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Abstract. The main purpose of this short article is to highlight the existence of a close connection between a family of complex integrals introduced by Blagouchine and some notable mathematical constants, namely the generalized Glaisher-Kinkelin constants (also known as the Bendersky-Adamchik constants) which occur quite naturally in number theory and analysis. Incidentely, we also point out the connection between these constants and the Ramanujan summation of certain divergent series.

Keywords. Glaisher-Kinkelin constants; Bendersky generalized gamma function; infinite series with zeta values; Blagouchine's integrals; Ramanujan summation of series.

1 Introduction

The aim of this article is to establish a link between the family of complex integrals $\{\mathcal{J}_{k,p}\}$ defined (for integers $k \geq 0$ and $p \geq 1$ with p odd) by

$$\mathcal{J}_{k,p} := \int_{-\infty}^{+\infty} \frac{\zeta(\frac{p}{2} + ix)}{(2k + p + 2ix)\cosh(\pi x)}\, dx\,,$$

and a famous sequence of mathematical constants, namely the generalized Glaisher-Kinkelin constants (which are also known as the Bendersky-Adamchik constants) arising quite naturally in the theory of special functions and number theory (see Definition 1 and Remark 1). Blagouchine [3] introduced these integrals in the cases $p = 1$ and $p = 3$. To show this close connection, we make use of a relation between the integral $\mathcal{J}_{k,p}$ and the alternating series

$$\mathcal{S}_{k,p} := \sum_{n=N_p}^{\infty} (-1)^n \frac{\zeta(n)}{n+k} \quad \text{with } N_p = \max(2, \frac{p+1}{2})$$

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which can be deduced from the residue theorem (see Proposition 1). These alternating series have been studied in detail in our previous article [7]. In particular, this study allows us to provide a general expression of the integrals $\mathcal{J}_{k,1}$ and $\mathcal{J}_{k,3}$ in terms of the generalized Glaisher-Kinkelin constants for all positive integers k (see Corollary 1).

Recently, using another method, Candelpergher [5] has generalized the relation between the integral $\mathcal{J}_{k,1}$ and the alternating series $\mathcal{S}_{k,1}$ by introducing a complex parameter s (see Proposition 3). As a corollary, this leads to an integral representation of the Dirichlet series

$$\sum_{n=2}^{\infty}(-1)^n\frac{\zeta(n)}{n^s}$$

which is valid for any complex number s with $\mathrm{Re}(s) > \frac{1}{2}$ (see Corollary 2).

2 Generalized Glaisher-Kinkelin constants

Definition 1. For any integer $k \geq 0$, the constant A_k are usually defined by

$$\ln A_0 = \lim_{N\to\infty}\left\{\sum_{n=1}^{N}\ln n - \left(N+\frac{1}{2}\right)\ln N + N\right\},$$

$$\ln A_1 = \lim_{N\to\infty}\left\{\sum_{n=1}^{N}n\ln n - \left(\frac{N^2}{2}+\frac{N}{2}+\frac{1}{12}\right)\ln N + \frac{N^2}{4}\right\},$$

$$\ln A_2 = \lim_{N\to\infty}\left\{\sum_{n=1}^{N}n^2\ln n - \left(\frac{N^3}{3}+\frac{N^2}{2}+\frac{N}{6}\right)\ln N + \frac{N^3}{9} - \frac{N}{12}\right\},$$

and more generally

$$\ln A_k = \lim_{N\to\infty}\left\{\sum_{n=1}^{N}n^k\ln n - U_{k+1}(N)\ln N + V_{k+1}(N)\right\},$$

where U_{k+1} and V_{k+1} are polynomials of degree $k+1$ that can be explicitely computed [12, Eq. (1.6)]. The numbers A_k for $k = 0, 1, 2, \ldots$ are called the *generalized Glaisher-Kinkelin constants* or the *Bendersky-Adamchik constants* [9, 10, 12]. Adamchik [1, Prop. 4] has given a nice expression of the constants A_k in terms of the derivatives of the Riemann zeta function. More precisely, this expression (that we will call Adamchik's formula) is the following:

$$\ln A_k = \frac{H_k B_{k+1}}{k+1} - \zeta'(-k)\,, \tag{1}$$

where H_n are the harmonic numbers (with the usual convention $H_0 = 0$) and B_n are the Bernoulli numbers.

Example 1. The constant A_0 is the Stirling constant:

$$A_0 = \exp(-\zeta'(0)) = \sqrt{2\pi}\,,$$

the constant A_1 is the classical Glaisher-Kinkelin constant [9, 12]:

$$A_1 = \exp\left\{\frac{1}{12} - \zeta'(-1)\right\} = \exp\left\{\frac{1}{12}(\gamma + \ln 2\pi) - \frac{\zeta'(2)}{2\pi^2}\right\}\,,$$

where γ denotes the Euler constant, and the constant A_2 is

$$A_2 = \exp\left(-\zeta'(-2)\right) = \exp\left\{\frac{\zeta(3)}{4\pi^2}\right\}\,.$$

More generally, from the following identities:

$$\zeta'(-2k) = (-1)^k \frac{(2k)!}{2(2\pi)^{2k}}\zeta(2k+1) \qquad (k \geq 1)\,,$$

and

$$\zeta'(1-2k) = (-1)^{k+1}\frac{(2k)!}{k(2\pi)^{2k}}\zeta'(2k) + \frac{B_{2k}}{2k}\left(H_{2k-1} - \gamma - \ln 2\pi\right) \qquad (k \geq 1)\,,$$

which are easily derived by differentiation of the functional equation for the zeta function, we can deduce, using Adamchik's formula (1), the expressions

$$\ln A_{2k-1} = (-1)^k \frac{(2k)!}{k(2\pi)^{2k}}\zeta'(2k) + \frac{B_{2k}}{2k}\left(\gamma + \ln 2\pi\right) \qquad (k \geq 1)\,, \tag{2}$$

and

$$\ln A_{2k} = (-1)^{k+1}\frac{(2k)!}{2(2\pi)^{2k}}\zeta(2k+1) \qquad (k \geq 1)\,. \tag{3}$$

Remark 1. Bendersky [2] introduced for the first time the sequence of numbers A_k without any consideration of their relation with the ζ-function. From the point of view of the summation of divergent series, the constants $\ln A_k$ can be interpreted as follows: if $\sum_{n\geq 1}^{\mathcal{R}} n^k \ln n$ denotes the $\mathcal{R}$-sum of the divergent series $\sum_{n\geq 1} n^k \ln n$ (i.e. the sum of the series in the sense of Ramanujan's summation method as outlined in [4]), then, for any integer $k \geq 0$, we have the equivalent expressions [4, p. 68], [2, Eq. (V'$_k$) p. 280]:

$$\begin{aligned}
\sum_{n\geq 1}^{\mathcal{R}} n^k \ln n &= -\zeta'(-k) - \frac{1}{(k+1)^2} \\
&= \ln A_k - \frac{H_k B_{k+1}}{k+1} - \frac{1}{(k+1)^2} \\
&= \int_0^1 \ln \Gamma_k(x+1)\,dx\,,
\end{aligned}$$

where Γ_k is the Bendersky generalized gamma function [2, p. 279]. This function verifies

$$\Gamma_k(n+1) = 1^{1^k} 2^{2^k} \cdots n^{n^k} \qquad \text{for any integer } n \geq 1\,.$$

In particular, we have $\Gamma = \Gamma_0$, and $K = \Gamma_1$, where K is the Kinkelin-Bendersky hyperfactorial function which can be defined as follows [2, Eq. (28') p. 302], [11, Def. 3]:

$$\ln K(x) = \int_0^x \ln \Gamma(u)\, du + \frac{x^2 - x}{2} - \frac{x}{2} \ln 2\pi \qquad (x \geq 0)\,.$$

Unaware of Bendersky's work and following an idea of Milnor, Kurokawa and Ochiai [8, Thm. 2] provided an expression of the function Γ_k in terms of the derivative of the Hurwitz zeta function $\zeta(s,x)$ at $s = -k$. Precisely, they established the formula

$$\ln \Gamma_k(x) = \zeta'(-k,x) - \zeta'(-k) \qquad (x > 0,\ k \geq 0)\,,$$

which generalizes a classical formula for Γ [6, Def. 9.6.13], and can be seen as the analogue of Adamchik's formula for A_k.

Example 2. For $k = 0$ and $k = 1$, we have

$$\sum_{n\geq 1}^{\mathcal{R}} \ln n = \int_0^1 \ln \Gamma(x+1)\, dx = \frac{1}{2} \ln 2\pi - 1\,,$$

$$\sum_{n\geq 1}^{\mathcal{R}} n \ln n = \int_0^1 \ln K(x+1)\, dx = \ln A_1 - \frac{1}{3}\,.$$

3 Blagouchine's integrals and series with zeta values

Definition 2. For any non-negative integer k and positive odd integer p, the integral $\mathcal{J}_{k,p}$ is defined by

$$\mathcal{J}_{k,p} = \int_{-\infty}^{+\infty} \frac{\zeta(\frac{p}{2} + ix)}{(2k + p + 2ix)\cosh(\pi x)}\, dx\,.$$

Proposition 1. We have the following relations:

$$\mathcal{J}_{k,1} = \frac{\gamma}{k+1} - \frac{1}{(k+1)^2} - \mathcal{S}_{k,1}\,, \tag{4}$$

and

$$\mathcal{J}_{k,p} = (-1)^{\frac{p+1}{2}} \mathcal{S}_{k,p} \qquad (p = 3, 5, 7, \cdots) \tag{5}$$

with

$$\mathcal{S}_{k,p} := \sum_{n=N_p}^{\infty} (-1)^n \frac{\zeta(n)}{n+k} \quad \text{with } N_p = \max(2, \frac{p+1}{2})\,.$$

Proof. For $k \geq 0$, let us consider the function

$$f_k(z) = \frac{\zeta(z)}{(k+z)\sin(\pi z)}\,.$$

The function f_k has poles at integers $n \in \mathbb{Z}$. For $n \geq 2$, the residue of f_k at $z = n$ is

$$\mathrm{Res}(f_k; n) = \frac{(-1)^n \zeta(n)}{(n+k)\pi}\,.$$

For $n = 1$, f_k has a double pole and

$$\mathrm{Res}(f_k; 1) = -\frac{1}{\pi}\left(\frac{\gamma}{k+1} - \frac{1}{(k+1)^2}\right).$$

Applying the residue theorem, we get

$$-\frac{1}{2i\pi}\int_{\mathrm{Re}(z)=p/2} f_k(z)\,dz = \sum_{n>\frac{p}{2}} \mathrm{Res}(f_k; n)\,.$$

This leads to formulas (4) and (5). □

Proposition 2. We have

$$\mathcal{S}_{k,1} = \frac{\gamma}{k+1} + \frac{1}{k} - \sum_{j=0}^{k-1}(-1)^j \binom{k}{j} \ln A_j \qquad (k \geq 1)\,. \tag{6}$$

Proof. It results from [7, Prop. 1] that

$$\sum_{n=2}^{\infty}(-1)^n \frac{\zeta(n)}{n+k} = \frac{\gamma}{k+1} + \sum_{j=0}^{k-1}(-1)^j \binom{k}{j}\zeta'(-j) + \frac{1}{k} + \sum_{j=0}^{k-1}\binom{k}{j}\frac{B_{j+1}\,H_j}{j+1} \qquad (k \geq 1)\,.$$

By Adamchik's formula (1), this later expression is equivalent to (6). □

Corollary 1. We have

$$\mathcal{J}_{k,1} = \sum_{j=0}^{k-1}(-1)^j \binom{k}{j} \ln A_j - \frac{1}{k} - \frac{1}{(k+1)^2} \qquad (k \geq 1)\,, \tag{7}$$

and

$$\mathcal{J}_{k,3} = \frac{\gamma}{k+1} - \frac{1}{(k+1)^2} - \mathcal{J}_{k,1} = \frac{\gamma}{k+1} + \frac{1}{k} - \sum_{j=0}^{k-1}(-1)^j \binom{k}{j} \ln A_j \qquad (k \geq 1)\,. \tag{8}$$

Example 3. For $p = 1$ and the first values of k, we have

$$\begin{aligned}
\mathcal{J}_{0,1} &= -1\,,\\
\mathcal{J}_{1,1} &= \frac{1}{2}\ln 2\pi - \frac{5}{4}\,,\\
\mathcal{J}_{2,1} &= \frac{1}{3}\ln 2\pi - \frac{1}{6}\gamma - \frac{11}{18} + \frac{\zeta'(2)}{\pi^2}\,,\\
\mathcal{J}_{3,1} &= \frac{1}{4}\ln 2\pi - \frac{1}{4}\gamma - \frac{19}{48} + \frac{3}{2\pi^2}\zeta'(2) + \frac{3}{4\pi^2}\zeta(3)\,.
\end{aligned}$$

More generally,

$$\mathcal{J}_{k,1} = \frac{\ln 2\pi}{k+1} - \frac{(k-1)\,\gamma}{2(k+1)} - \frac{k^2+3k+1}{k(k+1)^2}$$
$$-\sum_{j=1}^{[\frac{k}{2}]}(-1)^j\binom{k}{2j-1}\frac{(2j)!}{j(2\pi)^{2j}}\zeta'(2j) - \sum_{j=1}^{[\frac{k-1}{2}]}(-1)^j\binom{k}{2j}\frac{(2j)!}{2(2\pi)^{2j}}\zeta(2j+1) \qquad (k\geq 3)\,.$$

4 Further generalization

Using a Fourier transform method, Candelpergher [5, Eq. (7)] has recently established the following beautiful relation which is a rather natural generalization of (4).

Proposition 3. for $k \geq 0$ and $\mathrm{Re}(s) > \frac{1}{2}$, we have

$$2^{s-1}\mathcal{J}_{k,1}(s) = \frac{\gamma}{(k+1)^s} - \frac{s}{(k+1)^{s+1}} - \mathcal{S}_{k,1}(s)\,, \tag{9}$$

with

$$\mathcal{J}_{k,1}(s) := \int_{-\infty}^{+\infty}\frac{\zeta(\frac{1}{2}+ix)}{(2k+1+2ix)^s\cosh(\pi x)}\,dx\,,$$

and

$$\mathcal{S}_{k,1}(s) := \sum_{n=2}^{\infty}(-1)^n\frac{\zeta(n)}{(n+k)^s}\,.$$

Applying (9) with $k = 0$ allows us to deduce the following identity:

Corollary 2.

$$\sum_{n=2}^{\infty}(-1)^n\frac{\zeta(n)}{n^s} = \gamma - s - \frac{1}{2}\int_{-\infty}^{+\infty}\frac{\zeta(\frac{1}{2}+ix)}{(\frac{1}{2}+ix)^s\cosh(\pi x)}\,dx \qquad (\mathrm{Re}(s) > \frac{1}{2})\,. \tag{10}$$

Example 4. For $s = 1$, the representation

$$\sum_{n=2}^{\infty}(-1)^n \frac{\zeta(n)}{n} = \gamma$$

is regained (since $\mathcal{J}_{0,1}(1) = -1$), and for $s = 2$, formula (10) translates into the relation

$$\sum_{n=2}^{\infty}(-1)^n \frac{\zeta(n)}{n^2} = \gamma - 2 - \frac{1}{2}\int_{-\infty}^{+\infty} \frac{\zeta(\frac{1}{2}+ix)}{(\frac{1}{2}+ix)^2 \cosh(\pi x)}\, dx\,. \tag{11}$$

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