectors of coefficients of x^5 and x^8 of the polynomials $P^\diamond$, $xP^\diamond$, $x^2P^\diamond$ and $x^3P^\diamond$ equal $(A, 0)$, $(B, 0)$, $(C, 1)$ and (D, A) respectively. If $A \neq 0$ or $B \neq 0$, there are two noncollinear among the first three of these vectors and the choice of v_j^* and a^* is possible. If $A = B = 0$, then, as $F \neq 0$, either the polynomial $P^\diamond$ or one of its derivatives is not with all roots real which is a contradiction.

The analog of Lemma 3 reads: *Conditions* $p_5 < 0$*,* $p_8 < 0$ *fail for* $u_1 = \cdots = u_8 = 1$ *and any* $a > 0$*.*

Here's the proof of this. For the polynomial $(x+1)^8(x-a)^2$, we list its coefficients p_5, p_8 and their roots:

$$\begin{array}{ccccccc} p_5 & = & 28(2-5a+2a^2) & , & p_8 & = & 28-16a+a^2 \quad , \\ 0.5 & , & 2 & & 2 & , & 14 \end{array}$$

Hence for no value of $a \geq 0$ does one have $p_5 < 0$, $p_8 < 0$.

We remind that Lemma 4 is formulated for any $d \geq 4$.

4^0. In the analog of part 4^0 of the proof, one has $R = (x+1)^{s_1}(x+v)^{s_2}(x+w)^{s_3}$, $s_1+s_2+s_3 = 8$, and one has to consider the following cases of exactly three different quantities u_j:

$$\begin{array}{ccccccccccc} (6,1,1) & , & (5,2,1) & , & (4,3,1) & , & (4,2,2) & , & (3,4,1) & , & (3,3,2) \quad , \\ (2,5,1) & , & (2,4,2) & , & (2,3,3) & , & (1,6,1) & , & (1,5,2) & \text{and} & (1,4,3) \quad . \end{array}$$

The cases with exactly two different quantities u_j are treated in the same way. The exceptional case is the one with $R = (x+1)^7(x+w)$ (the case $(7, 1)$).

5^0. We compute Res (p_5, p_8, a) as a function of v and w and then set $v := 1+V$, $w := 1 + W$. Our aim is to show that in all 12 cases, the leading coefficients of p_5 and p_8 considered as quadratic polynomials in a are positive. The rest of the reasoning is done by analogy with part 5^0 of the proof of part (1) of Proposition 1.

6^0. It is in the analog of 6^0 that there is much more technical work to be done. Of the twelve cases listed in 4^0, in three there is a single monomial with a negative coefficient, and this is UV. We list the coefficients of the monomials UV, U^2 and V^2 of the cases $(6, 1, 1)$, $(5, 2, 1)$ and $(4, 2, 2)$ respectively:

$$(-10206, 35721, 35721) \ , \ (-20412, 61236, 35721) \ , \ (-40824, 61236, 61236) \ .$$

Everywhere in 6^0 quadratic and biquadratic polynomials have negative discriminants. There are four cases in which exactly two monomials have negative signs, namely $(4, 3, 1)$, $(3, 4, 1)$, $(3, 3, 2)$ and $(2, 4, 2)$ in which we give only the monomials forming quadratic homogeneous polynomials (multiplied by 1 or U):

$$(-30618UV + 76545U^2 + 35721V^2) + U(-10206UV + 221130U^2 + 91854V^2)$$

$$(-40824UV + 81648U^2 + 35721V^2) + U(-81648UV + 326592U^2 + 122472V^2)$$

$$(-61236UV + 76545U^2 + 61236V^2) + U(-20412UV + 221130U^2 + 81648V^2)$$

$$(-81648UV + 81648U^2 + 61236V^2) + U(-163296UV + 326592U^2 + 108864V^2)$$

In the cases $(2, 5, 1)$ and $(1, 6, 1)$ there are four and five negative monomials respectively. These cases are treated in a similar way:

$$(-51030UV + 76545U^2 + 35721V^2) + U(-187110UV + 391230U^2 + 153090V^2)$$

$$+U^2(-245430UV + 868725U^2 + 297270V^2)$$

$$+U^3(-86670UV + 1094472U^2 + 352350V^2)$$

and

$$(-6804UV + 6804U^2 + 3969V^2) + U(-22680UV + 28728U^2 + 12474V^2)$$

$$+U^2(-29052UV + 50436U^2 + 15849V^2) + U^4(-3252UV + 24628U^2 + 3672V^2)$$

$$+U^3(-16848UV + 47088U^2 + 10368V^2)\ .$$

In the case $(2, 3, 3)$, there are four negative monomials which we include in polynomials as follows:

$$(-91854UV + 76545U^2 + 76545V^2) + (-59778U^2V^2 + 273375U^4 + 273375V^4)$$

$$(-30618U^2V + 221130U^3 + 221130V^3 - 30618UV^2)\ .$$

For the third polynomial in brackets its corresponding inhomogeneous polynomial

$$-30618x^2 + 221130x^3 + 221130 - 30618x$$

has one negative and two complex conjugate roots. For a univariate real polynomial with positive leading coefficient and having only negative and complex conjugate roots we say that it is of *type P*. It is clear that the homogeneous polynomial corresponding to a type P univariate polynomial (we say that it is also of type P) is nonnegative.

In the case $(1, 5, 2)$, there are seven negative monomials:

$$(-102060UV+76545U^2+61236V^2)+U(-374220UV+391230U^2+136080V^2)$$
$$+(-490860U^3V+868725U^4+10530U^2V^2+369360UV^3+79704V^4)$$
$$+U^2(513540V^3-210600UV^2-173340U^2V+1094472U^3)$$
$$+U^2(-215190U^2V^2+855450U^4+372915V^4)$$
$$+U^4V(-64116UV+300060U^2+176760V^2)\ .$$

The third and fourth of the polynomials in brackets are of type P hence nonnegative.

Finally, in the case $(1,4,3)$ we have also seven negative monomials:

$$(-122472UV+81648U^2+76545V^2)+(-383940U^2V^2+565056U^4+273375V^4)$$
$$+(326592U^3-244944U^2V-40824UV^2+221130V^3)+$$
$$+U(-359649U^2V^2+552096U^4+557928V^4)$$
$$+(-75816U^3V^3+332928U^6+79065V^6)$$
$$+U(-15066U^3V^3+126720U^6+138096V^6)\ .$$

The third and the last two polynomials in brackets are of type P. □

4. Proof of Theorem 2

Consider the polynomial $P=(x+1)^{d-2}(x^2-zx+y)$, where the quadratic factor has no real roots, $z>0$, $y>0$ i.e. $z^2<4y$. Hence $y>0$ and $z>0$ (otherwise all coefficients of P must be positive). We expand P in powers of x:

$$P:=x^d+\Sigma_{j=1}^d p_jx^{d-j}$$

where $p_j=C_{d-2}^j-C_{d-2}^{j-1}z+C_{d-2}^{j-2}y$ and $C_{d-2}^{-1}=C_{d-2}^{d-1}=C_{d-2}^d=0$.
The coefficients of P define the SP $\Sigma_{m,n,q}$, so $p_j>0$ for $j=1,\ldots,m-1$ and for $j=m+n,\ldots,d$ and $p_j<0$ for $j=m,m+1,\ldots,m+n-1$. The latter inequalities (combined with $z<2\sqrt{y}$) :

$$C_{d-2}^j+C_{d-2}^{j-2}y<C_{d-2}^{j-1}z<2C_{d-2}^{j-1}\sqrt{y},\quad j=m,m+1,\ldots,m+n-1$$

This means that :

$$\frac{C_{d-2}^{j-1}-\sqrt{\delta_{j-1}}}{C_{d-2}^j}<\sqrt{y}<\frac{C_{d-2}^{j-1}+\sqrt{\delta_{j-1}}}{C_{d-2}^j}, \tag{4.7}$$

where $\delta_{j-1}:=(C_{d-2}^{j-1})^2-C_{d-2}^{j-2}C_{d-2}^j>0$. Indeed, the quadratic polynomial (in $\sqrt{y}$) $C_{d-2}^j-2C_{d-2}^{j-1}\sqrt{y}+C_{d-2}^{j-2}y$ has a positive discriminant δ_{j-1} hence its value is negative precisely when $\sqrt{y}$ is between its roots. Set

$$Q^{\pm}(k):=(C_{d-2}^k\pm\sqrt{\delta_k})/C_{d-2}^{k-1} \tag{4.8}$$

Lemma 5. *The quantities $Q^{\pm}(k)$ are decreasing functions in k (for $k=1,2,\ldots,[\frac{d}{2}]$).*

Lemma 6. *One has*

$$\begin{aligned} Q^{\pm}(k) &:= \tfrac{d-k-1}{k}\left(1 \pm \sqrt{1-\tfrac{k(d-k-2)}{(k+1)(d-k-1)}}\right) \\ &= \tfrac{d-k-1}{k}\left(1 \pm \sqrt{\tfrac{(d-1)}{(k+1)(d-k-1)}}\right) \ . \end{aligned}$$

Lemmas 5 and 6 are proved after the proof of Theorem 2. It follows from Lemma 5 that one can find a value of y satisfying conditions (4.7) if :

$$Q^{-}(m-1) < Q^{+}(m+n-2) \tag{4.9}$$

or equivalently

$$\tfrac{(d-m)}{m-1}\left(1-\sqrt{1-\tfrac{(m-1)(d-m-1)}{m(d-m)}}\right) \quad < \quad \tfrac{(d-m-n+1)}{m+n-2}\left(1+\sqrt{1-\tfrac{(m+n-2)(d-m-n)}{(m+n-1)(d-m-n+1)}}\right)$$

Setting

$$a = \frac{d-m}{m-1} \qquad\qquad B = \sqrt{1-\frac{(m-1)(d-m-1)}{m(d-m)}}$$

$$f = \frac{d-m-n+1}{m+n-2} \qquad\qquad G = \sqrt{1-\frac{(m+n-2)(d-m-n)}{(m+n-1)(d-m-n+1)}}$$

one transforms inequality (4.9) into $a - f < aB + fG$. One has

$$a - f = \frac{(n-1)(d-1)}{(m+n-2)(m-1)} > 0$$

which permits to take squares to obtain the condition :

$$H := \frac{(a-f)^2 - (aB)^2 - (fG)^2}{2af} < GB \tag{4.10}$$

which is equivalent to (4.9). Hence if condition (1.2) holds true then one has $H < 0 < GB$. If one has $H > 0$, then one transforms (4.10) into $H^2 < (GB)^2$, i.e. into

$$-dn^2 + 4dm + 4dn - 4m^2 - 4mn - 4d + 4m > 0$$

which is equivalent to (1.3). Theorem 2 is proved.

Proof of Lemma 5. Recall that $Q^{\pm}(k) = \dfrac{d-k-1}{k}(1 \pm A(k))$ where

$$A(k) = \sqrt{1-\frac{k(d-k-2)}{(k+1)(d-k-1)}} = \sqrt{\frac{(d-1)}{(k+1)(d-k-1)}}$$

Both factors $\dfrac{d-k-1}{k}$ and $1 + A$ of $Q^{+}(k)$ are decreasing in k hence $0 < A < 1$. All summands of $(Q^{+}(k))'$ are negative, so $(Q^{+}(k))' < 0$ for $k \geq 1$. We represent the quantity $Q^{-}(k)$ in the form

$$Q^{-}(k) = \left(\frac{d-k-2}{k+1}\right) \Big/ \left(1+\sqrt{\frac{d-1}{(k+1)(d-k-1)}}\right)$$

The inequality $Q^-(k) > Q^-(k+1)$ is equivalent to

$$\tfrac{d-k-2}{k+1} + \tfrac{d-k-2}{k+1}\sqrt{\tfrac{d-1}{(k+2)(d-k-2)}} \quad > \quad \tfrac{d-k-3}{k+2} + \tfrac{d-k-3}{k+2}\sqrt{\tfrac{d-1}{(k+1)(d-k-1)}}$$

This follows from $\dfrac{d-k-2}{k+1} > \dfrac{d-k-3}{k+2}$, $\dfrac{1}{(k+1)\sqrt{k+2}} > \dfrac{1}{(k+2)\sqrt{k+1}}$ and $(d-k-2)(d-k-1) > (d-k-3)^2$. □

Proof of Lemma 6.

$$\begin{aligned}
\delta_k &= (C_{d-2}^k)^2 - C_{d-2}^{k-1}C_{d-2}^{k+1} \\
&= \left(\frac{(d-2)\dots(d-k-1)}{k!}\right)^2 - \frac{(d-2)\dots(d-k)}{(k-1)!}\cdot\frac{(d-2)\dots(d-k-2)}{(k+1)!} \\
&= \left(\frac{(d-2)\dots(d-k)(d-k-1)(k+1)}{(k+1)!}\right)^2 \\
&\qquad\qquad - \frac{(d-2)\dots(d-k)k(k+1)(d-2)\dots(d-k-2)}{((k+1)!)^2} \\
&= \left(\frac{(d-2)\dots(d-k)}{(k+1)!}\right)^2 \big\{(d-k-1)^2(k+1)^2 \\
&\qquad\qquad -k(k+1)(d-k-1)(d-k-2)\big\} \\
&= \left(\frac{(d-2)!}{(d-k-1)!(k+1)!}\right)^2 \{(k+1)(d-1)(d-k-1)\}
\end{aligned}$$

We substitute this expression of δ_k in (C) to obtain

$$\begin{aligned}
Q_k^{\pm} &= \frac{(d-k-1)!(k-1)!}{(d-2)!}\cdot\left(\frac{(d-2)!}{(d-k-2)!k!} \pm \sqrt{\delta_k}\right) \\
&= \frac{(d-k-1)}{k} \pm \frac{1}{k(k+1)}\sqrt{(k+1)(d-1)(d-k-1)} \\
&= \frac{(d-k-1)}{k} \pm \sqrt{\frac{(d-k-1)^2}{k^2} - \frac{(d-k-1)(d-k-2)}{k(k+1)}} \\
&= \frac{(d-k-1)}{k}\left(1 \pm \sqrt{1 - \frac{k(d-k-2)}{(k+1)(d-k-1)}}\right)
\end{aligned}$$

□

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