Generalized Glaisher-Kinkelin constants, Blagouchine's integrals, and Ramanujan summation of series

Marc-Antoine Coppo

Generalized Glaisher-Kinkelin constants, Blagouchine's integrals, and Ramanujan summation of series

Marc-Antoine Coppo*

Université Côte d'Azur, CNRS, LJAD (UMR 7351), Nice, France

Abstract The main motivation for this article is to establish a connection between a sequence of constants of Glaisher-Kinkelin type (known as the Bendersky-Adamchik constants) which appear quite naturally in number theory, and a family of complex integrals recently introduced by Blagouchine. Furthermore, we also elucidate the close relation between these constants and the Ramanujan summation of certain divergent series, and we give a new convergent series expansion for the logarithm of these constants.

Keywords Generalized Glaisher-Kinkelin constants; generalized gamma functions; infinite series with zeta values; Blagouchine's integrals; Ramanujan summation of series.

1 Introduction

Introduced in 1933 by Bendersky [2], the sequence of mathematical constants called generalized Glaisher-Kinkelin constants or Bendersky-Adamchik constants (see Definition 1) arises quite naturally in number theory [10, 12, 14]. The initial purpose of this article is to link this sequence of real numbers $\{A_k\}$ to the family of complex integrals $\{\mathcal{J}_{k,p}\}$ defined (for nonnegative integers k and p, with p odd) by

$$\mathcal{J}_{k,p} := \int_{-\infty}^{+\infty} \frac{\zeta(\frac{p}{2}+ix)}{(2k+p+2ix)\cosh(\pi x)}\,dx\,.$$

In an unpublished note, Blagouchine [3] introduced these integrals in the cases $p=1$ and $p=3$ only. This connection is deduced from the results of a previous study [8] on one hand (see Lemma 1), and from the residue theorem on the other

*Corresponding author. *Email address:* `coppo@unice.fr`

hand (see Proposition 1). In particular, we obtain a nice expression of the integral $\mathcal{J}_k = \mathcal{J}_{k,1}$ in terms of the generalized Glaisher-Kinkelin constants A_j for $0 \leq j \leq k-1$ (see Corollary 1). More precisely, we show that for all $k \geq 1$,

$$\mathcal{J}_k = \int_{-\infty}^{+\infty} \frac{\zeta(\frac{1}{2}+ix)}{(2k+1+2ix)\cosh(\pi x)}\,dx = \sum_{j=0}^{k-1}(-1)^j \binom{k}{j} \ln A_j - \frac{1}{k} - \frac{1}{(k+1)^2}\,.$$

Furthermore, we completely elucidate the close relation between the logarithm of the generalized Glaisher-Kinkelin constant A_k and the sum of the divergent series $\sum_{n\geq 1} n^k \ln n$ in the sense of Ramanujan's summation method, following the exposition in [4]. More precisely, using the notations of [4], we show that for all $k \geq 0$,

$$\ln A_k = \sum_{n\geq 1}^{\mathcal{R}} n^k \ln n + \frac{H_k B_{k+1}}{k+1} + \frac{1}{(k+1)^2}\,,$$

where, in this expression, H_k and B_k denote respectively the kth harmonic number and the kth Bernoulli number.

Finally, we present, in the last section, a new convergent series expansion for the constant $\ln A_k$ involving the Bernoulli numbers of the second kind (see Corollary 2).

2 Constants of Glaisher-Kinkelin type and Ramanujan summation

Definition 1. Let $\{B_{2r}\}_{r\geq 0}$ be the sequence of (even) Bernoulli numbers. The sequence of numbers $\{A_k\}_{k\geq 0}$ can be defined as follows [14, Eq. (1.1)–(1.6)]: for all $k \geq 0$,

$$\ln A_k := \lim_{n\to\infty} \left\{ \sum_{\nu=1}^{n} \nu^k \ln \nu - P_k(n)\ln n + Q_k(n) \right\}\,,$$

where P_k and Q_k are polynomials of degree $k+1$ whose expressions are given by

$$P_0(x) = x + \frac{1}{2}\,, Q_0(x) = x\,,$$

and for $k \geq 1$,

$$P_k(x) = \frac{x^{k+1}}{k+1} + \frac{x^k}{2} + \sum_{r=1}^{[\frac{k+1}{2}]} \frac{B_{2r}}{(2r)!} \left(\prod_{j=1}^{2r-1} (k-j+1) \right) x^{k+1-2r}\,,$$

$$Q_k(x) = \frac{x^{k+1}}{(k+1)^2} - \sum_{r=1}^{[\frac{k+1}{2}]+\frac{(-1)^k-1}{2}} \frac{B_{2r}}{(2r)!} \left\{ \prod_{j=1}^{2r-1} (k-j+1) \sum_{j=1}^{2r-1} \frac{1}{k-j+1} \right\} x^{k+1-2r}\,.$$

The numbers A_k (for $k = 0, 1, 2, \ldots$) are called the *generalized Glaisher-Kinkelin constants* or the *Bendersky-Adamchik constants*. Adamchik [1, Prop. 4] has given a nice expression of these constants A_k in terms of the derivatives of the Riemann zeta function. More precisely, this expression (called Adamchik's formula in the remainder of this article) is the following:

$$\ln A_k = \frac{H_k B_{k+1}}{k+1} - \zeta'(-k)\,, \tag{1}$$

where H_k and B_k denote respectively the kth harmonic number (with the usual convention $H_0 = 0$) and the kth Bernoulli number.

Example 1. a) The constant A_0 is the Stirling constant

$$A_0 = \sqrt{2\pi} = \lim_{n\to\infty} \left\{ \frac{n!}{n^{n+\frac{1}{2}}\, e^{-n}} \right\} = \exp(-\zeta'(0))\,.$$

b) The constant A_1 is the classical Glaisher-Kinkelin constant [1, 10, 14]. We have

$$A_1 = \lim_{n\to\infty} \left\{ \frac{\prod_{\nu=1}^{n} \nu^{\nu}}{n^{\frac{n^2}{2}+\frac{n}{2}+\frac{1}{12}}\, e^{-\frac{n^2}{4}}} \right\} = \exp\left\{ \frac{1}{12} - \zeta'(-1) \right\}\,.$$

c) For $k = 2$, we have

$$A_2 = \lim_{n\to\infty} \left\{ \frac{\prod_{\nu=1}^{n} \nu^{\nu^2}}{n^{\frac{n^3}{3}+\frac{n^2}{2}+\frac{n}{6}}\, e^{-\frac{n^3}{9}+\frac{n}{12}}} \right\} = \exp\left(-\zeta'(-2)\right)\,.$$

Remark 1. Bendersky [2] introduced for the first time the sequence of numbers A_k without any consideration of their relation with the ζ-function. From the point of view of the summation of divergent series, the constants $\ln A_k$ can be interpreted as follows: for any non-negative integer k, if $\sum_{n\geq 1}^{\mathcal{R}} n^k \ln n$ denotes the $\mathcal{R}$-sum of the divergent series $\sum_{n\geq 1} n^k \ln n$ (i.e. the sum of the series in the sense of Ramanujan's summation method, following the notations of [4]), then $\ln A_k$ and $\sum_{n\geq 1}^{\mathcal{R}} n^k \ln n$ are linked by the simple relation:

$$\ln A_k = \sum_{n\geq 1}^{\mathcal{R}} n^k \ln n + \frac{H_k B_{k+1}}{k+1} + \frac{1}{(k+1)^2} \qquad (k \geq 0)\,. \tag{2}$$

This relation results both from [4, p. 68] and from Adamchik's formula (1). By [2, Eq. (V'$_k$), p. 280], we also deduce the equivalent relation

$$\sum_{n\geq 1}^{\mathcal{R}} n^k \ln n = \ln A_k - \frac{H_k B_{k+1}}{k+1} - \frac{1}{(k+1)^2} = \int_0^1 \ln \Gamma_k(x+1)\, dx\,, \tag{2 b}$$

where Γ_k is the Bendersky generalized gamma function [2, pp. 279–280]. This function verifies

$$\Gamma_k(n+1) = \prod_{\nu=1}^{n} \nu^{\nu^k} \quad \text{for any integer } n \geq 1\,,$$

and

$$\Gamma_k(n+1) \sim A_k\, n^{P_k(n)} e^{-Q_k(n)} \quad \text{as } n \to \infty\,.$$

In particular, $\Gamma_0 = \Gamma$, and Γ_1 is the Kinkelin-Bendersky hyperfactorial function [13, Def. 3]. Unaware of Bendersky's work and following an idea of Milnor, Kurokawa and Ochiai [11, Thm. 2] provided an expression of the function Γ_k in terms of the derivative of the Hurwitz zeta function $\zeta(s,x)$ at $s=-k$. Precisely, they established the relation[1]

$$\ln \Gamma_k(x) = \zeta'(-k,x) - \zeta'(-k) \qquad (x>0,\ k \geq 0)\,,$$

which is a generalization of the classical formula for Γ [7, Def. 9.6.13]

$$\ln \Gamma(x) = \zeta'(0,x) - \zeta'(0) \qquad (x>0)\,.$$

Remark 2. The following identities:

$$\zeta'(-2k) = (-1)^k \frac{(2k)!}{2(2\pi)^{2k}} \zeta(2k+1) \qquad (k \geq 1)\,,$$

and

$$\zeta'(1-2k) = (-1)^{k+1} \frac{(2k)!}{k(2\pi)^{2k}} \zeta'(2k) + \frac{B_{2k}}{2k} \left(H_{2k-1} - \gamma - \ln 2\pi\right) \qquad (k \geq 1)\,,$$

where γ denotes the Euler-Mascheroni constant, are easily derived (by differentiation) from the functional equation of the zeta function. Thanks to Adamchik's formula (1) and the celebrated Euler formula

$$\zeta(2k) = (-1)^{k+1} \frac{(2\pi)^{2k}}{2(2k)!} B_{2k} \qquad (k \geq 1),$$

we can deduce the following two ones:

$$\ln A_{2k-1} = \frac{B_{2k}}{2k} \left(\gamma + \ln 2\pi - \frac{\zeta'(2k)}{\zeta(2k)}\right) \qquad (k \geq 1)\,, \tag{3}$$

1. According to Kellner [10, Rem. 27], in the special case where $x = n+1$ is an integer, this expression of $\ln \Gamma_k$ is due to Alexeiewsky.

and

$$\ln A_{2k} = \frac{B_{2k}}{4}\left(\frac{\zeta(2k+1)}{\zeta(2k)}\right) \qquad (k \geq 1)\,. \tag{4}$$

These identities will be useful in the remainder (see Example 3 below).

Example 2. The simplest cases of these identities are the following:

$$\ln A_1 = \frac{1}{12}\left(\gamma + \ln 2\pi - \frac{\zeta'(2)}{\zeta(2)}\right) \quad \text{and} \quad \ln A_2 = \frac{1}{24}\left(\frac{\zeta(3)}{\zeta(2)}\right)\,.$$

In particular, for small values of k, formula (2 b) translates into

$$\sum_{n\geq 1}^{\mathcal{R}} \ln n = \frac{1}{2}\ln 2\pi - 1\,,$$

$$\sum_{n\geq 1}^{\mathcal{R}} n\ln n = \frac{1}{12}\left(\gamma + \ln 2\pi - \frac{\zeta'(2)}{\zeta(2)}\right) - \frac{1}{3}\,,$$

$$\sum_{n\geq 1}^{\mathcal{R}} n^2\ln n = \frac{1}{24}\left(\frac{\zeta(3)}{\zeta(2)}\right) - \frac{1}{9}\,.$$

3 Generalized Glaisher-Kinkelin constants and Blagouchine's integrals

Definition 2. For non-negative integers k and p, with p odd, the integral $\mathcal{J}_{k,p}$ and the series $\mathcal{S}_{k,p}$ are defined respectively by

$$\mathcal{J}_{k,p} := \int_{-\infty}^{+\infty} \frac{\zeta(\frac{p}{2}+ix)}{(2k+p+2ix)\cosh(\pi x)}\,dx\,,$$

and

$$\mathcal{S}_{k,p} := \sum_{n=N_p}^{\infty}(-1)^n\frac{\zeta(n)}{n+k} \quad \text{with } N_p = \max(2, \frac{p+1}{2})\,.$$

To lighten our notations, in the remainder of the text, we set

$$\mathcal{J}_k := \int_{-\infty}^{+\infty} \frac{\zeta(\frac{1}{2}+ix)}{(2k+1+2ix)\cosh(\pi x)}\,dx = \mathcal{J}_{k,1}\,,$$

and

$$\mathcal{S}_k := \sum_{n=2}^{\infty}(-1)^n\frac{\zeta(n)}{n+k} = \mathcal{S}_{k,1} = \mathcal{S}_{k,3}\,.$$

Lemma 1. For all $k \geq 1$, we have

$$\mathcal{S}_k = \frac{\gamma}{k+1} + \frac{1}{k} - \sum_{j=0}^{k-1} (-1)^j \binom{k}{j} \ln A_j \,. \tag{5}$$

Proof. It results from [8, Prop. 1] that

$$\mathcal{S}_k = \frac{\gamma}{k+1} + \sum_{j=0}^{k-1} (-1)^j \binom{k}{j} \zeta'(-j) + \frac{1}{k} + \sum_{j=0}^{k-1} \binom{k}{j} \frac{B_{j+1}\, H_j}{j+1} \qquad (k \geq 1)\,.$$

Using Adamchik's formula (1), this later expression is equivalent to (5). □

The following proposition includes and extends the results of [3].

Proposition 1. For all $k \geq 0$, we have the following relations:

$$\mathcal{J}_k = \frac{\gamma}{k+1} - \frac{1}{(k+1)^2} - \mathcal{S}_k \,, \tag{6}$$

and

$$\mathcal{J}_{k,p} = (-1)^{\frac{p+1}{2}} \mathcal{S}_{k,p} \qquad (p = 3, 5, 7, \cdots). \tag{7}$$

In particular, $\mathcal{J}_{k,3} = S_k$.

Proof. For $k \geq 0$, let us consider the function

$$f_k(z) = \frac{\zeta(z)}{(k+z)\sin(\pi z)} \,.$$

This function f_k has poles at integers $n \in \mathbb{Z}$. For $n \geq 2$, the residue of f_k at $z = n$ is

$$\mathrm{Res}(f_k; n) = \frac{(-1)^n \zeta(n)}{(n+k)\pi} \,.$$

For $n = 1$, f_k has a double pole and

$$\mathrm{Res}(f_k; 1) = -\frac{1}{\pi} \left(\frac{\gamma}{k+1} - \frac{1}{(k+1)^2} \right).$$

If p, q are positive odd integers with $p < q$, then, by the residue theorem, we have

$$\int_{\mathrm{Re}(z)=p/2} f_k(z)\, dz - \int_{\mathrm{Re}(z)=q/2} f_k(z)\, dz = -2i\pi \sum_{\frac{q}{2} > n > \frac{p}{2}} \mathrm{Res}(f_k; n)\,. \tag{$*$}$$

Moreover, there is a positive constant C such that

$$\left| \int_{\mathrm{Re}(z)=q/2} f_k(z)\, dz \right| \leq C \int_{-\infty}^{+\infty} \frac{1}{\left((k+\frac{q}{2})^2 + t^2\right)^{\frac{1}{2}} (e^{\pi t} - e^{-\pi t})}\, dt \,,$$

and thus, by the dominated convergence theorem, we have

$$\left|\int_{\mathrm{Re}(z)=q/2} f_k(z)\,dz\right| \to 0 \quad \text{as} \quad q \to +\infty\,.$$

Therefore, taking the limit in $(*)$, we obtain

$$\int_{\mathrm{Re}(z)=p/2} f_k(z)\,dz = -2i\pi \sum_{n>\frac{p}{2}} \mathrm{Res}(f_k;n)\,. \tag{$**$}$$

This last identity allows us to deduce formulas (6) and (7). □

From Lemma 1 and Proposition 1 above, we derive the following corollary.

Corollary 1. For all $k \geq 1$, we have

$$\mathcal{J}_k = \sum_{j=0}^{k-1} (-1)^j \binom{k}{j} \ln A_j - \frac{1}{k} - \frac{1}{(k+1)^2}\,. \tag{8}$$

Proof. Formula (8) is a direct consequence of (5) and (6). □

Example 3. For small positive values of k, we have

$$\begin{aligned}
\mathcal{J}_1 &= \frac{1}{2}\ln 2\pi - \frac{5}{4}\,,\\
\mathcal{J}_2 &= \frac{1}{3}\ln 2\pi - \frac{1}{6}\gamma - \frac{11}{18} + \frac{1}{6}\frac{\zeta'(2)}{\zeta(2)}\,,\\
\mathcal{J}_3 &= \frac{1}{4}\ln 2\pi - \frac{1}{4}\gamma - \frac{19}{48} + \frac{1}{4}\frac{\zeta'(2)}{\zeta(2)} + \frac{1}{8}\frac{\zeta(3)}{\zeta(2)}\,.
\end{aligned}$$

Moreover, using formulas (3) and (4), we obtain the following general formula:

$$\begin{aligned}
\mathcal{J}_k = \frac{1}{k+1}\ln 2\pi - \frac{(k-1)}{2(k+1)}\gamma - \frac{k^2+3k+1}{k(k+1)^2}\\
+ \sum_{j=1}^{[\frac{k}{2}]} \binom{k}{2j-1} \frac{B_{2j}\,\zeta'(2j)}{2j\,\zeta(2j)} + \frac{1}{4}\sum_{j=1}^{[\frac{k-1}{2}]} \binom{k}{2j} \frac{B_{2j}\,\zeta(2j+1)}{\zeta(2j)} \qquad (k \geq 3)\,.
\end{aligned} \tag{9}$$

Remark 3. Since $\mathcal{J}_0 = -1$ (by (6)), we can write (8) in a slightly different manner:

$$\sum_{j=0}^{k-1} (-1)^j \binom{k}{j} \ln A_j = \mathcal{J}_k - \frac{k^2+3k+1}{k(k+1)^2}\,\mathcal{J}_0 \qquad (k \geq 1)\,.$$

Remark 4. By means of a Fourier transform method, Candelpergher [5] has recently established a beautiful relation which is a natural generalization of (6) involving a complex parameter s. More precisely, for any non-negative integer k and any complex number s such that $\mathrm{Re}(s) > \frac{1}{2}$, we have the following relation [5, Eq. (7)]:

$$2^{s-1}\mathcal{J}_k(s) = \frac{\gamma}{(k+1)^s} - \frac{s}{(k+1)^{s+1}} - \mathcal{S}_k(s)\,. \tag{10}$$

with

$$\mathcal{J}_k(s) := \int_{-\infty}^{+\infty} \frac{\zeta(\frac{1}{2}+ix)}{(2k+1+2ix)^s\cosh(\pi x)}\,dx\,,$$

and

$$\mathcal{S}_k(s) := \sum_{n=2}^{\infty}(-1)^n\frac{\zeta(n)}{(n+k)^s}\,.$$

In particular, applying (10) to the special case $k=0$, allows us deduce the following interesting identity:

$$\sum_{n=2}^{\infty}(-1)^n\frac{\zeta(n)}{n^s} = \gamma - s - \frac{1}{2}\int_{-\infty}^{+\infty}\frac{\zeta(\frac{1}{2}+ix)}{(\frac{1}{2}+ix)^s\cosh(\pi x)}\,dx \qquad (\mathrm{Re}(s) > \frac{1}{2})\,. \tag{11}$$

4 New series expansions for the logarithm of the generalized Glaisher-Kinkelin constants

In his seminal work of 1933, Bendersky [2, p. 295–299] had presented two convergent series expansions of the logarithm $L_k = \ln A_k$ of the generalized Glaisher-Kinkelin constants. In this last section, we give a new one, of a different kind, involving the Bernoulli numbers of the second kind.

We first define a sequence of positive rational numbers $\{\lambda_n\}_{n\geq 1}$ (called non-alternating Cauchy numbers [6]) by

$$\lambda_n := \left|\sum_{k=1}^{n}\frac{s(n,k)}{k+1}\right| \qquad (n \geq 1)\,,$$

where $s(n,k)$ denotes the (signed) Stirling numbers of the first kind. Alternatively, these numbers can also be defined recursively by means of the relation

$$\sum_{k=1}^{n-1}\frac{\lambda_k}{k!\,(n-k)} = \frac{1}{n} \qquad (n \geq 2)\,.$$

The first ones are the following:

$$\lambda_1 = \frac{1}{2}\,,\ \lambda_2 = \frac{1}{6}\,,\ \lambda_3 = \frac{1}{4}\,,\ \lambda_4 = \frac{19}{30}\,,\ \lambda_5 = \frac{9}{4}\,,\ \lambda_6 = \frac{863}{84}\,,\ \text{etc.}$$

The numbers λ_n are closely related to the Bernoulli numbers of the second kind b_n [9] through the relation

$$\lambda_n = (-1)^{n-1} n!\, b_n = n!\, |b_n| \qquad (n \geq 1)\,.$$

We now give another interesting application of Adamchik's formula (1).

Proposition 2. Let $S(k,r)$ be the Stirling numbers of the second kind

$$S(k,r) = \frac{1}{r!} \sum_{j=0}^{r} (-1)^{r-j} \binom{r}{j} j^k \qquad (0 \leq r \leq k)\,,$$

and σ_r the shifted Mascheroni series

$$\sigma_r := \sum_{n=r+1}^{\infty} \frac{\lambda_n}{n!\,(n-r)} \qquad (r \geq 0)\,.$$

Then, for all integers $k \geq 1$, we have

$$\ln A_k = (-1)^k \sum_{r=1}^{k} (-1)^r r!\, S(k,r)\, \sigma_{r+1} + \frac{B_{k+1}}{k+1} \left(H_{k+1} + \gamma\right)\,. \tag{12}$$

Proof. Thanks to Adamchik's formula (1), formula (12) can be easily deduced from the decomposition of $\zeta'(-k)$ given by [9, Prop. 3]. □

Remark 5. This formula (12) also extends to the case $k = 0$ through the identity:

$$\ln A_0 = \frac{1}{2}\ln(2\pi) = \sigma_1 + \frac{1}{2}\gamma + \frac{1}{2} = \sum_{n=2}^{\infty} \frac{\lambda_n}{n!\,(n-1)} + \frac{1}{2}(1+\gamma)\,.$$

Corollary 2. For all integers $k \geq 1$, we have

$$\ln A_{2k} = \sum_{n=2k+2}^{\infty} \frac{\lambda_n}{n!} \left\{ \sum_{r=1}^{2k} \frac{(-1)^r r!\, S(2k,r)}{n-1-r} \right\} + C_{2k}\,, \tag{13}$$

and

$$\ln A_{2k-1} = \sum_{n=2k+1}^{\infty} \frac{\lambda_n}{n!} \left\{ \sum_{r=1}^{2k-1} \frac{(-1)^{r-1} r!\, S(2k-1,r)}{n-1-r} \right\} + \frac{B_{2k}}{2k}\left(H_{2k} + \gamma\right) + C_{2k-1}\,, \tag{14}$$

where the constants C_k are given by $C_1 = 0$, and

$$C_k = (-1)^k \sum_{r=1}^{k-1} (-1)^r r!\, S(k,r) \sum_{j=r+2}^{k+1} \frac{\lambda_j}{j!\,(j-1-r)} \qquad (k \geq 2)\,.$$

Example 4.

$$\begin{aligned}
\ln A_1 &= \sum_{n=3}^{\infty} \frac{\lambda_n}{n!(n-2)} + \frac{1}{12}\gamma + \frac{1}{8} \\
\ln A_2 &= \sum_{n=4}^{\infty} \frac{\lambda_n\,(n-1)}{n!\,(n-2)(n-3)} - \frac{1}{24} \\
\ln A_3 &= \sum_{n=5}^{\infty} \frac{\lambda_n\, n(n-1)}{n!\,(n-2)(n-3)(n-4)} - \frac{1}{120}\gamma - \frac{29}{240} \\
\ln A_4 &= \sum_{n=6}^{\infty} \frac{\lambda_n\,(n-1)^2(n+4)}{n!\,(n-2)(n-3)(n-4)(n-5)} - \frac{113}{480} \\
\ln A_5 &= \sum_{n=7}^{\infty} \frac{\lambda_n\, n(n-1)(n^2+13n-18)}{n!\,(n-2)(n-3)(n-4)(n-5)(n-6)} + \frac{1}{252}\gamma - \frac{55087}{80640}
\end{aligned}$$

Remark 6. Thanks to the simple relation linking $\ln A_k$ to the $\mathcal{R}$-sum $\sum_{n\geq 1}^{\mathcal{R}} n^k \ln n$ given by (2), we can also deduce from the previous corollary, the corresponding formulas:

$$\sum_{n\geq 1}^{\mathcal{R}} n^{2k}\ln n = \sum_{n=2k+2}^{\infty} \frac{\lambda_n}{n!} \left\{ \sum_{r=1}^{2k} \frac{(-1)^r r!\, S(2k,r)}{n-1-r} \right\} - \frac{1}{(2k+1)^2} + C_{2k}\,, \tag{15}$$

$$\sum_{n\geq 1}^{\mathcal{R}} n^{2k-1}\ln n = \sum_{n=2k+1}^{\infty} \frac{\lambda_n}{n!} \left\{ \sum_{r=1}^{2k-1} \frac{(-1)^{r-1} r!\, S(2k-1,r)}{n-1-r} \right\} + \frac{B_{2k}}{2k}\gamma + D_{2k}\,, \tag{16}$$

with

$$D_{2k} = \frac{B_{2k}-1}{(2k)^2} + C_{2k-1} \qquad (k \geq 1)\,.$$

References

[1] V. Adamchik, Polygamma functions of negative order, *J. Comput. Appl. Math.* **100** (1998), 191–199.

[2] L. Bendersky, Sur la function gamma généralisée, *Acta Math.* **61** (1933), 263–322.

[3] I. V. Blagouchine, A complement to a recent paper on some infinite sums with the zeta values, preprint, 2020. Available at https://arxiv.org/abs/2001.00108

[4] B. Candelpergher, *Ramanujan Summation of Divergent Series*, Lecture Notes in Math. 2185, Springer, 2017.

[5] B. Candelpergher, An expansion of the Riemann Zeta function on the critical line, preprint, 2021. Available at https://hal.archives-ouvertes.fr/hal-03271709

[6] B. Candelpergher, M-A. Coppo, A new class of identities involving Cauchy numbers, harmonic numbers and zeta values, *Ramanujan J.* **27** (2012), 305–328.

[7] H. Cohen, *Number Theory, Volume II: Analytic and Modern Tools*, Graduate Texts in Math., vol. 240, Springer, 2007.

[8] M-A. Coppo, A note on some alternating series involving zeta and multiple zeta values, *J. Math. Anal. Appl.* **475** (2019), 1831–1841.

[9] M-A. Coppo and P. T. Young, On shifted Mascheroni series and hyperharmonic numbers, *J. Number Theory*, **169** (2016), 1–20.

[10] B. C. Kellner, On asymptotic constants related to products of Bernoulli numbers, *Integers* **9**, (2009), 83–106.

[11] N. Kurokawa and H. Ochiai, Generalized Kinkelin's formulas *Kodai Math. J.* **30** (2007), 195–212.

[12] M. Perkins and R. A. Van Gorder, Closed-form calculation of infinite products of Glaisher-type related to Dirichlet series, *Ramanujan J.* **49** (2019), 371–389.

[13] J. Sondow and P. Hadjicostas, The generalized-Euler-constant function $\gamma(z)$ and a generalization of Somos's quadratic recurrence constant, *J. Math. Anal. Appl.* **332** (2007), 292–314.

[14] W. Wang, Some asymptotic expansions of hyperfactorial functions and generalized Glaisher-Kinkelin constants, *Ramanujan J.* **43** (2017), 513–533.