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Anticanonical geometry of the blow-up of \mathbb{P}^4 in 8 points and its Fano model

Zhixin Xie

Abstract

Building on the work of Casagrande-Codogni-Fanelli, we develop our study on the birational geometry of the Fano fourfold $Y = M_{S, -K_S}$ which is the moduli space of semi-stable rank-two torsion-free sheaves with $c_1 = -K_S$ and $c_2 = 2$ on a polarised degree-one del Pezzo surface $(S, -K_S)$. Based on the relation between Y and the blow-up of \mathbb{P}^4 in 8 points, we describe completely the base scheme of the anticanonical system $|-K_Y|$. We also prove that the Bertini involution ι_Y of Y , induced by the Bertini involution ι_S of S , preserves every member in $|-K_Y|$. In particular, we establish the relation between ι_Y and the anticanonical map of Y , and we describe the action of ι_Y by analogy with the action of ι_S on S .

1 Introduction

Fano manifolds are classified up to dimension three. There are 10 deformation families of 2-dimensional Fano manifolds, and 105 deformation families of 3-dimensional Fano manifolds (classified by Mori-Mukai and Iskovskih, see [Isk77, Isk78, Tak89, MM04]).

In dimension 4, toric Fano manifolds have been classified by Batyrev [Bat99] and Sato [Sat00], and Fano manifolds of index $r > 2$ have been classified. Among Fano fourfolds, those of index one have special positions: Küchle constructed a number of examples with Picard number one, and explained some known results with lists of related problems (see [Küc97]). To find and classify Fano fourfolds of index one, Coates, Corti and others have embarked on a program using mirror symmetry ([CCG⁺13], and see a list of examples in [CGKS20]), where heavy computer calculations are involved. A complete classification might not be desirable, but it is interesting to exhibit some Fano fourfolds with special geometric properties, for example, those with Picard number ρ close to the conjectural boundary $\rho \leq 18$, or those whose anticanonical system has non-empty base locus. In order to study Fano manifolds with large Picard number (see [Cas12]), Casagrande introduced the invariant called *Lefschetz defect*, and developed fruitful results in this direction ([Cas13, CR22]).

Let $Y := M_{S, -K_S}$ be the moduli spaces of semi-stable rank-two torsion-free sheaves with $c_1 = -K_S$, $c_2 = 2$ on a polarised degree-one del Pezzo surface $(S, -K_S)$. The moduli spaces $Y = M_{S, -K_S}$ form a remarkable family of smooth Fano fourfolds with Picard number 9. The study of this family

is motivated by two issues. Firstly, for Fano fourfolds with large Picard number (e.g. at least 7), only few examples which are not products of del Pezzo surfaces are known. As pointed out in [CCF19, Sect. 1,B], the family of Fano fourfolds Y is the only known example of Fano fourfolds with Picard number at least 9, which is not a product of surfaces. Secondly, it is delicate to find examples of Fano fourfolds whose anticanonical system has non-empty base locus, since most Fano fourfolds classified so far are toric, which implies that any ample line bundle on them is globally generated. Some examples are constructed in [Heu16, Chapter 6.3] as complete intersections of two hypersurfaces in weighted projective spaces; two families are identified in [Sec21] as Fano fourfolds with Picard number 3 and having some contraction onto a smooth Fano threefold. In [CCF19, Thm. 1.10], it is shown that the base locus of the anticanonical system $|-K_Y|$ has positive dimension. Therefore, the geometry of Y is worth detailed understanding.

A. The anticanonical system of the Fano model Y . The birational geometry of $Y = M_{S,-K_S}$ is related to the birational geometry of the blow-up X of \mathbb{P}^4 at 8 points. In [CCF19, Lem. 5.18], an explicit relation between X and Y is given: the Fano fourfold Y is obtained from X by flipping the strict transforms of the lines through all pairs of blown up points and of the quartic curves through 7 blown up points in \mathbb{P}^4 . Thanks to this relation between X and Y , it is shown in [CCF19, Lem. 7.5, Cor. 7.6] that the base locus of $|-K_Y|$ contains the strict transform R_Y of a smooth rational quintic curve passing through the 8 blown up points in \mathbb{P}^4 , and that $|-2K_Y|$ is base-point-free. We complete the study of the anticanonical system and show more precisely that:

Theorem 1.1. *For the Fano fourfold $Y := M_{S,-K_S}$, the base scheme of $|-K_Y|$ is the reduced smooth curve R_Y .*

As a direct application, we obtain the smoothness of a general member in the anticanonical system.

Corollary 1.2. *Let $D \in |-K_Y|$ be a general divisor. Then D is smooth.*

B. The Bertini involution of the Fano model Y . Now we turn our attention to the automorphism group of Y . In [CCF19, Sect. 4], a group morphism ρ between the Picard groups of the degree-one del Pezzo surface S and of the moduli spaces $Y = M_{S,-K_S}$ is defined. This morphism ρ induces an isomorphism between the automorphism groups of S and of Y (see [CCF19, 6.15]). In particular, there is an involution ι_Y of Y which is induced by the Bertini involution ι_S of S .

We mention here that another motivation behind the study of the Bertini involution ι_Y is the understanding of the corresponding birational involutions ι_X of X and $\iota_{\mathbb{P}^4}$ of \mathbb{P}^4 . These birational maps ι_X and $\iota_{\mathbb{P}^4}$ are classically known, as they can be defined via the Cremona action of the Weyl group $W(E_8)$ on sets of 8 points in \mathbb{P}^4 (see [DO88] and [DV81]). Nevertheless, the classical definitions of ι_X and $\iota_{\mathbb{P}^4}$ do not give a geometric description of these maps. In [CCF19, Prop. 8.9, Cor. 8.10], a factorisation of these maps is given as smooth blow-ups and blow-downs using the interpretation of X as a moduli space of vector bundles on S . In view of the relation among Y , X and \mathbb{P}^4 , understanding one of the involutions helps describe the behaviour of the others.

By the analogy of Y and S , one expects that the action of ι_Y on Y has similar properties as the action of ι_S on S , where the latter is well understood (see for example [Dol12, 8.8.2]). To emphasize their analogy, we recall that the Bertini involution ι_S on S can be described as follows. The bianticanonical system $|-2K_S|$ is base-point-free and defines a 2:1-cover with image a quadric cone in \mathbb{P}^3 . The Bertini involution ι_S is then defined to be the associated covering involution. By construction, the Bertini involution ι_S on S preserves every element of $|-2K_S|$. Since a divisor $D \in |-K_S|$ defines an element $2D \in |-2K_S|$, we see that ι_S preserves every divisor in $|-K_S|$. In view of the abstract construction of ι_Y on the Fano fourfold Y , the same method cannot be applied to decide whether ι_Y preserves every divisor in $|-K_Y|$. However, by analysing the anticanonical map of Y , we show that the same property holds for Y .

Theorem 1.3. *The Bertini involution ι_Y of the Fano fourfold $Y := M_{S,-K_S}$ preserves every divisor in $|-K_Y|$.*

To understand the Bertini involution ι_Y on the Fano fourfold, our approach is analysing its behaviour on a special surface W_Y which is invariant by ι_Y . This surface W_Y is the strict transform of the cubic scroll swept out by the pencil of elliptic normal quintics in \mathbb{P}^4 through the 8 blown up points; in particular, it contains the curve R_Y . Inspired by the similarity with degree-one del Pezzo surfaces, we study the morphism defined by the restricted bianticanonical system of Y on W_Y , and we give the following description of ι_Y restricted to W_Y .

Proposition 1.4. *The Bertini involution ι_Y of the Fano fourfold $Y := M_{S,-K_S}$ preserves the surface W_Y , and its restriction $\iota_Y|_{W_Y}$ on W_Y is the biregular involution defined by the double covering*

$$\phi_{|-2K_Y|_{W_Y}} : W_Y \rightarrow V_{2,4} \subset \mathbb{P}^7,$$

where $V_{2,4} \simeq \mathbb{F}_2$ is a rational normal scroll of bidegree $(2,4)$. In particular, the Bertini involution ι_Y acts as the identity on the curve R_Y and ι_Y induces an involution on each elliptic fibre F_Y of $W_Y \rightarrow \mathbb{P}^1$.

Furthermore, there exists a smooth curve $R' \sim 3(R_Y + F_Y)$ of genus 4 on the surface W_Y , such that R' is disjoint from R_Y and contained in the fixed locus of ι_Y .

Since R_Y is contained in the fixed locus of the Bertini involution ι_Y , the involution can be lifted to the blow-up \tilde{Y} of Y along the curve R_Y . We establish the relation between the resolved anticanonical map and the lifted involution on \tilde{Y} as follows.

Theorem 1.5. *Let $\mu: \tilde{Y} \rightarrow Y$ be the blow-up of $Y := M_{S,-K_S}$ along the base curve R_Y of $|-K_Y|$, and E be the exceptional divisor. Let $f: \tilde{Y} \rightarrow \mathbb{P}(H^0(Y, \mathcal{O}_Y(-K_Y))^\vee) \simeq \mathbb{P}^5$ be the morphism defined by the base-point-free linear system $|\mu^*(-K_Y) - E|$. Then f has generically degree 4 with image Q a smooth quadric hypersurface in \mathbb{P}^5 , and f contracts the strict transform of the surface W_Y to a conic in \mathbb{P}^5 . Moreover, $f|_E: E \rightarrow f(E)$ is a finite birational morphism such that the image $f(E)$ has degree 4 in \mathbb{P}^5 .*

Furthermore, the Bertini involution ι_Y of Y can be lifted to \tilde{Y} , and the lifted involution $\iota_{\tilde{Y}}$ acts as the identity on E . Moreover, f factors through the quotient $\tilde{Y}/\iota_{\tilde{Y}}$:

$$\begin{array}{ccc}
 & \tilde{Y} & \\
 & \downarrow \mu & \searrow \\
 f \swarrow & Y & \tilde{Y}/\iota_{\tilde{Y}} \\
 & \downarrow \text{---} & \swarrow \\
 & Q \subset \mathbb{P}^5 &
 \end{array}$$

As open questions, one may like to understand the quotient $\tilde{Y}/\iota_{\tilde{Y}}$, the geometric interpretation of $\tilde{Y}/\iota_{\tilde{Y}} \rightarrow Q$, and to describe completely the fixed locus of $\iota_{\tilde{Y}}$ (see Remark 4.15).

Plan. We briefly explain the organisation of the paper. In Section 2, we summarise some results in [CCF19], including the geometry of the Fano model $Y := M_{S,-K_S}$, the connection between the blow-up X of \mathbb{P}^4 at 8 points and the degree-one del Pezzo surface S , and the relation between X and Y . We finish by recalling some basic properties of the Bertini involution of a degree-one del Pezzo surface.

In Section 3, we investigate the anticanonical system $|-K_Y|$ and the bianticanonical system $|-2K_Y|$. We prove Proposition 1.1 by an additional analysis on the simplicial facets of the cone of effective divisors on Y . We also give some auxiliary results on $|-K_Y|$ and $|-2K_Y|$, which serve as key ingredients in the study of the Bertini involution of Y .

In Section 4, we study the action of the Bertini involution of Y . Subsection 4.1 is devoted to the proof of Proposition 1.4. We study the morphism defined by the bianticanonical system $|-2K_Y|$ restricted to the surface W_Y . Computations by `Macaulay2` show that the image of W_Y is a surface of degree 6 in \mathbb{P}^7 , which helps us to describe completely the morphism; in particular, the morphism is finite of degree 2 and gives an involution on the surface W_Y . By examining the action of this covering involution, we show that it coincides with the Bertini involution ι_Y restricted to the surface W_Y .

In Subsection 4.2, we study the geometry of the anticanonical map of Y . Computations by `Macaulay2` show that the image of Y by the anticanonical map is a smooth quadric hypersurface Q in \mathbb{P}^5 . We are then ready to prove Theorem 1.3. The strategy is to prove by contradiction: we suppose that ι_Y does not preserve every divisor in $|-K_Y|$. We show that in this case, ι_Y induces a non-trivial involution ι_Q on Q . We then obtain a contradiction by analysing the fixed locus of the induced involution ι_Q and by studying the geometry of a special sub-linear systems of $|-K_Y|$ consisting of divisors containing the surface W_Y . Theorem 1.5 is obtained as a consequence of the study to prove Theorem 1.3.

In Appendix A, we include the code for several computations in Section 3 and Section 4 using the software system `Macaulay2`.

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2 Preliminaries

We fix S a general del Pezzo surface of degree one. Let $M_{S,L}$ be the moduli space of semi-stable (with respect to $L \in \text{Pic}(S)$ ample) rank-two torsion free sheaves \mathcal{F} on S with $c_1(\mathcal{F}) = -K_S$ and $c_2(\mathcal{F}) = 2$. Then it follows from the classical properties of the determinant line bundle that for the polarisation $L = -K_S$, the moduli space $Y := M_{S,-K_S}$ is Fano.

For the degree-one del Pezzo surface S , we introduce the following notions (see [CCF19, Sect. 2.1]). A conic on S is a smooth rational curve such that $-K_S \cdot C = 2$ and $C^2 = 0$. Every such conic yields a conic bundle $S \rightarrow \mathbb{P}^1$ having C as fibre. There are 2160 conics (as classes of a curve) in $H^2(S, \mathbb{Z})$. A big divisor h on S which realises S as the blow-up $\sigma : S \rightarrow \mathbb{P}^2$ at 8 distinct points is called a cubic. We have $h = \sigma^* \mathcal{O}_{\mathbb{P}^2}(1)$. There are 17280 cubics (as classes of a curve) in $H^2(S, \mathbb{Z})$.

Notation 2.1. *Given a cubic h , we use the following notation:*

- $\sigma_h : S \rightarrow \mathbb{P}^2$ is the birational map defined by h
- $q_1, \dots, q_8 \in \mathbb{P}^2$ are the points blown up by σ_h
- $e_i \subset S$ is the exceptional curve over q_i , for $i = 1, \dots, 8$
- $C_i \subset S$ is the transform of a general line through q_i , so that $C_i \sim h - e_i$, for $i = 1, \dots, 8$
- $\ell_{ij} \subset S$ is the transform of the line $\overline{q_i q_j} \subset \mathbb{P}^2$, so that $\ell_{ij} \sim h - e_i - e_j$ and ℓ_{ij} is a (-1) -curve, for $1 \leq i < j \leq 8$.

2.1 The Fano model Y

By [CCF19, Prop. 1.6], the moduli space $Y := M_{S,-K_S}$ is a smooth, rational Fano fourfold with index one and Picard number 9. For such a moduli space Y , the determinant map $\rho : H^2(S, \mathbb{R}) \rightarrow H^2(Y, \mathbb{R})$ is an isomorphism (see [CCF19, Thm. 1.3]) and yields a completely explicit description of the relevant cones of divisors $\text{Eff}(Y)$, $\text{Mov}(Y)$ and $\text{Nef}(Y)$, as well as the cone of effective curves $\text{NE}(Y)$. We cite the following statements for the cone of effective divisors $\text{Eff}(Y)$ and the cone of

effective curves $\text{NE}(Y)$, and refer the readers to [CCF19, Sect. 6] for the description of the other relevant cones.

Proposition 2.2 ([CCF19], Sect. 2.3, Cor. 6.2). *The determinant map $\rho: H^2(S, \mathbb{R}) \rightarrow H^2(Y, \mathbb{R})$ yields an isomorphism between \mathcal{E} and $\text{Eff}(Y)$, where \mathcal{E} is the subcone of $\text{Nef}(S)$ generated by the conics:*

$$\mathcal{E} := \langle C \mid C \text{ a conic} \rangle \subset H^2(S, \mathbb{R}).$$

Hence, the cone $\text{Eff}(Y)$ has 2160 extremal rays, each generated by a fixed divisor E_C , where $C \subset S$ is a conic.

Moreover, given a cubic h , $(2h + K_S)^\perp \cap \mathcal{E}$ is a simplicial facet (i.e. a face of codimension one) of \mathcal{E} , generated by the conics C_i for $i = 1, \dots, 8$ (notations as in Notation 2.1). Hence, the fixed divisors E_{C_i} for $i = 1, \dots, 8$ generate a simplicial facet of $\text{Eff}(Y)$.

Proposition 2.3 ([CCF19], Prop. 1.7). *The cone of effective curves $\text{NE}(Y)$ has 240 extremal rays, and is isomorphic to $\text{NE}(S)$. If ℓ is a (-1) -curve on S , the corresponding extremal ray of $\text{NE}(Y)$ is generated by the class of a line Γ_ℓ in $P_\ell \cong \mathbb{P}^2 \subset Y$. The corresponding elementary contraction is a small contraction, sending P_ℓ to a point.*

The determinant map ρ also relates the two automorphism groups $\text{Aut}(Y)$ and $\text{Aut}(S)$. By [CCF19, Thm. 1.9], the map $\psi: \text{Aut}(S) \rightarrow \text{Aut}(Y)$ given by $\psi(\phi)[\mathcal{F}] = [(\phi^{-1})^*\mathcal{F}]$, for $\phi \in \text{Aut}(S)$ and $[\mathcal{F}] \in Y$, is a group isomorphism. In particular, $\text{Aut}(Y)$ is finite; if S is general, then

$$\text{Aut}(Y) = \{\text{Id}_Y, \iota_Y\},$$

where $\iota_Y: Y \rightarrow Y$ is induced by the Bertini involution of S . We still call the involution $\iota_Y := \psi(\iota_S)$ of Y the Bertini involution. Explicitly, $\iota_Y: Y \rightarrow Y$ is given (see [CCF19, Def.6.19]) by $\iota_Y([\mathcal{F}]) = [\iota_S^*\mathcal{F}]$. We have a commutative diagram:

$$\begin{array}{ccc} H^2(S, \mathbb{R}) & \xrightarrow{\iota_S^*} & H^2(S, \mathbb{R}) \\ \rho \downarrow & & \downarrow \rho \\ H^2(Y, \mathbb{R}) & \xrightarrow{\iota_Y^*} & H^2(Y, \mathbb{R}). \end{array} \tag{1}$$

Finally, motivated by the analogy with del Pezzo surface of degree one, the study of the base loci of the anticanonical and the bianticanonical linear systems of Y gives the following:

Theorem 2.4 ([CCF19], Thm. 1.10). *The linear system $|-K_Y|$ has a base locus of positive dimension, while the linear system $|-2K_Y|$ is base point free.*

2.2 The blow-up X of \mathbb{P}^4 at 8 general points

2.2.1 Degree one del Pezzo surfaces and blow-ups of \mathbb{P}^4 in 8 points

For $S = \text{Bl}_{q_1, \dots, q_8} \mathbb{P}^2$ and $X = \text{Bl}_{p_1, \dots, p_8} \mathbb{P}^4$ the blow-ups respectively of \mathbb{P}^2 and \mathbb{P}^4 at 8 general points, there is a classical association between these two varieties due to Gale duality. The following is summarised from [CCF19, 2.21]; for further details of the association, we refer to [CCF19, 2.18].

Let h be a cubic on S . We associate to (S, h) a blow-up X of \mathbb{P}^4 in 8 points in general linear position as follows.

Let $q_1, \dots, q_8 \in \mathbb{P}^2$ be the points blown up under the birational morphism $S \rightarrow \mathbb{P}^2$ defined by h (the points q_1, \dots, q_8 are in general linear position by [CCF19, Rem. 2.20]), and let $p_1, \dots, p_8 \in \mathbb{P}^4$ be the associated points to $q_1, \dots, q_8 \in \mathbb{P}^2$ (the points p_1, \dots, p_8 are in general linear position by [CCF19, Lem. 2.19]). Then we set

$$X = X_h = X_{(S, h)} := \text{Bl}_{p_1, \dots, p_8} \mathbb{P}^4.$$

We always assume that $q_1, \dots, q_8 \in \mathbb{P}^2$ and $p_1, \dots, p_8 \in \mathbb{P}^4$ are associated as ordered sets of point.

Conversely, let X be a blow-up of \mathbb{P}^4 in 8 general points. Differently from the case of surfaces, the blow-up map $X \rightarrow \mathbb{P}^4$ is unique and thus X determines $p_1, \dots, p_8 \in \mathbb{P}^4$ up to projective equivalence. The 8 points $p_1, \dots, p_8 \in \mathbb{P}^4$ in turn determine $q_1, \dots, q_8 \in \mathbb{P}^2$ up to projective equivalence, and thus a pair (S, h) such that $X \cong X_{(S, h)}$. The pair (S, h) is unique up to isomorphism, therefore S is determined up to isomorphism, and h is determined up to the action of the automorphism group $\text{Aut}(S)$ on cubics.

2.2.2 Notation for the blow-up X of \mathbb{P}^4 at 8 points

Let $p_1, \dots, p_8 \in \mathbb{P}^4$ be 8 points in general linear position, and set $X := \text{Bl}_{p_1, \dots, p_8} \mathbb{P}^4$. We use the following notation:

- $E_i \subset X$ is the exceptional divisor over $p_i \in \mathbb{P}^4$, for $i = 1, \dots, 8$
- $H \in \text{Pic}(X)$ is the pull-back of $\mathcal{O}_{\mathbb{P}^4}(1)$
- $L_{ij} \subset X$ is the transform of the line $\overline{p_i p_j} \subset \mathbb{P}^4$, for $1 \leq i < j \leq 8$
- $e_i \subset E_i$ is a line, for $i = 1, \dots, 8$
- $\gamma_i \subset \mathbb{P}^4$ is the rational normal quartic through $p_1, \dots, \check{p}_i, \dots, p_8$, for $i = 1, \dots, 8$
- $\Gamma_i \subset X$ is the transform of $\gamma_i \subset \mathbb{P}^4$, for $i = 1, \dots, 8$.

2.3 From the blow-up X to the Fano model Y

We recall the explicit relation between X and Y :

Lemma 2.5 ([CCF19], Lem. 5.18). *The birational map $\xi: X \dashrightarrow Y$ is the composition of 36 (K -positive) flips: first the flips of L_{ij} for $1 \leq i < j \leq 8$, and then the flips of Γ_k for $k = 1, \dots, 8$. There is a commutative diagram:*

$$\begin{array}{ccc} & \widehat{X} & \\ & \swarrow & \searrow \\ X & \overset{\xi}{\dashrightarrow} & Y \end{array}$$

where $\widehat{X} \rightarrow X$ is the blow-up of the curves L_{ij} and Γ_k , with every exceptional divisor isomorphic to $\mathbb{P}^1 \times \mathbb{P}^2$ with normal bundle $\mathcal{O}(-1, -1)$, and $\widehat{X} \rightarrow Y$ is the blow-up of 36 pairwise disjoint smooth rational surfaces.

Notation 2.6. *We use the following notation:*

- $P_{\ell_{ij}} \subset Y$ is the flipped surface replacing $L_{ij} \subset X$, for $1 \leq i < j \leq 8$
- $P_{e_k} \subset Y$ is the flipped surface replacing $\Gamma_k \subset X$, for $k = 1, \dots, 8$.

We will sometimes write $\xi_h: X_h \dashrightarrow Y$ to stress that X_h and ξ_h depend on the chosen cubic h (while Y does not). Denote by η_h the composition map:

$$\begin{array}{ccccc} & & \eta_h & & \\ & & \text{---} & & \\ Y & \overset{\xi_h^{-1}}{\dashrightarrow} & X_h & \longrightarrow & \mathbb{P}^4 \end{array}$$

2.4 The Bertini involution of S

We recall some basic properties of the Bertini involution of a del Pezzo surface of degree one.

Proposition 2.7 ([Dol12], Thm. 8.3.2). *Suppose that S is a del Pezzo surface of degree 1. Then*

- $|-K_S|$ is a pencil of genus 1 curves with smooth general member and one base point;
- $|-2K_S|$ is base-point-free and defines a morphism $\phi_{|-2K_S|}: S \rightarrow \mathbb{P}^3$ which is finite of degree 2 with image Q a quadric cone.

The Bertini involution $\iota_S: S \rightarrow S$ is the biregular involution defined by the double covering

$$\phi_{|-2K_S|}: S \rightarrow Q.$$

For S general, ι_S is the unique non-trivial automorphism of S . The pull-back ι_S^* acts on $\text{Pic}(S)$ (and on $H^2(S, \mathbb{R})$) by fixing K_S and acting as -1 on K_S^\perp (see [Dol12, §8.8.2]). This yields:

$$\iota_S^* \gamma = 2(\gamma \cdot K_S)K_S - \gamma \quad \text{for every } \gamma \in H^2(S, \mathbb{R}). \quad (2)$$

The fixed locus of ι_S is a smooth irreducible curve of genus 4 isomorphic to the branch curve of the double cover and the base point of $|-K_S|$. The fixed curve belongs to the linear system $|-3K_S|$.

3 Anticanonical and bianticanonical linear systems of the Fano model Y

Let S be a degree-one del Pezzo surface, and $Y := M_{S, -K_S}$ be the associated Fano fourfold. To analyse the anticanonical linear system $|-K_Y|$, we introduce a special surface as follows.

Lemma 3.1 ([CCF19], Lem. 7.2). *Let $p_1, \dots, p_8 \in \mathbb{P}^4$ be general points. Then there is a pencil of elliptic normal quintics in \mathbb{P}^4 through p_1, \dots, p_8 , which sweeps out a cubic scroll $W \subset \mathbb{P}^4$.*

Let moreover $q_1, \dots, q_8 \in \mathbb{P}^2$ be the associated points to $p_1, \dots, p_8 \in \mathbb{P}^4$. Then there is a birational map $\alpha: W \rightarrow \mathbb{P}^2$ such that $\alpha(p_i) = q_i$ for $i = 1, \dots, 8$, α sends the pencil of elliptic normal quintics to the pencil of plane cubics through q_1, \dots, q_8 , and α is the blow-up of the ninth base point $q_0 \in \mathbb{P}^2$ of the pencil of plane cubics.

Let $W' \subset X$ be the transform of the cubic scroll $W \subset \mathbb{P}^4$. By [CCF19, (7.3)], we have the following diagram:

$$\begin{array}{ccc}
 & W' \subset X & \\
 \eta \swarrow & \downarrow \alpha' & \searrow \\
 W \subset \mathbb{P}^4 & & S \\
 \alpha \searrow & & \swarrow \sigma \\
 & \mathbb{P}^2 &
 \end{array} \tag{3}$$

where $\eta: W' \rightarrow W$ is the blow-up of p_1, \dots, p_8 , so the composition $\alpha' := \alpha \circ \eta: W' \rightarrow \mathbb{P}^2$ is the blow-up of q_0, \dots, q_8 . Thus W' is isomorphic to the blow-up of S in the base point of $|-K_S|$. Hence, there is an elliptic fibration $\pi: W' \rightarrow \mathbb{P}^1$, where the smooth fibres are the transforms of the elliptic normal quintics through p_1, \dots, p_8 in \mathbb{P}^4 , and every fibre is integral.

Lemma 3.2 ([CCF19], Lem. 7.4). *The surface $W' \subset X$ is disjoint from L_{ij} for $1 \leq i < j \leq 8$ and from Γ_k for $k = 1, \dots, 8$, and W' is contained in the open subset where $\xi: X \dashrightarrow Y$ is an isomorphism.*

We denote by W_Y the strict transform of W' in Y . Then $W_Y \simeq W'$.

Lemma 3.3 ([CCF19], Lem. 7.5, Lem. 7.7, Rem. 7.10). *We have $(-K_X)|_{W'} = \mathcal{O}_{W'}(R + 2F)$ and $R = \text{Bs}|-K_X|_{W'}$, where $F \subset W'$ is a fibre of the elliptic fibration, and $R \subset W'$ is a (-1) -curve and a section of the elliptic fibration. The curves R and F satisfy $-K_X \cdot R = -K_X \cdot F = 1$ and $E_i \cdot R = E_i \cdot F = 1$ for every $i = 1, \dots, 8$, so $R \equiv F$ in X and $\xi(R) \equiv \xi(F)$ in Y .*

Moreover, let $R_4 \subset \mathbb{P}^4$ be the images of R under $\eta: W' \subset X \rightarrow W \subset \mathbb{P}^4$ (see diagram (3)). Then R_4 is a smooth rational quintic curve through p_1, \dots, p_8

Corollary 3.4 ([CCF19], Cor. 7.6). *The base locus of $|-K_X|$ contains the smooth rational curve R , and the base locus of $|-K_Y|$ contains the smooth rational curve $\xi(R)$.*

We denote by R_Y the smooth rational curve $\xi(R)$ contained in the base locus of $|-K_Y|$, and F_Y a fibre of the elliptic fibration $W_Y \rightarrow \mathbb{P}^1$.

Lemma 3.5. *The normal bundle $\mathcal{N}_{R_Y/Y} \cong \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus 2}$.*

Proof. Since R_4 is a rational quintic curve in \mathbb{P}^4 , one has

$$\mathcal{N}_{R_4/\mathbb{P}^4} \cong \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b) \oplus \mathcal{O}_{\mathbb{P}^1}(c)$$

with $a \leq b \leq c$ and $a + b + c = 23$. Since $\mathcal{T}_{\mathbb{P}^4}|_{R_4} \rightarrow \mathcal{N}_{R_4/\mathbb{P}^4} \rightarrow 0$, one has that $\mathcal{N}_{R_4/\mathbb{P}^4}$ is ample. Hence, we deduce that $a, b, c > 0$. Moreover, by `Macaulay2` (see Listing 2),

$$h^0(R_4, \mathcal{N}_{R_4/\mathbb{P}^4}^* \otimes \mathcal{O}_{\mathbb{P}^4}(1) \otimes \omega_{R_4}^*) = 1,$$

we deduce that $a = 7$ and $b, c > 7$. Hence, $b = c = 8$. Therefore, by [Ful98, B.6.10], one has

$$\mathcal{N}_{R/X} \cong \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus 2}.$$

As R is disjoint from the indeterminacy locus of the map ξ_h , we deduce that

$$\mathcal{N}_{R_Y/Y} \cong \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus 2}.$$

□

Remark 3.6. (see also [CCF19, Remark 7.8]) *In \mathbb{P}^4 , let \mathcal{M} be the linear system of quintic hyper-surfaces with multiplicity at least 3 at 8 general points. Then by `Macaulay2` (see Listing 1) the base ideal $\mathfrak{b}(\mathcal{M})$ is the intersection of the ideals of 28 line $\overline{p_i p_j}$ for $1 \leq i < j \leq 8$, the ideals of 8 rational normal quartic curves γ_k for $k = 1, \dots, 8$ and the ideal of the rational quintic curve R_4 . This shows that the base scheme of $|-K_Y|$, in the open subset of Y where $\eta_h : Y \dashrightarrow \mathbb{P}^4$ is an isomorphism, is R_Y (reduced) minus the 8 points of intersection with the 8 exceptional divisors $\xi(E_1), \dots, \xi(E_8)$. The base locus of $|-K_Y|$ is thus given by R_Y , possibly union some other components contained in $\xi(E_1), \dots, \xi(E_8)$.*

Lemma 3.7. *The base locus of the anticanonical system $|-K_Y|$ is disjoint from the surfaces $P_{\ell_{ij}}$ and P_{e_k} , for $1 \leq i < j \leq 8$ and $k = 1, \dots, 8$.*

Proof. Consider the commutative diagram in Lemma 2.5:

$$\begin{array}{ccc} & \hat{X} & \\ p \swarrow & & \searrow q \\ X & \xrightarrow{\xi} & Y \end{array}$$

where $p: \hat{X} \rightarrow X$ is the blow-up of X along the curves L_{ij} and Γ_k with every exceptional divisor isomorphic to $\mathbb{P}^1 \times \mathbb{P}^2$, and $q: \hat{X} \rightarrow Y$ is the blow-up of 36 pairwise disjoint smooth rational surfaces $P_{\ell_{ij}}$ and P_{e_k} , for $1 \leq i < j \leq 8$ and $k = 1, \dots, 8$.

Suppose by contradiction that there exists a base point y of $|-K_Y|$ contained in some flipped surface that we denote by P (which is one of the surfaces $P_{\ell_{ij}}$ or P_{e_k}). Denote by $C \subset X$ the corresponding flipping curve (which is one of the curves L_{ij} or Γ_k).

Let E be the sum of exceptional divisors over L_{ij} for $1 \leq i < j \leq 8$ and over Γ_k for $k = 1, \dots, 8$. Since

$$p^*(-K_X) - 2E = -K_{\hat{X}} = q^*(-K_Y) - E,$$

one has

$$q^*(-K_Y) = p^*(-K_X) - E.$$

Let $E_y \simeq \mathbb{P}^1$ be the exceptional fibre in \hat{X} above y . Then E_y is contained in $\text{Bs}|q^*(-K_Y)| = \text{Bs}|p^*(-K_X) - E|$ and E_y is mapped surjectively onto C .

Since the blow-up of the 8 points $X = \text{Bl}_{p_1, \dots, p_8} \rightarrow \mathbb{P}^4$ is an isomorphism near a general point of C , the base scheme of $|-K_X|$ is generically reduced along C by Remark 3.6. Hence, the linear system $|p^*(-K_X) - E|$ is base-point-free above the generic point of C . This contradicts the fact that $\text{Bs}|p^*(-K_X) - E|$ contains a curve which is mapped surjectively onto C . \square

Remark 3.8. *More generally, the proof of Lemma 3.7 shows the following. Let X, Y be smooth projective fourfolds. Let $\xi: X \dashrightarrow Y$ be an anti-flip. In [Kaw89, Thm. 1.1], Kawamata showed that for smooth projective fourfolds, there exists only one type of flip and it is obtained by blowing up a \mathbb{P}^2 with normal bundle $\mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2}$ (the exceptional locus of the blowing up is $\mathbb{P}^2 \times \mathbb{P}^1$) and blowing down this $\mathbb{P}^2 \times \mathbb{P}^1$ to \mathbb{P}^1 . Thus ξ (anti-)flips a smooth curve $C \subset X$ to a smooth surface $P \subset Y$. If $\text{Bs}|-K_X|$ is reduced in the generic point of C , then $|-K_Y|$ is base-point-free on P .*

Corollary 3.9. *The curve R_Y is the unique base curve in $\text{Bs}|-K_Y|$ of anticanonical degree 1. Therefore, R_Y is independent of the choice of cubic h .*

Proof. Let $C \subset \text{Bs}|-K_Y|$ be a base curve contained in some exceptional divisor $\xi(E_i)$, for $i = 1, \dots, 8$. Let \tilde{C} be its strict transform in X . By Lemma 3.7, the curve C is disjoint from the indeterminacy locus of ξ^{-1} . Hence, one has $-K_Y \cdot C = -K_X \cdot \tilde{C}$ and $\tilde{C} \subset E_i$.

Since $-K_X = 5H - 3\sum_{j=1}^8 E_j$, $H \cdot \tilde{C} = 0$, $E_j \cdot \tilde{C} = 0$ for $j \neq i$, and $E_i \cdot \tilde{C} \leq -1$, one has

$$-K_Y \cdot C = -K_X \cdot \tilde{C} \geq 3.$$

Therefore, the curve R_Y is the unique base curve satisfying $-K_Y \cdot R_Y = -K_X \cdot R = 1$. \square

Corollary 3.10. *Let $B \subset Y$ be an irreducible component of the (set-theoretic) base locus of $|-K_Y|$, which is distinct from R_Y . Then for every simplicial facet $\langle E_{C_1}, \dots, E_{C_8} \rangle$ of $\text{Eff}(Y)$ (notation as in Notation 2.1 and Proposition 2.2), there exists a unique E_{C_i} for $i = 1, \dots, 8$ such that $B \subset E_i$.*

Proof. Given a cubic h , consider the simplicial facet $\langle E_{C_1}, \dots, E_{C_8} \rangle$ of $\text{Eff}(Y)$, where $C_i \sim h - e_i$ for $i = 1, \dots, 8$ (notation as in Notation 2.1). Then E_{C_i} are the strict transforms of the exceptional divisors $E_i \simeq \mathbb{P}^3 \subset X_h = X$ under $\xi_h = \xi: X \dashrightarrow Y$.

Since B is distinct from R_Y , we deduce that B is contained in some fixed divisor E_{C_i} by Remark 3.6. By the construction of the composition of flips ξ (see Lemma 2.5), the intersection of two fixed divisors E_{C_j} and E_{C_k} (for $k \neq j$) is the union of the flipped surfaces $P_{\ell_{jk}}$ and P_{e_l} for $l \neq j, k$. Hence, by Lemma 3.7, the fixed divisor E_i containing B is unique. \square

Proof of Theorem 1.1. We first show that R_Y is the unique irreducible component of the (set-theoretic) base locus of $|-K_Y|$.

Let h be a cubic. Let C_i be a conic such that $C_i \sim h - e_i$ for $i = 1, \dots, 8$ (notation as in Notation 2.1). Let $E_i := E_{C_i}$, where we use the notation of Proposition 2.2. By the same proposition, E_1, \dots, E_8 generate a simplicial facet of $\text{Eff}(Y)$. Suppose by contradiction that there exists another component B distinct from R_Y of the base locus of $|-K_Y|$. Then by Corollary 3.10, we may suppose that $B \subset E_1$ and $B \not\subset E_2, E_3, \dots, E_8$.

Let i, j, k, l be distinct indices in $\{1, \dots, 8\}$. Consider the conics C'_l such that $C'_l \sim 2h - e_i - e_j - e_k - e_l$ and the corresponding fixed divisors $F_{ijkl} := E_{2h - e_i - e_j - e_k - e_l}$.

Claim. The fixed divisors E_i, E_j, E_k and F_{ijkl} for $l \in \{1, \dots, 8\}$ distinct from i, j, k generate a simplicial facet of $\text{Eff}(Y)$.

Indeed, by Proposition 2.2, it is enough to find a cubic h' such that $2h' + K_S$ is orthogonal to the 8 conics C_i, C_j, C_k and C'_l for $l \in \{1, \dots, 8\}$ distinct from i, j, k .

We take $h' \sim 2h - e_i - e_j - e_k$. Then we can check that

$$\begin{aligned} C_i &\sim h' - \ell_{jk} \\ C_j &\sim h' - \ell_{ik} \\ C_k &\sim h' - \ell_{ij} \\ C'_l &\sim h' - e_l \end{aligned}$$

and $2h' + K_S$ is orthogonal to the above 8 conics. Moreover, h' is nef and big, and the corresponding birational map $\sigma_{h'} : S \rightarrow \mathbb{P}^2$ contracts the 8 pairwise disjoint (-1) -curves $\ell_{jk}, \ell_{ik}, \ell_{ij}, e_l$ for $l \neq i, j, k$. Hence, h' is a cubic. This proves the claim.

We will repeatedly use Corollary 3.10 in the following.

- Consider the simplicial facet generated by $E_1, E_2, E_3, F_{1234}, F_{1235}, F_{1236}, F_{1237}, F_{1238}$. Then $B \not\subset F_{1234}, F_{1235}, F_{1236}, F_{1237}, F_{1238}$.
- Consider the simplicial facet generated by $E_2, E_3, E_4, F_{1234}, F_{2345}, F_{2346}, F_{2347}, F_{2348}$. Then B is contained in one of the fixed divisors $F_{2345}, F_{2346}, F_{2347}, F_{2348}$. We may suppose that $B \subset F_{2345}$. Then $B \not\subset F_{2346}, F_{2347}, F_{2348}$.
- Consider the simplicial facet generated by $E_2, E_3, E_5, F_{1235}, F_{2345}, F_{2356}, F_{2357}, F_{2358}$. Then $B \not\subset F_{2356}, F_{2357}, F_{2358}$.

- Consider the simplicial facet generated by $E_2, E_3, E_6, F_{1236}, F_{2346}, F_{2356}, F_{2367}, F_{2368}$. Then by what precedes, we know that B is contained in one of the fixed divisors F_{2367}, F_{2368} . We may suppose that $B \subset F_{2367}$. Then $B \not\subset F_{2368}$.
- Consider the simplicial facet generated by $E_2, E_3, E_7, F_{1237}, F_{2347}, F_{2357}, F_{2367}, F_{2378}$. Then $B \not\subset F_{2378}$.
- Finally, consider the simplicial facet generated by $E_2, E_3, E_8, F_{1238}, F_{2348}, F_{2358}, F_{2368}, F_{2378}$. Then by what precedes, we know that B is contained in none of these 8 fixed divisors, which contradicts Corollary 3.10.

Therefore, the curve R_Y is the unique irreducible component of the base locus of $|-K_Y|$.

Now we show that the base scheme of $|-K_Y|$ is the reduced curve R_Y , i.e. there are no embedded points. Indeed, given a cubic h , consider the birational map $\eta_h: Y \dashrightarrow \mathbb{P}^4$. By Remark 3.6, the base scheme of $|-K_Y|$ is the reduced curve R_Y with possible embedded points which have support in the 8 points of intersection with the 8 exceptional divisors of η_h . By varying h , we may consider another map $\eta_{h'}: Y \dashrightarrow \mathbb{P}^4$ with other 8 exceptional divisors, so that we get 8 different points of intersection on R_Y . Such a cubic h' exists because otherwise, there is a base point y on R_Y such that for every simplicial facet $\langle E_{C_1}, \dots, E_{C_8} \rangle$ of $\text{Eff}(Y)$ the point y is contained in a unique E_{C_i} , and thus we obtain a contradiction by replacing B with y in the above paragraph. Hence, there is no embedded base point on R_Y . \square

Proof of Corollary 1.2. Since the base scheme $\text{Bs } |-K_Y|$ is the smooth curve R_Y by Proposition 1.1, we can apply [MM86, Prop. 6.8] which implies that a general member in $|-K_Y|$ is smooth. \square

In the rest of this section, we collect some auxiliary results which will be used in Section 4.

Lemma 3.11. *For a general point $x \in R_4$ (notation as in Lemma 3.3), there exists a unique divisor in \mathcal{M} which has multiplicity 3 at x : it is the secant variety of the elliptic normal quintic through the nine points p_1, \dots, p_8 and x .*

By varying x on R_4 , one obtains a one-dimensional family Sec of divisors in \mathcal{M} with scheme-theoretic intersection $\text{Bs } \text{Sec}$ defined by the ideal $\mathfrak{b}(\text{Sec})$. Then

$$(\mathfrak{b}(\text{Sec}) : \mathfrak{b}(\mathcal{M})) : \mathfrak{I}_W = \mathfrak{I}_W,$$

where the scheme defined by the ideal \mathfrak{I}_W is the reduced surface W .

Proof. We choose a random point x on R_4 which is not one of the 8 blown up points. Let $\mathcal{M}_{x,3}$ be the linear subspace of divisors in \mathcal{M} having multiplicity at least 3 at the point x . Then $\dim \mathcal{M}_{x,3} = 0$ by `Macaulay2` (see Listing 3) and thus the unique element in $\mathcal{M}_{x,3}$ is the secant variety $\text{Sec}(E_x)$, where E_x is the elliptic normal quintic in W passing through the point x and the 8 blown up points.

Let Sec be the family of secant varieties $Sec(E_x)$ for x varying on R_4 and $\mathfrak{b}(Sec)$ be the ideal associated to the scheme-theoretic intersection $Bs\ Sec$ of the family Sec . Let $\mathfrak{b}_3(Sec)$ be the ideal associated to the scheme-theoretic intersection of three general secant varieties in Sec (obtained by choosing three distinct random points on R_4 and intersecting the corresponding secant varieties).

By `Macaulay2` (see Listing 4), the quotient $\mathfrak{I}_S := (\mathfrak{b}_3(Sec) : \mathfrak{b}(\mathcal{M}))$ has degree 6 and dimension 2. Let \mathfrak{I}_W be the ideal of singular locus of the variety defined by \mathfrak{I}_S . Then by `Macaulay2` (see Listing 4), \mathfrak{I}_W has dimension 2 and degree 3; moreover, the variety defined by \mathfrak{I}_W is smooth and one has $(\mathfrak{I}_S : \mathfrak{I}_W) = \mathfrak{I}_W$. Since each of these secant varieties in Sec contains the cubic scroll W , we deduce that the variety defined by \mathfrak{I}_W is indeed the surface W .

Let \mathcal{M}_W be the sub-linear system of effective divisors in \mathcal{M} containing the surface W . By `Macaulay2` (see Listing 5), the base ideal $\mathfrak{b}(\mathcal{M}_W)$ is equal to $\mathfrak{b}_3(Sec)$. Since Sec is a family of divisors in \mathcal{M}_W , we deduce that $\mathfrak{b}_3(Sec) = \mathfrak{b}(Sec) = \mathfrak{b}(\mathcal{M}_W)$. \square

Lemma 3.12. *The surface W_Y is unique, i.e. W_Y is independent of the choice of cubic h . Therefore, W_Y is disjoint from every one of the loci P_ℓ of the small extremal rays of Y .*

Proof. Let Sec_Y be the family of the strict transforms in Y of the secant varieties in Sec . Let $\mathcal{M}_{Y,3}$ be the family of divisors in $|-K_Y|$ having multiplicity 3 at some point on R_Y . Then the two families $\mathcal{M}_{Y,3}$ and Sec_Y are equal, as $\dim \mathcal{M}_{x,3} = 0$ for a general point $x \in R_4$ by Lemma 3.11 and η_h is an isomorphism at the generic point of R_4 .

Suppose by contradiction that W_Y depends on h . Then there exist two distinct surfaces $W_{Y,h}$ and $W_{Y,h'}$. Let $Bs\ \mathcal{M}_{Y,3}$ be the scheme-theoretic intersection of the family $\mathcal{M}_{Y,3}$. Then by Lemma 3.11, one has the following set-theoretic inclusion:

$$Bs\ \mathcal{M}_{Y,3} \supset W_{Y,h} \cup W_{Y,h'}.$$

Since $W_{Y,h'}$ contains the curve R_Y which is generically in the locus where $\xi_h^{-1}: Y \dashrightarrow X_h$ is an isomorphism, we deduce that $W_{Y,h'}$ is not contracted by ξ_h^{-1} .

Since the surface $\xi_h^{-1}(W_{Y,h'})$ contains the curve R , this surface cannot be contained in any exceptional locus E_i , $i = 1, \dots, 8$ of $X_h \rightarrow \mathbb{P}^4$, and thus it cannot be contracted; we denote by $W_{h'}$ its image in \mathbb{P}^4 . Therefore, $Bs\ Sec$ contains two distinct surfaces W and $W_{h'}$, which contradicts Lemma 3.11.

Since by Lemma 3.2 the surface W_Y is disjoint from the indeterminacy locus of the map $\xi_h^{-1}: Y \dashrightarrow X_h$, which is a union of some of the loci P_ℓ (depending on h), and W_Y is the same for all h , we deduce that W_Y is disjoint from every one of the loci P_ℓ . \square

Lemma 3.13. (i) *We have $h^0(W_Y, \mathcal{O}_{W_Y}(-K_Y)) = 3$. The restriction*

$$r_1: H^0(Y, \mathcal{O}_Y(-K_Y)) \rightarrow H^0(W_Y, \mathcal{O}_{W_Y}(-K_Y))$$

is surjective.

(ii) We have $h^0(W_Y, \mathcal{O}_{W_Y}(-2K_Y)) = 8$. The restriction

$$r_2: H^0(Y, \mathcal{O}_Y(-2K_Y)) \rightarrow H^0(W_Y, \mathcal{O}_{W_Y}(-2K_Y))$$

is surjective.

Proof. Since $-K_{W_Y} \sim F_Y$ and $-K_Y|_{W_Y} \sim R_Y + 2F_Y$ by Lemma 3.3, by the Riemann-Roch formula one has $\chi(W_Y, -K_Y|_{W_Y}) = 3$. Since $-K_Y|_{W_Y}$ is ample on W_Y and $-K_{W_Y}$ is nef, by Kodaira vanishing theorem one has

$$h^j(W_Y, \mathcal{O}_{W_Y}(-K_Y)) = h^j(W_Y, \mathcal{O}_{W_Y}(K_{W_Y} - K_{W_Y} + (-K_Y))) = 0$$

for $j = 1, 2$. Therefore, $h^0(W_Y, \mathcal{O}_{W_Y}(-K_Y)) = 3$. The same argument can be applied to obtain $h^0(W_Y, \mathcal{O}_{W_Y}(-2K_Y)) = 8$.

(i) By Macaulay2 (see Listing 5),

$$h^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(5) \otimes \mathcal{I}_{p_1, \dots, p_8}^3 \otimes \mathcal{I}_W) = 3.$$

Since $H^0(Y, \mathcal{O}_Y(-K_Y)) \simeq H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(5) \otimes \mathcal{I}_{p_1, \dots, p_8}^3)$, and the surface W_Y is disjoint from the indeterminacy locus of η_h by Lemma 3.2 and W_Y is not contained in the exceptional locus of η_h , we deduce that

$$H^0(Y, \mathcal{O}_Y(-K_Y) \otimes \mathcal{I}_{W_Y}) \simeq H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(5) \otimes \mathcal{I}_{p_1, \dots, p_8}^3 \otimes \mathcal{I}_W).$$

Hence,

$$h^0(Y, \mathcal{O}_Y(-K_Y) \otimes \mathcal{I}_{W_Y}) = 3.$$

As $h^0(Y, \mathcal{O}_Y(-K_Y)) = 6$ and $h^0(W_Y, \mathcal{O}_{W_Y}(-K_Y)) = 3$, we deduce that the restriction morphism

$$H^0(Y, \mathcal{O}_Y(-K_Y)) \rightarrow H^0(W_Y, \mathcal{O}_{W_Y}(-K_Y))$$

is surjective.

(ii) By Macaulay2 (see Listing 7),

$$h^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(10) \otimes \mathcal{I}_{p_1, \dots, p_8}^6 \otimes \mathcal{I}_W) = 21.$$

Since $H^0(Y, \mathcal{O}_Y(-2K_Y)) \simeq H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(10) \otimes \mathcal{I}_{p_1, \dots, p_8}^6)$ and by the same argument as above, we deduce that

$$H^0(Y, \mathcal{O}_Y(-2K_Y) \otimes \mathcal{I}_{W_Y}) \simeq H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(10) \otimes \mathcal{I}_{p_1, \dots, p_8}^6 \otimes \mathcal{I}_W).$$

Hence,

$$h^0(Y, \mathcal{O}_Y(-2K_Y) \otimes \mathcal{I}_{W_Y}) = 21.$$

As $h^0(Y, \mathcal{O}_Y(-2K_Y)) = 29$ and $h^0(W_Y, \mathcal{O}_{W_Y}(-2K_Y)) = 8$, we deduce that the restriction morphism

$$H^0(Y, \mathcal{O}_Y(-2K_Y)) \rightarrow H^0(W_Y, \mathcal{O}_{W_Y}(-2K_Y))$$

is surjective. □

4 The Bertini involution of the Fano model Y

Let S be a degree-one del Pezzo surface, and $Y := M_{S, -K_S}$ be the associated Fano fourfold. In this section, we study the action of the Bertini involution ι_Y on the Fano fourfold Y , which is analogous to the action of the Bertini involution ι_S on the surface S . We first notice that by the diagram (1) and the behaviour of ι_S described in (2), the invariant part of $H^2(Y, \mathbb{R})$ by the action of ι_Y is $\mathbb{R}K_Y$.

4.1 Action of the Bertini involution on the surface W_Y

In this subsection, we further our study of the involution ι_Y by looking at its action on the surface W_Y (which is the strict transform of the cubic scroll swept out by the pencil of elliptic normal quintics in \mathbb{P}^4). The aim of this subsection is to prove Proposition 1.4.

We start by showing that the surface W_Y is invariant by the Bertini involution ι_Y .

Lemma 4.1. *The Bertini involution ι_Y preserves the curve R_Y and the surface W_Y . Moreover, $(\iota_Y|_{W_Y})^*(e_i) \sim -2K_Y|_{W_Y} - e_i$ and $(\iota_Y|_{W_Y})^*(F_Y) \sim F_Y$, where e_i is the exceptional curve of $\eta_h|_{W_Y}: W_Y \rightarrow W$ for $i = 1, \dots, 8$.*

Proof. Since ι_Y preserves the family of divisors in the anticanonical system $|-K_Y|$, the involution ι_Y preserves the base locus of $|-K_Y|$. Thus $\iota_Y(R_Y) = R_Y$ by Proposition 1.1.

Let x be a general point in R_Y . Then by Lemma 3.11, there exists a unique divisor in $|-K_Y|$ having multiplicity 3 at x : it is the strict transform in Y of the secant variety of the elliptic normal quintic through p_1, \dots, p_8 and $\eta_h(x)$ in \mathbb{P}^4 . In particular, this divisor has multiplicity 3 along the elliptic fibre of W_Y through x . By varying x in R_Y , this gives a one-dimensional family $\mathcal{M}_{Y,3}$ of divisors in $|-K_Y|$, which is preserved by ι_Y . On the other hand, the intersection of these divisors is the surface W_Y , so W_Y is preserved by ι_Y . Let $D_1 \in \mathcal{M}_{Y,3}$ and $D_2 = \iota_Y(D_1) \in \mathcal{M}_{Y,3}$. Let F_1 (resp. F_2) be the elliptic fibre of W_Y along which D_1 (resp. D_2) has multiplicity 3. Then $\iota_Y(F_1) = F_2$, and thus ι_Y preserves the family of elliptic fibres of W_Y , i.e. $(\iota_Y|_{W_Y})^*(F_Y) \sim F_Y$.

By [CCF19, 7.12], one has $\iota_Y^*(\xi(E_i)) \sim -2K_Y - \xi(E_i)$. Hence, $(\iota_Y|_{W_Y})^*(e_i) \sim -2K_Y|_{W_Y} - e_i$. \square

Now we investigate the morphism defined by the linear system $|-2K_Y|_{W_Y}|$.

Proposition 4.2. *The linear system $|-2K_Y|_{W_Y}|$ defines a finite morphism $\phi: W_Y \rightarrow V \subset \mathbb{P}^7$ of degree 2, where $V = V_{2,4} \simeq \mathbb{F}_2$ is a rational normal scroll of bidegree $(2, 4)$. There is a non-trivial involution i of W_Y such that $\phi = \phi \circ i$. Moreover, i is the identity on R_Y and i induces an involution on each elliptic fibre of W_Y .*

Proof. Since $h^0(W_Y, \mathcal{O}_{W_Y}(-2K_Y)) = 8$ (see Lemma 3.13), and $|-2K_Y|$ is base-point-free by Theorem 2.4, the linear system $|-2K_Y|_{W_Y}|$ defines a morphism $\phi: W_Y \rightarrow V \subset \mathbb{P}^7$, where V is the image of W_Y .

Claim. V is a surface of degree 6 in \mathbb{P}^7 , the image of an elliptic fibre F_Y by ϕ is a line and the image of R_Y by ϕ is a conic.

Since the restriction morphism $H^0(Y, \mathcal{O}_Y(-2K_Y)) \rightarrow H^0(W_Y, \mathcal{O}_{W_Y}(-2K_Y))$ is surjective by Lemma 3.13 (ii), the restriction of the morphism $\phi_{|-2K_Y|}$ defined by $|-2K_Y|$ to the surface W_Y coincides with the morphism ϕ , i.e. $\phi = \phi_{|-2K_Y|}|_{W_Y}$.

In \mathbb{P}^4 , let $2\mathcal{M}$ be the linear system of hypersurfaces of degree 10 with multiplicity at least 6 at the 8 general points p_1, \dots, p_8 . Consider the map $\phi_{2\mathcal{M}}$ defined by the linear system $2\mathcal{M}$. Then by `Macaulay2` (see Listing 8), the image of the surface W by $\phi_{2\mathcal{M}}$ is a surface of degree 6, the image of an elliptic normal quintic through the 8 points by $\phi_{2\mathcal{M}}$ is a line and the image of the rational quintic R_4 through the 8 points by $\phi_{2\mathcal{M}}$ is a conic. This proves the claim.

Since $(-2K_Y|_{W_Y})^2 = 4(R_Y + 2F_Y)^2 = 12$, and the image of W_Y by ϕ is of degree 6, we deduce that ϕ is of degree 2. As $-K_Y$ is ample, the morphism ϕ does not contract any curve and thus it is a finite morphism of degree 2.

Since the linear system $|-2K_Y|_{W_Y}|$ has no fixed divisor, the image V is not contained in any hyperplane of \mathbb{P}^7 (see for example [Bea96, II.6]), i.e. V is non-degenerate. Hence, V is a non-degenerate irreducible surface of degree 6 (variety of minimal degree) in \mathbb{P}^7 , and by [GH94, p. 525] we deduce that V is a rational normal scroll $V_{k,l}$ of bidegree (k, l) , with $0 \leq k \leq l$ and $k + l = 6$. In particular, V is isomorphic to one of the following: a cone over a rational normal curve of degree 6, $\mathbb{P}^1 \times \mathbb{P}^1$, or a Hirzebruch surface \mathbb{F}_{l-k} , where the minimal section is mapped to the rational normal curve of degree k , and the fibres are mapped to lines. Therefore, ϕ is a finite morphism between two normal surfaces and by [Fuj83, (2.3)], there is a non-trivial involution i of W_Y such that $\phi = \phi \circ i$ and $V \simeq W_Y/i$.

Since the restriction of ϕ to a general fibre F_Y induces a finite morphism from an elliptic curve to a line $l \subset V$, which cannot be an isomorphism, we deduce that $\phi^{-1}(l) = F_Y$ as ϕ is of degree 2. Hence, i induces an involution on F_Y .

Since $-2K_Y|_{W_Y}$ is i -invariant, one has $2(i^*(-K_Y|_{W_Y}) - (-K_Y|_{W_Y})) \sim 0$. As $\text{Pic}(W_Y)$ is torsion-free (this is because W_Y is isomorphic to \mathbb{P}^2 blown up at 9 points), we deduce that $i^*(-K_Y|_{W_Y}) \sim -K_Y|_{W_Y}$. Since R_Y is the base locus of $|-K_Y|_{W_Y}|$, the curve R_Y is preserved by i . We claim that R_Y is contained in the ramification locus of ϕ . Indeed, suppose that R_Y is not contained in the ramification locus of ϕ . Then there exists a curve $C \subset V$ such that $R_Y = \phi^*(C)$. As R_Y is a (-1) -curve on W_Y , one has

$$-1 = R_Y^2 = (\phi^*(C))^2 = \deg \phi \cdot C^2,$$

i.e. $C^2 = -\frac{1}{2}$. Hence, C is not Cartier on V , i.e. V is singular. In view of the classification of minimal degree varieties, we see that V is a cone. But there is no curve with negative self-intersection number on a cone, which leads to a contradiction. Therefore, R_Y is in the ramification locus. As ϕ is a double cover, we deduce that i is the identity on R_Y .

Let $C = \phi(R_Y)$. Since R_Y is contained in the ramification locus of ϕ , and every point in R_Y

has ramification index 2, one has

$$R_Y^2 = \left(\frac{1}{2}\phi^*(C)\right)^2 = \frac{1}{2}C^2.$$

Since R_Y is a (-1) -curve on W_Y , one has $C^2 = -2$. Therefore, $V = V_{2,4} \simeq \mathbb{F}_2$, and $\phi(R_Y)$ is minimal section of \mathbb{F}_2 which is a conic. \square

Remark 4.3. *Since ϕ is a finite morphism of degree 2 between smooth surfaces, the ramification locus is a smooth divisor on W_Y (see [Fuj83, (2.5)]). Let e be the minimal section of $V \simeq \mathbb{F}_2$ and f be a fibre of V . Let D be the ramification divisor. Then*

$$K_{W_Y} \sim \phi^*(K_S) + D.$$

As $K_{W_Y} \sim -F_Y = -\phi^*(f)$, and $K_S \sim -2e - 4f$, one has

$$D \sim \phi^*(2e + 3f).$$

Let $B \subset V$ be the branch locus. Then $D = \frac{1}{2}\phi^*B$ and thus $B \sim 4e + 6f$. As e is contained in the branch locus, we can write $B = e + B_1$, where B_1 is a smooth curve disjoint from e . Then $B_1 \sim 3e + 6f$. Notice that B_1 is irreducible. Indeed, suppose that B_1 has at least two disjoint irreducible components. Then we can decompose B_1 as

$$B_1 \sim (e + bf) + (2e + (6 - b)f)$$

with $0 \leq b \leq 6$ and $(e + bf) \cdot (2e + (6 - b)f) = 0$. Hence $b = -2$, which leads to a contradiction.

Hence $D = R_Y + R'$, where $R' \sim \frac{1}{2}\phi^*(3e + 6f) = 3(R_Y + F_Y)$ is a smooth curve of genus 4 which is disjoint from R_Y .

Finally, we compare the action of the two automorphisms i and $\iota_Y|_{W_Y}$ on W_Y .

Lemma 4.4. *Let e_i be the exceptional curves of $\eta_h|_{W_Y}: W_Y \rightarrow W$ for $i = 1, \dots, 8$. Then*

$$i^*(e_i) \sim -2K_Y|_{W_Y} - e_i.$$

Proof. For $i = 1, \dots, 8$, by `Macaulay2` (see Listing 9), there exists a unique hypersurface of degree 10 with multiplicity at least 7 at the point p_i and multiplicity at least 6 at p_j for $j \neq i$. Moreover, this hypersurface does not contain the surface W . Therefore, the linear system $|-2K_Y|_{W_Y} - e_i|$ is non-empty.

Let $R_i \in |-2K_Y|_{W_Y} - e_i|$. Since $-K_Y|_{W_Y} \sim R_Y + 2F_Y$, and $R_Y \cdot e_i = F_Y \cdot e_i = 1$, one has $R_i^2 = -1$, and $R_i \cdot F_Y = R_i \cdot R_Y = 1$. Hence, R_i is a (-1) -curve on W_Y .

Since $e_i + R_i \in |-2K_Y|_{W_Y} = \phi^*|\mathcal{O}_V(1)|$, one has $R_i \sim i^*(e_i)$. \square

Proposition 4.5. *The involution i coincides with the restriction of the Bertini involution ι_Y on the surface W_Y , i.e. $\iota_Y|_{W_Y} = i$.*

Proof. We first show that $(\iota_Y|_{W_Y})^* = i^*$. By Lemma 4.2, Lemma 4.4 and Lemma 4.1, it is enough to show that R_Y, F_Y and e_i for $i = 1, \dots, 8$ form a basis of $H^2(W_Y, \mathbb{R})$.

Since W' is disjoint from the indeterminacy locus of ξ_h , it is equivalent to show that R, F and e_i for $i = 1, \dots, 8$ form a basis of $H^2(W', \mathbb{R})$. We have the following diagram (see (3)):

$$\begin{array}{ccc}
 & W' \subset X & \\
 \eta \swarrow & & \downarrow \alpha' \\
 W \subset \mathbb{P}^4 & & \mathbb{P}^2 \\
 \searrow \alpha & &
 \end{array}$$

where α is the blow-up of \mathbb{P}^2 at one point and η is the blow-up of W at p_1, \dots, p_8 . Moreover, let $e_0 \subset W$ be the (-1) -curve and $f_0 \subset W$ be a fibre of the \mathbb{P}^1 -bundle on W , then by Lemma 3.1 and Lemma 3.3, one has $F \sim \eta^*(2e_0 + 3f_0) - \sum_{i=1}^8 e_i$ and $R \sim \eta^*(e_0 + 4f_0) - \sum_{i=1}^8 e_i$. Therefore, R, F and e_i for $i = 1, \dots, 8$ form a basis of $H^2(W', \mathbb{R})$.

We have a group homomorphism $\rho_1: \text{Aut}(W_Y) \rightarrow \text{Aut}(H^2(W_Y, \mathbb{R}))$ given by $g \mapsto (g^{-1})^*$. Let $\text{Aut}(R_Y, W_Y)$ be the subgroup of automorphisms in $\text{Aut}(W_Y)$ fixing the curve R_Y . We show that the restriction $\rho_1: \text{Aut}(R_Y, W_Y) \rightarrow \text{Aut}(H^2(W_Y, \mathbb{R}))$ is injective, which implies that $\iota_Y|_{W_Y} = i$.

Since R_Y is a (-1) -curve on W_Y , by blowing down R_Y , we obtain a rational surface S' with $(-K_{S'})^2 = 1$, and the curve R_Y is contracted to a point $x_0 \in S'$. We denote by $\beta: W_Y \rightarrow S'$ the blow-up of S' at x_0 . Since $-K_{W_Y}$ is nef, we obtain that $-K_{S'}$ is nef by the projection formula (see for example [Har77, Appendix A, A4]). Moreover, since every fibre of $W_Y \rightarrow \mathbb{P}^1$ is integral, there is no $K_{S'}$ -trivial curve. Hence, S' is a del Pezzo surface of degree one. By [Dol12, Prop. 8.2.39], the homomorphism $\rho_2: \text{Aut}(S') \rightarrow \text{Aut}(H^2(S', \mathbb{R}))$ is injective.

Let $\text{Aut}(x_0, S')$ be the subgroup of automorphisms in $\text{Aut}(S')$ fixing the point x_0 . Then $\text{Aut}(x_0, S') \simeq \text{Aut}(R_Y, W_Y)$. Since $\text{Pic}(W_Y) \simeq \beta^* \text{Pic}(S') \oplus \mathbb{Z}[R_Y]$, the image $\rho_1(\text{Aut}(R_Y, W_Y))$ is contained in a subgroup G_1 of $\text{Aut}(H^2(W_Y, \mathbb{R}))$ such that $G_1 \simeq \text{Aut}(H^2(S', \mathbb{R}))$. Hence, we have the following diagram:

$$\begin{array}{ccc}
 \text{Aut}(R_Y, W_Y) & \xrightarrow{\rho_1} & G_1 \\
 \downarrow \simeq & & \downarrow \simeq \\
 \text{Aut}(x_0, S') & \xrightarrow{\rho_2} & \text{Aut}(H^2(S', \mathbb{R}))
 \end{array}$$

Since ρ_2 is injective, the restriction $\rho_1: \text{Aut}(R_Y, W_Y) \rightarrow G_1 \subset \text{Aut}(H^2(W_Y, \mathbb{R}))$ is injective. \square

Proof of Proposition 1.4. The first paragraph follows from Lemma 4.1, Proposition 4.2 and Proposition 4.5. The second paragraph follows from Remark 4.3. \square

4.2 Action of the Bertini involution on the anticanonical system

In this subsection, we study the action of the involution ι_Y on the anticanonical system $|-K_Y|$. This is closely related to the anticanonical map of $Y = M_{S, -K_S}$.

Lemma 4.6. *Let $\mu: \tilde{Y} \rightarrow Y$ be the blow-up of Y along the curve R_Y which is the base scheme of $|-K_Y|$. Let E be the exceptional divisor and \tilde{D} be the strict transform of a general member $D \in |-K_Y|$. Then $|\tilde{D}| = |\mu^*(-K_Y) - E|$ is base-point-free and induces a morphism $f: \tilde{Y} \rightarrow \mathbb{P}(H^0(Y, \mathcal{O}_Y(-K_Y)))^\vee \simeq \mathbb{P}^5$ with image Q a smooth quadric hypersurface, and f has generically degree 4. We have the following commutative diagram:*

$$\begin{array}{ccc}
 & \tilde{Y} & \\
 \mu \swarrow & & \searrow f \\
 Y & \xrightarrow{\phi_{|-K_Y|}} & Q \subset \mathbb{P}^5
 \end{array} \tag{4}$$

The following statements hold:

- (i) *The Bertini involution ι_Y can be lifted to an involution $\iota_{\tilde{Y}}$ of \tilde{Y} , which preserves the exceptional divisor E and induces an involution on each \mathbb{P}^2 above a point of R_Y .*
- (ii) *The Bertini involution ι_Y induces an involution $\iota_{\mathbb{P}^5}$ of \mathbb{P}^5 , which preserves the quadric hypersurface Q . Denote by ι_Q its restriction on Q . Then $\iota_Q \circ f = f \circ \iota_{\tilde{Y}}$.*

Proof. In \mathbb{P}^4 , let \mathcal{M} be the linear system of quintic hypersurfaces with multiplicity at least 3 at 8 general points. Then by Macaulay2, the image of \mathbb{P}^4 by the map defined by \mathcal{M} is a smooth quadric hypersurface Q in \mathbb{P}^5 .

Let E be the exceptional divisor of f . Since

$$\begin{aligned}
 \mu^*(-K_Y)^4 &= (-K_Y)^4 = 13, \\
 \mu^*(-K_Y).E^3 &= -K_Y.R_Y = 1, \\
 \mu^*(-K_Y)^3.E &= \mu^*(-K_Y)^2.E^2 = 0, \\
 E^4 &= -K_Y.R_Y + 2g(R_Y) - 2 = -1,
 \end{aligned}$$

one has $\tilde{D}^4 = 8$. Hence $\phi_{|-K_Y|}$ (and also f) has generically degree 4.

(i) Follows from the fact that R_Y is contained in the fixed locus of ι_Y (see Proposition 1.4).

(ii) The pull-back ι_Y^* induces an involution on $H^0(-K_Y, \mathcal{O}_Y(-K_Y))$, and thus an involution of $\mathbb{P}(H^0(Y, \mathcal{O}_Y(-K_Y)))^\vee \simeq \mathbb{P}^5$ preserving $\phi_{|-K_Y|}(Y) = Q$.

Let $s \in H^0(\tilde{Y}, \mathcal{O}_{\tilde{Y}}(\tilde{D}))$ be a global section which is zero at the point $\iota_{\tilde{Y}}(x)$, where x is a point in \tilde{Y} . Then for $s' := \iota_{\tilde{Y}}^*(s) \in H^0(\tilde{Y}, \mathcal{O}_{\tilde{Y}}(\iota_{\tilde{Y}}^*\tilde{D})) \simeq H^0(\tilde{Y}, \mathcal{O}_{\tilde{Y}}(\tilde{D}))$, one has

$$s'(x) = (\iota_{\tilde{Y}}^*(s))(x) = s(\iota_{\tilde{Y}}(x)) = 0.$$

Hence,

$$\begin{aligned}\phi_{|\tilde{D}|}(\iota_{\tilde{Y}}(x)) &= \{s \in H^0(\tilde{Y}, \mathcal{O}_{\tilde{Y}}(\tilde{D})) \mid s(\iota_{\tilde{Y}}(x)) = 0\}, \\ \phi_{|\iota_{\tilde{Y}}^* \tilde{D}|}(x) &= \{s' \in H^0(\tilde{Y}, \mathcal{O}_{\tilde{Y}}(\iota_{\tilde{Y}}^* \tilde{D})) \mid s'(x) = 0\}.\end{aligned}$$

Therefore, we obtain the following commutative diagram:

$$\begin{array}{ccc}\tilde{Y} & \xrightarrow{f} & \mathbb{P}(H^0(\tilde{Y}, \mathcal{O}_{\tilde{Y}}(\iota_{\tilde{Y}}^* \tilde{D})))^\vee \\ \downarrow \iota_{\tilde{Y}} & & \downarrow \iota_{\mathbb{P}^5} \\ \tilde{Y} & \xrightarrow{f} & \mathbb{P}(H^0(\tilde{Y}, \mathcal{O}_{\tilde{Y}}(\tilde{D})))^\vee.\end{array}$$

Thus, $\iota_Q \circ f = f \circ \iota_{\tilde{Y}}$. □

Remark 4.7. *The following statements are equivalent:*

- (a) *The Bertini involution ι_Y preserves every divisor in $|-K_Y|$.*
- (b) *The action $\iota_Y^*: H^0(Y, \mathcal{O}_Y(-K_Y)) \rightarrow H^0(Y, \mathcal{O}_Y(-K_Y))$ is Id or $-\text{Id}$.*
- (c) *The involution $\iota_{\mathbb{P}^5}$ of \mathbb{P}^5 (resp. ι_Q of Q) is the identity.*

Recall that we have a special surface $W_Y \subset Y$ containing R_Y , which is an elliptic fibration $W_Y \rightarrow \mathbb{P}^1$ with fibre F_Y . With the same notation as in Lemma 4.6, we describe the image of W_Y in $Q \subset \mathbb{P}^5$.

Lemma 4.8. *Every elliptic fibre F_Y (resp. its strict transform $\tilde{F}_Y \subset \tilde{Y}$) is contracted by $\phi_{|-K_Y|}$ (resp. by f). Moreover, the image of the surface W_Y (resp. its strict transform $\tilde{W}_Y \subset \tilde{Y}$) is a conic C in $Q \subset \mathbb{P}^5$.*

Furthermore, the curve $\tilde{R}_Y := \tilde{W}_Y \cap E$ is contained in the fixed locus of $\iota_{\tilde{Y}}$, and the conic C is contained in the fixed locus of ι_Q .

Proof. Since $-K_Y \cdot F_Y = 1$, one has $\tilde{D} \cdot \tilde{F}_Y = 0$, where \tilde{D} is the strict transform of a general member $D \in |-K_Y|$. Hence f contracts the elliptic fibres of \tilde{W}_Y .

As $-K_Y|_{W_Y} = R_Y + 2F_Y$, one has $\tilde{D}|_{\tilde{W}_Y} = (\mu^*(-K_Y) - E)|_{\tilde{W}_Y} = 2\tilde{F}_Y$. Therefore, the morphism f sends \tilde{W}_Y to a conic in \mathbb{P}^5 .

By Lemma 4.6 (i), $\iota_{\tilde{Y}}$ induces an involution on each fibre \mathbb{P}^2 of $\mu|_E: E \rightarrow R_Y$. Since W_Y is preserved by ι_Y by Proposition 1.4, its transform $\tilde{W}_Y \subset \tilde{Y}$ is preserved by $\iota_{\tilde{Y}}$. Therefore, the curve $\tilde{R}_Y := \tilde{W}_Y \cap E$ (which is a section of $\mu|_E$) is invariant. Since R_Y is in the fixed locus of ι_Y by Proposition 1.4, it follows that \tilde{R}_Y is contained in the fixed locus of $\iota_{\tilde{Y}}$. By Lemma 4.6 (ii), $f(\tilde{R}_Y) = f(\tilde{W}_Y) = C$ is contained in the fixed locus of ι_Q . □

The rest of this subsection is devoted to the proofs of Theorems 1.3 and 1.5. To show that ι_Y preserves every divisor in $|-K_Y|$, our strategy is to exclude the other remaining case by analysing the anticanonical map.

Lemma 4.9. *If the action ι_Y^* on $H^0(Y, \mathcal{O}_Y(-K_Y))$ is not $\pm \text{Id}$, then*

$$H^0(Y, \mathcal{O}_Y(-K_Y)) = V_1 \oplus V_2,$$

where V_1 is the sub-vector space of global sections vanishing on the surface W_Y , and V_2 is uniquely determined as eigenspace corresponding to the eigenvalue 1 or -1 of ι_Y^* , with $\dim V_1 = \dim V_2 = 3$. More precisely, ι_Y^* acts as Id or $-\text{Id}$ on V_i for $i = 1, 2$.

Proof. In \mathbb{P}^4 , let \mathcal{M} be the linear system of quintic hypersurfaces with multiplicity at least 3 at 8 general points. Let \mathcal{M}_W be the sub-linear system of effective divisors in \mathcal{M} containing the surface W . By `Macaulay2` (see Listing 5), one has $\mathfrak{b}(\mathcal{M}_W) = \mathfrak{b}(\text{Sec})$ (see Lemma 3.11 for notation) for the base ideals. Moreover, by `Macaulay2` (see Listing 6), a random divisor in \mathcal{M}_W is singular along two elliptic normal quintic curves E_p, E_q through the 8 blown up points (the two elliptic curves may coincide, in which case the divisor has multiplicity at least 3 along this elliptic curve, and in fact the divisor is the secant variety of the elliptic curve). Moreover, there exists a unique divisor in \mathcal{M}_W which is singular along E_p and E_q , as $H^0(\mathbb{P}^4, \mathcal{I}_W \otimes \mathcal{O}_{\mathbb{P}^4}(5) \otimes \mathcal{I}_{p_1, \dots, p_8}^3) \simeq H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2)) \simeq \mathbb{C}^3$.

Let $V_1 \subset H^0(Y, \mathcal{O}_Y(-K_Y))$ be the sub-vector space of global sections vanishing on the surface W_Y . Let $|V_1|$ be the corresponding sub-linear system (i.e. the linear system of effective divisors in $|-K_Y|$ containing the surface W_Y). Then ι_Y preserves the family of divisors in $|V_1|$, as ι_Y preserves the surface W_Y by Proposition 1.4. Since W_Y is disjoint from the indeterminacy locus of the map $\eta_h: Y \dashrightarrow \mathbb{P}^4$, and the intersection of W_Y with the exceptional locus of η_h is the union of 8 points, we deduce that a general member in $|V_1|$ is singular along two elliptic fibres $F_{Y,1}, F_{Y,2}$ of W_Y , and there exists a unique divisor in $|V_1|$ which is singular along $F_{Y,1}$ and $F_{Y,2}$. Since ι_Y preserves every elliptic fibre F_Y of W_Y (see Proposition 1.4), we deduce that ι_Y preserves every divisor in $|V_1|$, i.e. the action of ι_Y^* on V_1 is Id or $-\text{Id}$.

By Lemma 3.13(i), we have the following short exact sequence:

$$0 \rightarrow H^0(Y, \mathcal{O}_Y(-K_Y) \otimes \mathcal{I}_{W_Y}) \rightarrow H^0(Y, \mathcal{O}_Y(-K_Y)) \xrightarrow{r_1} H^0(W_Y, \mathcal{O}_{W_Y}(-K_Y)) \rightarrow 0.$$

Hence, $H^0(Y, \mathcal{O}_Y(-K_Y)) = V_1 \oplus V_2$ with $V_1 \simeq \text{Ker } r_1$ and $V_2 \simeq \text{Im } r_1$.

By Proposition 1.4, every elliptic fibre F_Y is preserved by ι_Y and R_Y is fixed by ι_Y . Since $-K_Y|_{W_Y} = 2F_Y + R_Y$, we deduce that ι_Y preserves every divisor in $|-K_Y|_{W_Y}$. Thus the action of $(\iota_Y|_{W_Y})^*$ on $H^0(W_Y, \mathcal{O}_{W_Y}(-K_Y))$ is Id or $-\text{Id}$. As ι_Y^* is not $\pm \text{Id}$ on $H^0(Y, \mathcal{O}_Y(-K_Y))$, we deduce that V_2 can be uniquely determined as the eigenspace corresponding to the eigenvalue 1 or -1 of ι_Y^* . \square

Let $\{s_{11}, s_{12}, s_{13}\}$ (resp. $\{s_{21}, s_{22}, s_{23}\}$) be a basis of V_1 (resp. of V_2). Suppose that ι_Q is not the identity. Then for $y \in Y \setminus R_Y$, one has

$$\iota_Q(\phi_{|-K_Y|}(y)) = [s_{11}(y) : s_{12}(y) : s_{13}(y) : -s_{21}(y) : -s_{22}(y) : -s_{23}(y)] \quad (5)$$

by Lemma 4.9. Moreover, if y is a fixed point of ι_Y , then by Lemma 4.6 (ii), $\phi_{|-K_Y|}(y)$ is fixed by ι_Q . Thus by (5), one has $s_{11}(y) = s_{12}(y) = s_{13}(y) = 0$ or $s_{21}(y) = s_{22}(y) = s_{23}(y) = 0$, i.e. $y \in \text{Bs } |V_1|$ or $y \in \text{Bs } |V_2|$.

Now for $i = 1, 2$, let $\tilde{V}_i \subset H^0(\tilde{Y}, \mathcal{O}_{\tilde{Y}}(\mu^*(-K_Y) - E))$ be the sub-vector space of global sections which are the linear spans of \tilde{s}_{ij} with $j = 1, 2, 3$, where \tilde{s}_{ij} is the strict transform of the global section $s_{ij} \in V_i$. Hence, if a point $y \in \tilde{Y}$ is fixed by $\iota_{\tilde{Y}}$, then by repeating the argument in above paragraph, we obtain $y \in \text{Bs } |\tilde{V}_1|$ or $y \in \text{Bs } |\tilde{V}_2|$. To summarise, we have the following corollary.

Corollary 4.10. *Suppose that ι_Q is not the identity. If a point $y \in \tilde{Y}$ is fixed by $\iota_{\tilde{Y}}$, then $y \in \text{Bs } |\tilde{V}_1|$ or $y \in \text{Bs } |\tilde{V}_2|$.*

Recall that we have the normal bundle $\mathcal{N}_{R_Y/Y} \cong \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus 2}$ by Lemma 3.5. Hence $E = \mathbb{P}(\mathcal{N}_{R_Y/Y}^*) \simeq \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus 2})$. Denote by ξ a tautological divisor associated to $\mathcal{O}_{\mathbb{P}(\mathcal{N}_{R_Y/Y}^*)}(1)$, and F_E a fibre of the projection $E = \mathbb{P}(\mathcal{N}_{R_Y/Y}^*) \rightarrow R_Y \simeq \mathbb{P}^1$. Let l be an exceptional curve of μ and γ be the curve which generates the other extremal ray Γ of $\text{NE}(E)$ such that $-K_E \cdot \gamma$ is the length of Γ . Then

$$\begin{aligned} F_E \cdot l &= 0, & F_E \cdot \gamma &= 1, \\ \xi \cdot l &= 1, & \xi \cdot \gamma &= 0. \end{aligned}$$

Moreover, $\tilde{R}_Y \sim l + \gamma$.

With the same notation as in Lemma 4.6, we may describe the image $f(E)$ as follows.

Remark 4.11. The exceptional divisor E is isomorphic to the blow-up B of \mathbb{P}^3 at a line (and B is embedded in $\mathbb{P}^1 \times \mathbb{P}^3$ with bidegree $(1, 1)$).

Let $\tilde{D} \subset \tilde{Y}$ be the strict transform of a general member $D \in |-K_Y|$. Then $(\tilde{D}|_E)^3 = 4$, hence $\tilde{D}|_E \sim \xi + F_E$ is very ample, with $h^0(E, \mathcal{O}_E(\tilde{D})) = 7$. Thus the corresponding linear system embeds B in \mathbb{P}^6 as a hyperplane section of the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^3$ in \mathbb{P}^7 , and B has degree 4. Hence, $f|_E$ is given by the projection of B from a point x outside B in \mathbb{P}^6 (in fact, it is given by a sub-linear system of $|\tilde{D}|$ of dimension 5, which is still base-point-free).

If the point x is general, then the projection is birational and the image has degree 4 in \mathbb{P}^5 . There could be special point x such that the projection has degree 2, and the image is a 3-dimensional quadric in \mathbb{P}^5 . In any case, the image of a fibre F_E is a plane in \mathbb{P}^5 .

Lemma 4.12. *Suppose that ι_Q is not the identity. Then $\iota_{\tilde{Y}}|_E$ is not the identity, and the following statements hold:*

- (i) *The fixed locus of $\iota_{\tilde{Y}}|_E$ is the disjoint union $S_E \cup C_2$, where $S_E = \text{Bs } |\tilde{V}_1| \cap E$ is the unique member in $|\xi - F_E|$ isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, and $C_2 = \text{Bs } |\tilde{V}_2| \cap E$ is a curve satisfying $C_2 \sim l + \gamma$ which is mapped surjectively to R_Y .*
- (ii) *The fixed locus of $\iota_{\mathbb{P}^5}$ is two disjoint planes $\mathbb{P}_1^2 \cup \mathbb{P}_2^2$ such that $f(S_E) = \mathbb{P}_1^2$ and that $f(C_2)$ is a conic contained in \mathbb{P}_2^2 . Furthermore, $f(E)$ is a 3-dimensional quadric in \mathbb{P}^5 .*

Proof. Suppose by contradiction that $\iota_{\tilde{Y}}|_E$ is the identity. Then by Corollary 4.10, one has $E \subset \text{Bs } |\tilde{V}_1|$ or $E \subset \text{Bs } |\tilde{V}_2|$. Since $|\tilde{V}_i| \subset |\mu^*(-K_Y) - E|$ for $i = 1, 2$, this contradicts to the fact that there is no divisor in $|-K_Y|$ having multiplicity at least 2 along R_Y by `Macaulay2` (see Listing 11).

(i) Since $\iota_{\tilde{Y}}|_E$ is not the identity and $\iota_Y|_{R_Y}$ is the identity, we have that $\iota_{\tilde{Y}}|_{F_E}$ is not the identity. Thus the fixed locus of $\iota_{\tilde{Y}}|_{F_E}$ is the disjoint union of a point and a line.

We first describe the base locus of $|\tilde{V}_1|$. Since by `Macaulay2` (see Listing 5), one has $\mathfrak{b}(\mathcal{M}_W) = \mathfrak{b}(\text{Sec})$ (see Lemma 3.11 for notation) for the base ideals. Thus the base locus of $|\tilde{V}_1|$ contains the surface W_Y with multiplicity 2 by Lemma 3.11. Therefore, the base locus of $|\tilde{V}_1|$ contains the strict transform $\tilde{W}_Y \subset \tilde{Y}$. Moreover, since a general member in $|\tilde{V}_1|$ is singular along two elliptic fibres of W_Y , a local computation shows that every member in $|\tilde{V}_1|$ contains two fibres F_E above the two points on R_Y where it is singular. As $\tilde{D}|_E \sim \xi + F_E$, we deduce that the unique member $S_E \in |\xi - F_E|$ is contained in the base locus of $|\tilde{V}_1|$. Therefore, $\text{Bs } |\tilde{V}_1| \cap E = S_E \simeq \mathbb{P}^1 \times \mathbb{P}^1$, and $S_E \cap \tilde{W}_Y = E \cap \tilde{W}_Y = \tilde{R}_Y$.

Now we describe the base locus of $|\tilde{V}_2|$. Since $|\mu^*(-K_Y) - E|$ is base-point-free, $\text{Bs } |\tilde{V}_2|$ is disjoint from the surface S_E . Let D_2 be a general member in $|\tilde{V}_2|$. Since D_2 does not contain the surface W_Y , and $D_2|_{W_Y} = R_Y + 2F_Y$, we deduce that the intersection of the singular locus $\text{Sing } D_2$ with the curve R_Y contains at most one point (which is a singularity of multiplicity two). Hence a general member in $|\tilde{V}_2|$ contains at most one fibre F_E of $E \rightarrow R_Y$.

Claim. $\text{Bs } |\tilde{V}_2| \cap E$ has dimension at most one.

Suppose that there is a surface $S_2 \subset \text{Bs } |\tilde{V}_2| \cap E$. Since $\tilde{D}|_E \sim \xi + F_E$, one has $S_2 \in |\xi|$ or $S_2 \in |\xi + F_E|$. As $\xi \cdot \tilde{R}_Y = F_E \cdot \tilde{R}_Y = 1$, one has $S_2 \cdot \tilde{R}_Y > 0$, which contradicts the fact that S_2 is disjoint from S_E . This proves the claim.

Note that $\text{Bs } |\tilde{V}_2| \cap E$ has dimension one. This is because the fixed locus of $\iota_{\tilde{Y}}|_{F_E}$ is the disjoint union of a point and a line, and $\text{Bs } |\tilde{V}_1| \cap F_E$ is a line. Thus by Corollary 4.10, the fixed point disjoint from the fixed line is contained in $\text{Bs } |\tilde{V}_2| \cap F_E$. Therefore, $\text{Bs } |\tilde{V}_2| \cap E$ is a curve which is mapped surjectively to R_Y .

Denote by C_2 the curve $\text{Bs } |\tilde{V}_2| \cap E$. Since C_2 is disjoint from $S_E \in |\xi - F_E|$, one has $(\xi - F_E) \cdot C_2 = 0$ and thus $C_2 \sim m(\gamma + l)$ with $m \geq 1$. As $(\tilde{D}|_E)^2 \sim (\xi + F)^2 \sim \gamma + 3l$, we deduce that $C_2 \sim \gamma + l$. Hence C_2 is an irreducible curve which is mapped surjectively to R_Y .

(ii) Since ι_Q is not the identity (i.e. $\iota_{\mathbb{P}^5}$ is not the identity), the fixed locus of $\iota_{\mathbb{P}^5}$ is the union of two disjoint sub-linear spaces $\mathbb{P}^i \cup \mathbb{P}^j$ with $i + j = 4$. Therefore, the fixed locus of $\iota_{\mathbb{P}^5}$ is two disjoint planes \mathbb{P}_1^2 and \mathbb{P}_2^2 by the equation (5).

By Remark 4.11, $f(S_E)$ has dimension 2. Hence, $f(S_E) \subset Q$ is one of the two planes \mathbb{P}_1^2 and \mathbb{P}_2^2 contained in the fixed locus of $\iota_{\mathbb{P}^5}$. We may denote $f(S_E) = \mathbb{P}_1^2$. Following the discussion in Remark 4.11, we now describe the map $f|_{S_E}$: the restricted linear system $|\tilde{D}|_{S_E}$ embeds the surface S_E in \mathbb{P}^3 as the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ in \mathbb{P}^3 , and the image S'_E has degree 2. Hence, $f|_{S_E}$ is given by the projection of S'_E from a point outside S'_E in \mathbb{P}^3 . The projection has degree 2 and the image is the plane \mathbb{P}_1^2 . Note that in Remark 4.11, the projection of B from a point x

outside B in \mathbb{P}^6 cannot be birational, as the projection of $S'_E \subset B$ from the point x in $\mathbb{P}^3 \subset \mathbb{P}^6$ has degree 2. Therefore, $f(E)$ is a 3-dimensional quadric in \mathbb{P}^5 .

Since $\tilde{D} \cdot C_2 = (\xi + F_E) \cdot (\gamma + l) = 2$, the image $f(C_2)$ is a conic. As $f(C_2)$ is disjoint from $f(S_E)$, we deduce that $f(C_2)$ is contained in \mathbb{P}_1^2 . \square

Corollary 4.13. *The involution ι_Q is the identity, and thus the Bertini involution ι_Y preserves every divisor in $|-K_Y|$ and f factors through the quotient $\tilde{Y}/\iota_{\tilde{Y}}$ via the lifted involution $\iota_{\tilde{Y}}$.*

Proof. Suppose by contradiction that ι_Q is not the identity. We use the notation as in Lemma 4.12.

Since the restricted linear system $|\tilde{D}|_{E|_{S_E}}$ embeds the surface S_E in \mathbb{P}^3 as the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ in \mathbb{P}^3 with image S'_E a quadric surface, the map $f|_{S_E}$ is given by the projection of S'_E from a point outside S'_E in \mathbb{P}^3 . The projection has degree 2, and $f(S_E) = \mathbb{P}_1^2$ by Lemma 4.12. Therefore, $f|_{S_E}: S_E \simeq \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}_1^2$ is a double cover branched over a non-singular conic Δ in \mathbb{P}_1^2 ; moreover, the image of any line on S_E is a tangent line to Δ , and conversely the preimage of each tangent line on Δ is two lines on S_E , one from each ruling.

Let $D \in |V_1|$ be a general member and $\tilde{D} \in |\tilde{V}_1|$ be its strict transform. Then by Lemma 4.12, $\tilde{D} \cap E$ contains the surface S_E and two distinct \mathbb{P}^2 (denoted by F_{E_1} and F_{E_2}) above the two points on R_Y where D is singular. Thus $f(\tilde{D})$ contains $f(F_{E_1}) =: \Pi_1$ and $f(F_{E_2}) =: \Pi_2$ which are two planes in \mathbb{P}^5 by Remark 4.11. Moreover, Π_1 and Π_2 are distinct. This is because $F_{E_1} \cap S_E$ and $F_{E_2} \cap S_E$ are two distinct lines of a same ruling of $S_E \simeq \mathbb{P}^1 \times \mathbb{P}^1$, and thus their images in $\mathbb{P}_1^2 = f(S_E)$ are two distinct tangent lines to Δ by the above discussion. Therefore, $f(\tilde{D}) \cap f(E)$ contains three distinct planes $\mathbb{P}_1^2, \Pi_1, \Pi_2$. This contradicts to the fact that $f(\tilde{D})$ is a hyperplane in \mathbb{P}^5 and $f(E)$ is a 3-dimensional quadric in \mathbb{P}^5 (so that their intersection is a surface of degree 2 in \mathbb{P}^5).

Therefore, ι_Q is the identity. By Lemma 4.6 (ii), one has $f = f \circ \iota_{\tilde{Y}}$. Thus f factors through the quotient $\tilde{Y}/\iota_{\tilde{Y}}$. \square

Corollary 4.14. *The morphism $f|_E: E \rightarrow f(E)$ is birational, and $f(E)$ has degree 4 in \mathbb{P}^5 . Moreover, the restricted involution $\iota_{\tilde{Y}}|_E$ is the identity.*

Proof. By Remark 4.11, either $f|_E$ has degree 2 and the image is a 3-dimensional quadric in \mathbb{P}^5 , or $f|_E$ is finite birational and the image has degree 4 in \mathbb{P}^5 . We will show that the first case cannot happen.

Suppose that $f|_E$ has degree 2 and $f(E)$ is a 3-dimensional quadric in \mathbb{P}^5 . We will follow the same argument as in the proof of Corollary 4.13. By Lemma 4.12 (with the same notation), for a general member $D \in |V_1|$, its strict transform $\tilde{D} \in |\tilde{V}_1|$ contains the surface $S_E \subset E$. Moreover, \tilde{D} contains the two distinct fibres (denoted by F_{E_1} and F_{E_2}) of $\mu|_E: E \rightarrow R_Y$ above the two points on R_Y where D is singular. Hence, $f(\tilde{D}) \cap f(E)$ contains the surface $f(S_E)$ and the two planes $f(F_{E_1}), f(F_{E_2})$ (which may coincide).

- (a) If $f|_{S_E}$ has degree 2, then the same argument as in the proof of Corollary 4.13 shows that $f(S_E), f(F_{E_1}), f(F_{E_2})$ are 3 distinct planes.
- (b) If $f|_{S_E}$ has degree 1, then $f(S_E)$ is either a non-normal surface or isomorphic to $S_E \simeq \mathbb{P}^1 \times \mathbb{P}^1$. Thus $f(S_E)$ has degree at least 2 in \mathbb{P}^5 .

This contradicts to the fact that $f(\tilde{D}) \cap f(E)$ is a surface of degree 2 in \mathbb{P}^5 . Therefore, $f|_E$ is finite birational and $f(E)$ has degree 4 in \mathbb{P}^5 .

Now suppose that $\iota_{\tilde{Y}}|_E$ is not the identity. Since $f|_E = f|_E \circ \iota_{\tilde{Y}}|_E$ by Corollary 4.13, we deduce that $f|_E$ has degree 2, which leads to a contradiction. \square

Proof of Theorem 1.3. Follows from Corollary 4.13. \square

Proof of Theorem 1.5. Follows from Lemmas 4.6, 4.8, and Corollaries 4.13, 4.14. \square

Remark 4.15. *The fixed locus of $\iota_{\tilde{Y}}$ is $E \cup Res$, where Res has dimension at most 2 and its intersection with every \tilde{P}_ℓ is non-empty and zero-dimensional, where $\tilde{P}_\ell \subset \tilde{Y}$ is the strict transform of P_ℓ (see notation in Proposition 2.3).*

Proof. Let $P_\ell \simeq \mathbb{P}^2$ be the exceptional locus of a small extremal contraction of Y . Then $\iota_Y(P_\ell) = P_{\iota_Y^*(\ell)}$ is also the exceptional locus of some small extremal contraction of Y and P_ℓ intersects $\iota_Y(P_\ell)$ transversally at 3 points by [CCF19, Rem. 2.15 (c), Lem. 6.4]. Therefore, the intersection of P_ℓ with the fixed locus of ι_Y is non-empty and zero-dimensional.

As $R_Y \subset W_Y$ is disjoint from P_ℓ by Lemma 3.12, we deduce that E is disjoint from \tilde{P}_ℓ . Therefore, $Res \cap \tilde{P}_\ell$ is non-empty and zero-dimensional. As every non-zero effective divisor in Y must have positive intersection with some extremal ray of $NE(Y)$, we deduce that Res has dimension at most 2. \square

A Computations by Macaulay2

```
restart
k = ZZ/67
```

We set up the projective space \mathbb{P}^4 :

```
R = k[x_0 .. x_4]
```

We choose 8 points in \mathbb{P}^4 :

```
I_0 = ideal(x_1, x_2, x_3, x_4)
I_1 = ideal(x_0, x_2, x_3, x_4)
I_2 = ideal(x_1, x_0, x_3, x_4)
I_3 = ideal(x_1, x_2, x_0, x_4)
I_4 = ideal(x_1, x_2, x_3, x_0)
I_5 = ideal(x_1-x_2, x_2-x_3, x_3-x_4, x_0-x_4)
I_6 = ideal(x_0-3*x_1, x_1-7*x_2, x_2-11*x_3, x_3-13*x_4)
I_7 = ideal(x_0-17*x_1, x_1-23*x_2, x_2-29*x_3, x_3-31*x_4)
```

We compute the ideal II defined by the 6 quintics through the 8 points with multiplicity at least 3:

```
J = I_0; for j from 1 to 7 do J = intersect(J, I_j);
H = saturate J^3;
G = gens (H);
betti G
G1 = submatrix (G, {0..5});
II = ideal(G1);
III = sheaf module II;
HH^0(III(5))
```

We check that II is the intersection of the ideal of the 28 lines, the ideal of the 8 quartics and the ideal I_5 of a smooth rational quintic curve:

```
LL = ideal(1_R);
for i from 0 to 7 do for j from 0 to i-1 do
LL = intersect(LL, ideal submatrix(gens intersect (I_i, I_j), {0..2}));
isSubset(II, LL)
RN = ideal(1_R);
for i from 0 to 7 do
RN = intersect(RN, minors(2, submatrix((res (J: I_i)).dd_4, {3..6}, {0..1})));
isSubset(II, RN)
I5 = ((II:LL):RN);
```

```

degree I5 , genus I5 , ideal singularLocus variety I5
II == intersect(intersect(LL,RN) , I5)

```

Listing 1: Base scheme

We compute the normal bundle of the smooth rational quintic curve:

```

RI5 = R/I5
N5 = (module I5)**RI5
PI5 = Proj RI5
SN5 = sheaf N5
HH^0(SN5)
HH^0(sheaf dual N5)
KI5 = Ext^3(R^1/I5 , R^{-5})**RI5
HH^0(SN5**OO_PI5(1)**(sheaf dual KI5))

```

Listing 2: Normal bundle

We choose three points on the smooth rational quintic curve:

```

P1 = ideal(x_3-14*x_4 , x_2-x_4 , x_1+x_4 , x_0-12*x_4)
P2 = ideal(x_3+17*x_4 , x_2-22*x_4 , x_1+20*x_4 , x_0+2*x_4)
P3 = ideal(x_3-26*x_4 , x_2+27*x_4 , x_1-30*x_4 , x_0+21*x_4)

```

We compute the quintic with multiplicity 3 at the 8 points and the point P1 (resp. P2 and resp. P3):

```

J13 = intersect(J^3,P1^3);
H13 = saturate J13;
G13 = gens(H13);
beti G13
GP1 = submatrix(G13,{0});
Q1 = ideal(GP1);

```

```

J23 = intersect(J^3,P2^3);
H23 = saturate J23;
G23 = gens(H23);
beti G23
GP2 = submatrix(G23,{0});
Q2 = ideal(GP2);

```

```

J33 = intersect(J^3,P3^3);
H33 = saturate J33;
G33 = gens(H33);

```

```

betti G33
GP3 = submatrix(G33, {0});
Q3 = ideal(GP3);

```

Listing 3: Three secant varieties

We compute the elliptic normal quintic curve along which Q1 is singular:

```

SingQ1 = ideal singularLocus variety Q1;
dim SingQ1, degree SingQ1
SSingQ1 = ideal singularLocus variety SingQ1;
dim SSingQ1, degree SSingQ1
E1 = (SingQ1:SSingQ1);
dim E1, degree E1, genus E1
ideal singularLocus variety E1

```

We compute the intersection of the three quintics and obtain the cubic scroll W:

```

SS3 = Q1 + Q2 + Q3;
SS = (SS3:II);
dim SS, degree SS
W = ideal singularLocus variety SS;
dim W, degree W, genus W, ideal singularLocus variety W
W == (SS:W)

```

Listing 4: Scheme-theoretic intersection of secant varieties

We compute the quintics through the 8 points with multiplicity at least 3 containing the surface W:

```

JW = intersect(J^3,W);
HW = saturate JW;
GW = gens(HW);
betti GW
GW1 = submatrix(GW, {0..2});
IIW = ideal(GW1);
IIIW = sheaf module IIW;
HH^0(IIIW(5))
IIW == SS3

```

Listing 5: Quintics containing W

We look at the singular locus of a quintic hypersurface through the 8 points with multiplicity at least 3 containing the surface W:

```

QW = ideal(11*GW1_(0,0)+7*GW1_(0,1)+19*GW1_(0,2));
SingQW = ideal singularLocus variety QW;

```

```

dim SingQW, degree SingQW, genus SingQW
SingW1 = (SingQW:J);
SingW2 = (SingW1:J);
SingW3 = (SingW2:J);
LSingQW = decompose SingW3;
EE = ideal(LSingQW);
degree EE, genus EE, dim EE
SingEE = ideal singularLocus variety EE;
dim SingEE, degree SingEE
degree (E1 + EE)
degree (I5 + EE)

```

Listing 6: Singular locus of a quintic containing W

We compute the hypersurfaces of degree 10 through the 8 points with multiplicity at least 6:

```

JI5 = intersect(I_5^6, intersect(I_6^6, I_7^6));
HI5 = saturate JI5;
GI5 = gens HI5;
beti GI5
GI15 = submatrix(GI5, {0..105});
II5 = ideal(GI15);
JI4 = intersect(I_4^6, II5);
HI4 = saturate JI4;
GI4 = gens HI4;
beti GI4
GI14 = submatrix(GI4, {0..132});
II4 = ideal(GI14);
JI3 = intersect(I_3^6, II4);
HI3 = saturate JI3;
GI3 = gens HI3;
beti GI3
GI13 = submatrix(GI3, {0..154});
II3 = ideal(GI13);
JI2 = intersect(I_2^6, II3);
HI2 = saturate JI2;
GI2 = gens HI2;
beti GI2
GI12 = submatrix(GI2, {0..123});
II2 = ideal(GI12);
JI1 = intersect(I_1^6, II2);
HI1 = saturate JI1;

```

```

GI1 = gens HI1;
beti GI1
GI11=submatrix(GI1,{0..136});
II1 = ideal(GI11);
JI0 = intersect(I_0^6,II1);
HI0 = saturate JI0;
GI0 = gens HI0;
beti GI0
GG = submatrix(GI0,{0..28});
IGG = ideal(GG);

```

We compute the hypersurfaces of degree 10 through the 8 points with multiplicity at least 6 containing the surface W :

```

JW2 = intersect(W,IGG);
HW2 = saturate JW2;
GW2 = gens HW2;
beti GW2
GGW = submatrix(GW2,{0..20});
IW2 = ideal(GGW);
IIW2 = sheaf module IW2;
HH^0(IIW2(10))

```

Listing 7: Decics containing W

We compute the image of the elliptic normal quintic $E1$ via the map defined by the linear system of hypersurfaces of degree 10 through the 8 points with multiplicity at least 6:

```

S2 = k[u_0..u_28];
ImE1 = ker map(R/E1,S2,GG);
dim ImE1, degree ImE1

```

We compute the image of the rational quintic curve $I5$ via the map defined by the linear system of hypersurfaces of degree 10 through the 8 points with multiplicity at least 6:

```

ImI5 = ker map(R/I5,S2,GG);
dim ImI5, degree ImI5

```

We compute the image of the surface W via the map defined by the linear system of hypersurfaces of degree 10 through the 8 points with multiplicity at least 6:

```

ImW = ker map(R/W,S2,GG);
dim ImW, degree ImW

```

Listing 8: Some images by the bianticanonical map

We compute the hypersurfaces of degree 10 with multiplicity at least 7 at the point I_0 and multiplicity at least 6 at the other 7 points:

```

JI00 = intersect( $I_0^7$ , II1);
HI00 = saturate JI00;
GI00 = gens HI00;
betti GI00
GG0 = submatrix(GI00, {0});
IGG0 = ideal(GG0);

```

And we obtain a unique such hypersurface of degree 10; now we check if this hypersurface contains the surface W :

```

JW0 = intersect(W, IGG0);
HW0 = saturate JW0;
GW0 = gens HW0;
betti GW0

```

Listing 9: Special member in the bianticanonical system

We compute the image of \mathbb{P}^4 via the map defined by the linear system of quintic hypersurfaces through the 8 points with multiplicity at least 3.

```

JJ = minors(2, random( $R^{\{4:0\}}$ ,  $R^{\{-2, -3\}}$ ));
degree JJ
genus JJ
betti res JJ == betti res ideal(G1)
S =  $k[y_0 \dots y_5]$  ; g = map(R, S, gens JJ);
K = ker g;
dim K
degree K
singularLocus variety K

```

Listing 10: Image by the anticanonical map

We check that there is no quintic hypersurfaces through the 8 points with multiplicity at least 3 and having multiplicity at least 2 along the smooth rational quintic curve I_5 :

```

JRR = intersect( $J^3$ ,  $I_5^2$ );
HRR = saturate JRR;
GRR = gens HRR;
betti GRR

```

Listing 11: Quintics having multiplicity 2 along the smooth rational quintic base curve

References

- [Bat99] V. V. Batyrev. On the classification of toric Fano 4-folds. volume 94, pages 1021–1050. 1999. Algebraic geometry, 9.
- [Bea96] Arnaud Beauville. *Complex algebraic surfaces*, volume 34 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, second edition, 1996. Translated from the 1978 French original by R. Barlow, with assistance from N. I. Shepherd-Barron and M. Reid.
- [Cas12] Cinzia Casagrande. On the Picard number of divisors in Fano manifolds. *Ann. Sci. Éc. Norm. Supér. (4)*, 45(3):363–403, 2012.
- [Cas13] Cinzia Casagrande. Numerical invariants of Fano 4-folds. *Math. Nachr.*, 286(11-12):1107–1113, 2013.
- [CCF19] Cinzia Casagrande, Giulio Codogni, and Andrea Fanelli. The blow-up of \mathbb{P}^4 at 8 points and its Fano model, via vector bundles on a del Pezzo surface. *Rev. Mat. Complut.*, 32(2):475–529, 2019.
- [CCG⁺13] Tom Coates, Alessio Corti, Sergey Galkin, Vasily Golyshev, and Alexander Kasprzyk. Mirror symmetry and Fano manifolds. In *European Congress of Mathematics*, pages 285–300. Eur. Math. Soc., Zürich, 2013.
- [CGKS20] Tom Coates, Sergey Galkin, Alexander Kasprzyk, and Andrew Strangeway. Quantum periods for certain four-dimensional Fano manifolds. *Exp. Math.*, 29(2):183–221, 2020.
- [CR22] Cinzia Casagrande and Eleonora A. Romano. Classification of Fano 4-folds with Lefschetz defect 3 and Picard number 5. *J. Pure Appl. Algebra*, 226(3):Paper No. 106864, 13, 2022.
- [DO88] Igor Dolgachev and David Ortland. Point sets in projective spaces and theta functions. *Astérisque*, 165, 1988.
- [Dol12] Igor V. Dolgachev. *Classical algebraic geometry*. Cambridge University Press, Cambridge, 2012. A modern view.
- [DV81] Patrick Du Val. Crystallography and Cremona transformations. In *The geometric vein*, pages 191–201. Springer, New York-Berlin, 1981.
- [Fuj83] Takao Fujita. On hyperelliptic polarized varieties. *Tohoku Math. J. (2)*, 35(1):1–44, 1983.

- [Ful98] William Fulton. *Intersection theory*, volume 2 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, second edition, 1998.
- [GH94] Phillip Griffiths and Joseph Harris. *Principles of algebraic geometry*. Wiley Classics Library. John Wiley & Sons, Inc., New York, 1994. Reprint of the 1978 original.
- [Har77] Robin Hartshorne. *Algebraic geometry*. Springer-Verlag, New York-Heidelberg, 1977. Graduate Texts in Mathematics, No. 52.
- [Heu16] Liana Heuberger. *Deux points de vue sur les variétés de Fano : géométrie du diviseur anticanonique et classification des surfaces à singularités $1/3(1,1)$* . PhD thesis, 2016. <http://www.theses.fr/2016PA066129/document>.
- [Isk77] V. A. Iskovskih. Fano threefolds. I. *Izv. Akad. Nauk SSSR Ser. Mat.*, 41(3):516–562, 717, 1977.
- [Isk78] V. A. Iskovskih. Fano threefolds. II. *Izv. Akad. Nauk SSSR Ser. Mat.*, 42(3):506–549, 1978.
- [Kaw89] Yujiro Kawamata. Small contractions of four-dimensional algebraic manifolds. *Math. Ann.*, 284(4):595–600, 1989.
- [Küc97] Oliver Küchle. Some remarks and problems concerning the geography of Fano 4-folds of index and Picard number one. *Quaestiones Math.*, 20(1):45–60, 1997.
- [MM86] Shigefumi Mori and Shigeru Mukai. Classification of Fano 3-folds with $B_2 \geq 2$. I. In *Algebraic and topological theories (Kinosaki, 1984)*, pages 496–545. Kinokuniya, Tokyo, 1986.
- [MM04] Shigefumi Mori and Shigeru Mukai. Extremal rays and Fano 3-folds. In *The Fano Conference*, pages 37–50. Univ. Torino, Turin, 2004.
- [Sat00] Hiroshi Sato. Toward the classification of higher-dimensional toric Fano varieties. *Tohoku Math. J. (2)*, 52(3):383–413, 2000.
- [Sec21] Saverio Andrea Secci. Fano 4-folds having a prime divisor of Picard number 1. *arXiv preprint arXiv: 2103.16140*, 2021.
- [Tak89] Kiyohiko Takeuchi. Some birational maps of Fano 3-folds. *Compositio Math.*, 71(3):265–283, 1989.

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