Generalized Glaisher-Kinkelin constants and Blagouchine's integrals

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On generalized Glaisher-Kinkelin's constants and Blagouchine's integrals

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Abstract The main purpose of this article is to establish a close connection between a sequence of complex integrals introduced by Blagouchine and some important mathematical constants, namely the generalized Glaisher-Kinkelin constants (also known as the Bendersky constants) which occur quite naturally in analysis and number theory.

Keywords Generalized Glaisher-Kinkelin constants, Bendersky generalized gamma function, infinite series with zeta values, complex integration.

1 Introduction

The main purpose of this article is to highlight the link between the sequence of complex integrals $\{\mathcal{J}_{k,p}\}$ (for integers $k \geq 0$ and $p \geq 1$ with p odd) defined by

$$\mathcal{J}_{k,p} = \int_{-\infty}^{+\infty} \frac{\zeta(\frac{p}{2} + ix)}{(2k + p + 2ix)\cosh(\pi x)}\, dx\,,$$

and some important mathematical constants, namely the generalized Glaisher-Kinkelin constants (also known as the Bendersky constants) which occur quite naturally in analysis and number theory [1, 9, 11]. Blagouchine [3] introduced these integrals in the cases $p = 1$ and $p = 3$. To establish this close connection, we make use of a relation between the integral $\mathcal{J}_{k,p}$ and the alternating series

$$\sum_{n=N_p}^{\infty} (-1)^n \frac{\zeta(n)}{n+k} \quad \text{with } N_p = \max(2, \frac{p+1}{2})$$

that we deduce from the residue theorem (see Proposition 1). Previously, these series were studied in detail in [7]. In particular, this enables us to give a general

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expression of the integrals $\mathcal{J}_{k,1}$ and $\mathcal{J}_{k,3}$ in terms of the generalized Glaisher-Kinkelin constants for all positive integers k (see Corollary 1).

Recently, the relation between $\mathcal{J}_{k,1}$ and $\mathcal{J}_{k,3}$ has been generalized by Candelpergher [5] (see Proposition 2). This allows us to give, as a corollary, a representation of the Dirichlet series

$$\sum_{n=2}^{\infty}(-1)^n\frac{\zeta(n)}{n^s}$$

which is valid for any complex number s with $\mathrm{Re}(s) > \frac{1}{2}$ (see Corollary 2).

2 Generalized Glaisher-Kinkelin constants

Definition 1 ([9, 11])**.** For any integer $k \geq 0$, the constant A_k are usually defined by

$$\begin{aligned}
\ln A_0 &= \lim_{N\to\infty}\left\{\sum_{n=1}^{N}\ln n - \left(N+\frac{1}{2}\right)\ln N + N\right\},\\
\ln A_1 &= \lim_{N\to\infty}\left\{\sum_{n=1}^{N}n\ln n - \left(\frac{N^2}{2}+\frac{N}{2}+\frac{1}{12}\right)\ln N + \frac{N^2}{4}\right\},\\
\ln A_2 &= \lim_{N\to\infty}\left\{\sum_{n=1}^{N}n^2\ln n - \left(\frac{N^3}{3}+\frac{N^2}{2}+\frac{N}{6}\right)\ln N + \frac{N^3}{9}-\frac{N}{12}\right\},
\end{aligned}$$

and more generally

$$\ln A_k = \lim_{N\to\infty}\left\{\sum_{n=1}^{N}n^k\ln n - P_k(N)\ln N + Q_k(N)\right\},$$

where P_k and Q_k are polynomials of degree $k+1$ that can be explicitely computed (see e.g. [11, Eq. (1.1)]). The numbers A_k for $k = 0, 1, 2, \ldots$ are called the *generalized Glaisher-Kinkelin constants* (sometimes called the *Bendersky constants*). Adamchik [1, Proposition 4] has given an alternative expression of the constants A_k in terms of the derivatives of the Riemann zeta function. More precisely, we have the expression

$$\ln A_k = \frac{H_k B_{k+1}}{k+1} - \zeta'(-k)\,, \tag{1}$$

where $H_k = \sum_{j=1}^{k}\frac{1}{j}$ is the k-th harmonic number with the usual convention $H_0 = 0$.

Example 1. The constant A_0 is the Stirling constant:

$$A_0 = \exp(-\zeta'(0)) = \sqrt{2\pi}\,,$$

the constant A_1 is the classical Glaisher-Kinkelin constant:

$$A_1 = \exp\left\{\frac{1}{12} - \zeta'(-1)\right\} = \exp\left\{\frac{1}{12}(\gamma + \ln 2\pi) - \frac{\zeta'(2)}{2\pi^2}\right\}\,,$$

and, for $k = 2$, we have

$$A_2 = \exp\left(-\zeta'(-2)\right) = \exp\left\{\frac{\zeta(3)}{4\pi^2}\right\}\,.$$

More generally, the following relations :

$$\zeta'(-2k) = (-1)^k \frac{(2k)!}{2(2\pi)^{2k}}\zeta(2k+1) \qquad (k \geq 1)\,,$$

and

$$\zeta'(1-2k) = (-1)^{k+1}\frac{(2k)!}{k(2\pi)^{2k}}\zeta'(2k) + \frac{B_{2k}}{2k}\left(H_{2k-1} - \gamma - \ln 2\pi\right) \qquad (k \geq 1)$$

which are easily derived by differentiation of the Riemann functional equation for the zeta function enable to deduce from Adamchik's formula (1) the expressions

$$\ln A_{2k-1} = (-1)^k \frac{(2k)!}{k(2\pi)^{2k}}\zeta'(2k) + \frac{B_{2k}}{2k}\left(\gamma + \ln 2\pi\right) \qquad (k \geq 1)\,, \tag{2}$$

and

$$\ln A_{2k} = (-1)^{k+1}\frac{(2k)!}{2(2\pi)^{2k}}\zeta(2k+1) \qquad (k \geq 1)\,. \tag{3}$$

Remark 1. Bendersky [2] introduced for the first time the sequence of numbers A_k without any consideration of their relation with the ζ-function. From the point of view of the summation of divergent series, the constants A_k can be interpreted as follows: let $\sum_{n\geq 1}^{\mathcal{R}} n^k \ln n$ denotes the $\mathcal{R}$-sum of the divergent series $\sum_{n\geq 1} n^k \ln n$ (i.e. the sum of the series in the sense of Ramanujan's summation method [4]), then, for any integer $k \geq 0$, we have (see [4, p. 68] and [2, p. 280]):

$$\begin{aligned}\sum_{n\geq 1}^{\mathcal{R}} n^k \ln n &= -\zeta'(-k) - \frac{1}{(k+1)^2} \\ &= \ln A_k - \frac{H_k B_{k+1}}{k+1} - \frac{1}{(k+1)^2} \\ &= \int_0^1 \ln \Gamma_k(x+1)\,dx\,,\end{aligned}$$

where Γ_k is the Bendersky generalized gamma function [2, p. 279]. This function verifies

$$\Gamma_k(n+1) = 1^{1^k} 2^{2^k} \cdots n^{n^k} \qquad \text{for any integer } n \geq 1\,.$$

In particular, we have $\Gamma_0 = \Gamma$, and $\Gamma_1 = K$, where K denotes the Kinkelin-Bendersky hyperfactorial function which can be defined (see e.g. [10, Definition 3]) by the relation

$$\ln K(x) = \frac{x^2 - x}{2} - \frac{x}{2}\ln 2\pi + \int_0^x \ln\Gamma(u)\,du \qquad (x \geq 0)\,.$$

Unaware of Bendersky's work and following an idea of Milnor, Kurokawa and Ochiai [8, Theorem 2] have given an expression of the function Γ_k in terms of the derivative of the Hurwitz zeta function $\zeta(s,x)$ at $s = -k$. Precisely, they show that

$$\ln\Gamma_k(x) = \zeta'(-k,x) - \zeta'(-k) \qquad \text{for } x > 0 \text{ and } k \geq 0\,,$$

a formula that generalizes a classical formula for Γ (see [6, Definition 9.6.13]) and can be seen as the analogue of Adamchik's formula for A_k.

3 Blagouchine's integrals and series with zeta values

Definition 2. For any non-negative integer k and positive odd integer p, the integral $\mathcal{J}_{k,p}$ is defined by

$$\mathcal{J}_{k,p} = \int_{-\infty}^{+\infty} \frac{\zeta(\frac{p}{2} + ix)}{(2k + p + 2ix)\cosh(\pi x)}\,dx\,.$$

Proposition 1. We have the following relations:

$$\mathcal{J}_{k,1} = \frac{\gamma}{k+1} - \frac{1}{(k+1)^2} - \sum_{n=2}^{\infty}(-1)^n \frac{\zeta(n)}{n+k}\,, \tag{4}$$

and

$$\mathcal{J}_{k,p} = (-1)^{\frac{p+1}{2}} \sum_{n=\frac{p+1}{2}}^{\infty} (-1)^n \frac{\zeta(n)}{n+k} \quad \text{for } p = 3, 5, 7, \cdots \tag{5}$$

Proof. For $k \geq 0$, let us consider the function

$$f_k(z) = \frac{\zeta(z)}{(k+z)\sin(\pi z)}\,.$$

The function f_k has poles at integers $n \in \mathbb{Z}$. For $n \geq 2$, the residue of f_k at $z = n$ is

$$\operatorname{Res}(f_k; n) = \frac{(-1)^n \zeta(n)}{(n+k)\pi}\,.$$

For $n = 1$, f_k has a double pole and

$$\operatorname{Res}(f_k; 1) = -\frac{1}{\pi}\left(\frac{\gamma}{k+1} - \frac{1}{(k+1)^2}\right).$$

Applying the residue theorem, we get

$$-\frac{1}{2i\pi}\int_{\operatorname{Re}(z)=p/2} f_k(z)\,dz = \sum_{n>\frac{p}{2}} \operatorname{Res}(f_k; n)\,.$$

This leads to formulas (4) and (5). □

Corollary 1. We have

$$\mathcal{J}_{k,1} = \sum_{j=0}^{k-1}(-1)^j\binom{k}{j}\ln A_j - \frac{1}{k} - \frac{1}{(k+1)^2} \qquad (k \geq 1)\,, \tag{6}$$

and

$$\mathcal{J}_{k,3} = \frac{\gamma}{k+1} - \frac{1}{(k+1)^2} - \mathcal{J}_{k,1} = \frac{\gamma}{k+1} + \frac{1}{k} - \sum_{j=0}^{k-1}(-1)^j\binom{k}{j}\ln A_j \qquad (k \geq 1)\,. \tag{7}$$

Proof. From [7, Proposition 1], we have

$$\sum_{n=2}^{\infty}(-1)^n\frac{\zeta(n)}{n+k} = \frac{\gamma}{k+1} + \sum_{j=0}^{k-1}(-1)^j\binom{k}{j}\zeta'(-j) + \frac{1}{k} + \sum_{j=0}^{k-1}\binom{k}{j}\frac{B_{j+1}\,H_j}{j+1} \qquad (k \geq 1)\,.$$

By Adamchik's formula (1), this expression may be rewritten as follows:

$$\sum_{n=2}^{\infty}(-1)^n\frac{\zeta(n)}{n+k} = \frac{\gamma}{k+1} + \frac{1}{k} - \sum_{j=0}^{k-1}(-1)^j\binom{k}{j}\ln A_j \qquad (k \geq 1)\,.$$

Then, using the relation (4) and (5), we get the expressions (6) and (7). □

Example 2. We have

$$\begin{aligned}
\mathcal{J}_{0,1} &= -1\,,\\
\mathcal{J}_{1,1} &= \frac{1}{2}\ln 2\pi - \frac{5}{4}\,,\\
\mathcal{J}_{2,1} &= \frac{1}{3}\ln 2\pi - \frac{1}{6}\gamma - \frac{11}{18} + \frac{\zeta'(2)}{\pi^2}\,,\\
\mathcal{J}_{3,1} &= \frac{1}{4}\ln 2\pi - \frac{1}{4}\gamma - \frac{19}{48} + \frac{3}{4\pi^2}\left(\zeta(3) + 2\zeta'(2)\right)\,.
\end{aligned}$$

4 Further generalization

Using a Fourier transform method, Candelpergher [5, Eq. (7)] recently proved the following beautiful relation which is a natural generalization of (4).

Proposition 2. for $k \geq 0$ and $\mathrm{Re}(s) > \frac{1}{2}$, we have

$$2^{s-1}\mathcal{J}_{k,1}(s) = \frac{\gamma}{(k+1)^s} - \frac{s}{(k+1)^{s+1}} - \sum_{n=2}^{\infty}(-1)^n \frac{\zeta(n)}{(n+k)^s}\,. \tag{8}$$

with

$$\mathcal{J}_{k,1}(s) := \int_{-\infty}^{+\infty} \frac{\zeta(\frac{1}{2}+ix)}{(2k+1+2ix)^s \cosh(\pi x)}\, dx$$

Applying (8) with $k = 0$ allows us to deduce the following identity:

Corollary 2.

$$\sum_{n=2}^{\infty}(-1)^n \frac{\zeta(n)}{n^s} = \gamma - s - \frac{1}{2}\int_{-\infty}^{+\infty} \frac{\zeta(\frac{1}{2}+ix)}{(\frac{1}{2}+ix)^s \cosh(\pi x)}\, dx \qquad (\mathrm{Re}(s) > \frac{1}{2})\,. \tag{9}$$

Example 3. For $s = 1$, the representation $\sum_{n=2}^{\infty}(-1)^n \frac{\zeta(n)}{n} = \gamma$ is regained (since $\mathcal{J}_{0,1}(1) = -1$), and for $s = 2$, formula (9) gives the relation

$$\sum_{n=2}^{\infty}(-1)^n \frac{\zeta(n)}{n^2} = \gamma - 2 - \frac{1}{2}\int_{-\infty}^{+\infty} \frac{\zeta(\frac{1}{2}+ix)}{(\frac{1}{2}+ix)^2 \cosh(\pi x)}\, dx\,. \tag{10}$$

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