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Abstract We provide a proof and give some applications of an amazing formula discovered by Blagouchine.

Keywords Complex integration, generalized Glaisher-Kinkelin constants, infinite series with zeta values.

1 Introduction

The purpose of this short note is twofold: first, we provide a complete proof of a complex valued integral formula recently discovered by Blagouchine [2, Theorem 2], and then we relate this integral to some important mathematical constants, namely the Euler-Mascheroni constant, the Cohen-Boyadzhiev constant, the generalized Glaisher-Kinkelin constants (also known as Bendersky's constants) which occur quite naturally in analysis and number theory [9, 10]. Let us note in passing that a special case of Blagouchine's formula has already been mentioned (without proof) on page 1836 of [7].

2 Blagouchine's integral

Proposition 1. For any integer $k \geq 0$, let μ_k be the infinite sum

$$\mu_k := \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\zeta(n+1)}{n+k}$$

and $\mathcal{I}_k$ be the complex valued integral

$$\mathcal{I}_k := \int_{-\infty}^{+\infty} \frac{\zeta(\frac{3}{2}+ix)}{(2k+1+2ix)\cosh(\pi x)}\, dx\,.$$

Then we have the identity

$$\mathcal{I}_k = \mu_k\,. \tag{1}$$

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Proof. For $k \geq 0$, let us consider the function

$$f_k(z) = \frac{\zeta(\frac{3}{2} + iz)}{(\frac{1}{2} + k + iz)\cosh(\pi z)} \,.$$

We have $\cosh(\pi z) = 0$ if and only if $z = i/2 + in$ with $n \in \mathbb{Z}$. For $n \geq 1$, the residue of f_k at $z = i/2 - in$ is

$$\frac{\zeta(1+n)}{(n+k)\pi \sinh(i\pi(\frac{1}{2} - n))} = \frac{\zeta(1+n)}{(n+k)i\pi \sin(\pi(\frac{1}{2} - n))} = \frac{(-1)^n \zeta(1+n)}{(n+k)i\pi} \,.$$

We integrate on a closed contour composed of the interval $D_R = [-R, R]$ and the lower semicircle C_R of radius R with center at 0. Using the Cauchy residue theorem, we can then write the following relation:

$$\frac{1}{2i\pi}\int_{C_R} f_k(z)\,dz + \frac{1}{2i\pi}\int_{D_R} f_k(z)\,dz = -\sum_{n=1}^{N_R} \mathrm{Res}(f_k; \frac{i}{2} - in) \,,$$

which, from the foregoing, translates into the identity

$$\int_{C_R} f_k(z)\,dz + \int_{D_R} f_k(z)\,dz = 2\sum_{n=1}^{N_R} (-1)^{n+1} \frac{\zeta(1+n)}{(n+k)} \,. \tag{2}$$

For $z \in C_R$, the parameterization $iz = Re^{it}$ with $-\pi/2 < t < \pi/2$, enables us to write

$$\left|\int_{C_R} f_k(z)dz\right| = \left|\int_{-\pi/2}^{+\pi/2} \frac{\zeta(\frac{3}{2} + Re^{it})}{(\frac{1}{2} + k + Re^{it})\cosh(i\pi Re^{it})} Re^{it}\,dt\right|$$
$$\leq \int_{-\pi/2}^{+\pi/2} \left|\frac{\zeta(\frac{3}{2} + Re^{it})}{(\frac{1}{2} + k + Re^{it})\cosh(i\pi Re^{it})}\right| R\,dt \,.$$

Since $\frac{3}{2} + Re^{it}$ is in the half-plane $\mathrm{Re}(z) > 3/2$, its absolute value is bounded by $\zeta(\frac{3}{2})$, i.e.

$$\left|\zeta(\frac{3}{2} + Re^{it})\right| \leq \zeta(\frac{3}{2}) \,.$$

Hence, when R increases towards infinity, we have the following limits:

$$\lim_{R\to\infty} \int_{C_R} f_k(z)\,dz = 0 \,,$$

$$\lim_{R\to\infty} \int_{D_R} f_k(z)\,dz = \int_{-\infty}^{+\infty} \frac{\zeta(\frac{3}{2} + ix)}{(\frac{1}{2} + k + ix)\cosh(\pi x)}\,dx = 2\,\mathcal{I}_k \,,$$

and

$$\lim_{R\to\infty} \sum_{n=1}^{N_R} (-1)^{n+1} \frac{\zeta(1+n)}{(n+k)} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\zeta(1+n)}{(n+k)} = \mu_k \,.$$

This allows us to deduce formula (1) by passing to the limit in (2). ◻

Remark 1. The constant

$$\mu_0 = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\zeta(n+1)}{n} = \sum_{n=1}^{\infty} \frac{1}{n} \ln\left(1 + \frac{1}{n}\right) = 1.257746\ldots$$

has been thoroughly studied by Boyadzhiev [4] (see also [6, p. 142]). This constant is noted M in [4], K in [6], and also appears as ν_{-1} in [7]. By a well-known series representation of Euler's constant γ, we also have

$$\mu_1 = \sum_{n=2}^{\infty} (-1)^n \frac{\zeta(n)}{n} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \ln\left(1 + \frac{1}{n}\right)\right) = \gamma = 0.577215\ldots.$$

Example 1. For $k = 0$ and $k = 1$ respectively, formula (1) translates into

$$\mathcal{I}_0 = \int_{-\infty}^{+\infty} \frac{\zeta(\frac{3}{2} + ix)}{(1 + 2ix)\cosh(\pi x)}\, dx = \int_0^1 \frac{\psi(x+1) + \gamma}{x}\, dx\,, \tag{3}$$

where ψ is the digamma function, and

$$\mathcal{I}_1 = \int_{-\infty}^{+\infty} \frac{\zeta(\frac{3}{2} + ix)}{(3 + 2ix)\cosh(\pi x)}\, dx = \gamma = -\psi(1)\,. \tag{4}$$

3 Link with the generalized Glaisher-Kinkelin constants

Definition 1 ([1, 9, 10])**.** For any integer $k \geq 0$, the constant A_k is defined by

$$\ln(A_k) = \lim_{N\to\infty} \left(\sum_{n=1}^{N} n^k \ln n - P_k(N)\right)\,,$$

where $P_k(N)$ is given by $P_0(N) = \left(N + \frac{1}{2}\right)\ln N - N$, and

$$P_k(N) = \left(\frac{N^{k+1}}{k+1} + \frac{N^k}{2} + k! \sum_{j=1}^{k} \frac{N^{k-j} B_{j+1}}{(j+1)!(k-j)!}\right) \ln N$$
$$- \frac{N^{k+1}}{(k+1)^2} + k! \sum_{j=1}^{k} \frac{N^{k-j} B_{j+1}}{(j+1)!(k-j)!} \left\{(1 - \delta_{k,j}) \sum_{i=1}^{j} \frac{1}{k-i+1}\right\} \qquad (k \geq 1)\,,$$

where B_j is the j-th Bernoulli number and $\delta_{k,j}$ the Kronecker symbol. The numbers A_k for $k \geq 0$ are the generalized Glaisher-Kinkelin constants (sometimes called the Bendersky-Adamchik constants). Adamchik [1, Proposition 4] has shown that theses constants admit a nice expression in terms of the derivatives of the Riemann zeta function:

$$A_k = \exp\left\{\frac{H_k B_{k+1}}{k+1} - \zeta'(-k)\right\} \qquad (k \geq 0)\,, \tag{5}$$

where $H_k = \sum_{j=1}^{k} \frac{1}{j}$ is the k-th harmonic number with the usual convention $H_0 = 0$.

Remark 2. Bendersky [3] introduced for the first time the sequence of numbers $L_k := \ln(A_k)$ without any consideration of their relation with the ζ-function. From the point of view of the summation of divergent series, the constants L_k should be interpreted as follows: if $\sum_{n\geq 1}^{\mathcal{R}} n^k \ln n$ denotes the $\mathcal{R}$-sum (i.e. the sum in the sense of Ramanujan's summation method [5]) of the divergent series $\sum_{n\geq 1} n^k \ln n$, then

$$\sum_{n\geq 1}^{\mathcal{R}} n^k \ln n = \int_0^1 \ln \Gamma_k(x+1)\, dx = L_k - \frac{H_k B_{k+1}}{k+1} - \frac{1}{(k+1)^2} \qquad (k \geq 0)\,,$$

where Γ_k is Bendersky's generalized gamma function [3]. Unaware of Bendersky's work, Kurokawa and Ochoai [8, Theorem 2] have shown that

$$\Gamma_k(x) = \exp\left\{\zeta'(-k,x) - \zeta'(-k)\right\} \qquad \mathrm{Re}(x) > 0\,,$$

where $\zeta(s,x)$ is the Hurwitz zeta function. This expression is consistent with Adamchik's formula (5).

Remark 3. The following relations are easily deduced by differentiation of Riemann's functional equation for the zeta function:

$$\zeta'(-2k) = (-1)^k \frac{(2k)!}{2(2\pi)^{2k}} \zeta(2k+1) \qquad (k \geq 1),$$

and

$$\zeta'(1-2k) = (-1)^{k+1} \frac{(2k)!}{k(2\pi)^{2k}} \zeta'(2k) + \frac{B_{2k}}{2k}\left(H_{2k-1} - \gamma - \ln 2\pi\right) \qquad (k \geq 1)\,.$$

Hence, it follows from Adamchik's formula (5) that

$$A_{2k-1} = \exp\left\{(-1)^k \frac{(2k)!}{k(2\pi)^{2k}} \zeta'(2k) + \frac{B_{2k}}{2k}\left(\gamma + \ln 2\pi\right)\right\} \qquad (k \geq 1)\,, \tag{6}$$

and

$$A_{2k} = \exp\left\{(-1)^{k+1} \frac{(2k)!}{2(2\pi)^{2k}} \zeta(2k+1)\right\} \qquad (k \geq 1)\,. \tag{7}$$

Example 2. The constant $A_0 = \exp(-\zeta'(0)) = \sqrt{2\pi}$ is the Stirling constant,

$$A_1 = \exp\left(\frac{1}{12} - \zeta'(-1)\right) = \exp\left(-\frac{\zeta'(2)}{2\pi^2} + \frac{\ln(2\pi)+\gamma}{12}\right)$$

is the Glaisher-Kinkelin constant, and

$$A_2 = \exp\left(-\zeta'(-2)\right) = \exp\left(\frac{\zeta(3)}{4\pi^2}\right).$$

Proposition 2. For any integer $k \geq 2$, we have the following identity:

$$\int_{-\infty}^{+\infty} \frac{\zeta(\frac{3}{2}+ix)}{(2k+1+2ix)\cosh(\pi x)}\,dx = \frac{\gamma}{k} + \frac{1}{k-1} - \sum_{j=0}^{k-2}(-1)^j\binom{k-1}{j}\ln(A_j). \quad (8)$$

Proof. We have shown [7, Proposition 1] that

$$\mu_{k+1} = \frac{\gamma}{k+1} + \sum_{j=0}^{k-1}(-1)^j\binom{k}{j}\zeta'(-j) + \frac{1}{k} + \sum_{j=0}^{k-1}\binom{k}{j}\frac{B_{j+1}\,H_j}{j+1} \qquad (k \geq 1).$$

By means of (5), this expression may be rewritten as follows:

$$\mu_{k+1} = \frac{\gamma}{k+1} + \frac{1}{k} - \sum_{j=0}^{k-1}(-1)^j\binom{k}{j}\ln(A_j) \qquad (k \geq 1).$$

Hence, formula (8) results from (1). □

Example 3. For small values of k, formula (8) translates into the following identities:

$$\mathcal{I}_2 = \int_{-\infty}^{+\infty} \frac{\zeta(\frac{3}{2}+ix)}{(5+2ix)\cosh(\pi x)}\,dx = \frac{1}{2}\gamma + 1 - \frac{1}{2}\ln(2\pi) \quad (9)$$

$$\mathcal{I}_3 = \int_{-\infty}^{+\infty} \frac{\zeta(\frac{3}{2}+ix)}{(7+2ix)\cosh(\pi x)}\,dx = \frac{1}{2}\gamma + \frac{1}{2} - \frac{1}{3}\ln(2\pi) - \frac{\zeta'(2)}{\pi^2} \quad (10)$$

$$\mathcal{I}_4 = \int_{-\infty}^{+\infty} \frac{\zeta(\frac{3}{2}+ix)}{(9+2ix)\cosh(\pi x)}\,dx = \frac{1}{2}\gamma + \frac{1}{3} - \frac{1}{4}\ln(2\pi) - \frac{3\zeta'(2)}{2\pi^2} - \frac{3\zeta(3)}{4\pi^2} \quad (11)$$

In the general case, we obtain the following expression which, by means of formulas (6)–(7), is equivalent to (8):

Corollary 1. For any integer $k \geq 4$, we have

$$\mathcal{I}_k = \int_{-\infty}^{+\infty} \frac{\zeta(\frac{3}{2}+ix)}{(2k+1+2ix)\cosh(\pi x)}\,dx = \frac{1}{2}\gamma + \frac{1}{k-1} - \frac{1}{k}\ln(2\pi)$$
$$+ \sum_{j=1}^{[\frac{k-1}{2}]}(-1)^j\binom{k-1}{2j-1}\frac{(2j)!}{j(2\pi)^{2j}}\zeta'(2j) + \sum_{j=1}^{[\frac{k}{2}]-1}(-1)^j\binom{k-1}{2j}\frac{(2j)!}{2(2\pi)^{2j}}\zeta(2j+1). \quad (12)$$

Remark 4. Blagouchine [2, Theorem 1] also established the relation

$$\mathcal{I}_{k+1} = \frac{\gamma}{k+1} - \frac{1}{(k+1)^2} - \int_{-\infty}^{+\infty} \frac{\zeta(\frac{1}{2}+ix)}{(2k+1+2ix)\cosh(\pi x)}\,dx \qquad (k \geq 0)\,,$$

which allows us to deduce from (8) the following identity:

$$\sum_{j=0}^{k-1}(-1)^j\binom{k}{j}\ln(A_j) = \frac{1}{k} + \frac{1}{(k+1)^2} + \int_{-\infty}^{+\infty} \frac{\zeta(\frac{1}{2}+ix)}{(2k+1+2ix)\cosh(\pi x)}\,dx \qquad (k \geq 1). \tag{13}$$

In particular, in the simplest case, this reduces to

$$\ln(\sqrt{2\pi}) = \frac{5}{4} + \int_{-\infty}^{+\infty} \frac{\zeta(\frac{1}{2}+ix)}{(3+2ix)\cosh(\pi x)}\,dx\,.$$

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