On the generalized Glaisher-Kinkelin constants and Blagouchine's integrals

Marc-Antoine Coppo

On the generalized Glaisher-Kinkelin constants and Blagouchine's integrals

Marc-Antoine Coppo*

Université Côte d'Azur, CNRS, LJAD (UMR 7351), Nice, France

Abstract The main purpose of this article is to establish a close connection between a sequence of complex integrals introduced by Blagouchine and some important mathematical constants, namely the generalized Glaisher-Kinkelin constants (also known as the Bendersky constants) which occur quite naturally in analysis and number theory.

Keywords Generalized Glaisher-Kinkelin constants, infinite series with zeta values, complex integration.

1 Introduction

The main purpose of this article is to highlight the link between the sequence of complex integrals $\{\mathcal{J}_{k,p}\}$ (for integers $k \geq 0$ and $p \geq 1$ with p odd) defined by

$$\mathcal{J}_{k,p} = \int_{-\infty}^{+\infty} \frac{\zeta(\frac{p}{2} + ix)}{(2k + p + 2ix)\cosh(\pi x)}\, dx\,,$$

and some important mathematical constants, namely the generalized Glaisher-Kinkelin constants (also known as the Bendersky constants) which occur quite naturally in analysis and number theory [1, 8, 10]. Blagouchine [3] introduced these integrals in the cases $p = 1$ and $p = 3$. To establish this close connection, we make use of a relation between the integral $\mathcal{J}_{k,p}$ and the alternating series

$$\sum_{n=N_p}^{\infty} (-1)^n \frac{\zeta(n)}{n+k} \quad \text{with } N_p = \max(2, \frac{p+1}{2})$$

that we deduce from the residue theorem (see Proposition 1). Previously, these series were studied in detail in [6]. In particular, this enables us to give a general

*Corresponding author. *Email address:* `coppo@unice.fr`

expression of the integrals $\mathcal{J}_{k,1}$ and $\mathcal{J}_{k,3}$ for all positive integers k (see Theorem 1 and Corollary 1).

Recently, the relation between $\mathcal{J}_{k,1}$ and $\mathcal{J}_{k,3}$ has been generalized by Candelpergher [5] (see Theorem 2). This allows us to give, as a corollary, an expression of the Dirichlet series

$$\sum_{n=2}^{\infty}(-1)^n\frac{\zeta(n)}{n^s}$$

which is valid for any complex number s with $\mathrm{Re}(s) > \frac{1}{2}$ (see Corollary 2).

2 Generalized Glaisher-Kinkelin constants

Definition 1 ([8, 10])**.** For any integer $k \geq 0$, the constant A_k are usually defined by

$$\begin{aligned}
\ln A_0 &= \lim_{N\to\infty}\left\{\sum_{n=1}^{N}\ln n - \left(N+\frac{1}{2}\right)\ln N + N\right\},\\
\ln A_1 &= \lim_{N\to\infty}\left\{\sum_{n=1}^{N}n\ln n - \left(\frac{N^2}{2}+\frac{N}{2}+\frac{1}{12}\right)\ln N + \frac{N^2}{4}\right\},\\
\ln A_2 &= \lim_{N\to\infty}\left\{\sum_{n=1}^{N}n^2\ln n - \left(\frac{N^3}{3}+\frac{N^2}{2}+\frac{N}{6}\right)\ln N + \frac{N^3}{9} - \frac{N}{12}\right\},
\end{aligned}$$

and more generally

$$\ln A_k = \lim_{N\to\infty}\left\{\sum_{n=1}^{N}n^k\ln n - P_k(N)\ln N + Q_k(N)\right\},$$

where P_k and Q_k are polynomials of degree $k+1$ that can be explicitely computed (see e.g. [10, Eq. (1.1)]). The numbers A_k for $k = 0, 1, 2, \ldots$ are called the *generalized Glaisher-Kinkelin constants* (sometimes called the *Bendersky constants*). Adamchik [1, Proposition 4] has given an alternative expression of the constants A_k in terms of the derivatives of the Riemann zeta function. More precisely, we have

$$A_k = \exp\left\{\frac{H_k B_{k+1}}{k+1} - \zeta'(-k)\right\}, \tag{1}$$

where $H_k = \sum_{j=1}^{k}\frac{1}{j}$ is the k-th harmonic number with the usual convention $H_0 = 0$.

Example 1. The constant $A_0 = \exp(-\zeta'(0)) = \sqrt{2\pi}$ is the Stirling constant,

$$A_1 = \exp\left(\frac{1}{12} - \zeta'(-1)\right)$$

is the classical Glaisher-Kinkelin constant, and for $k=2$, we have

$$A_2 = \exp\left(-\zeta'(-2)\right) = \exp\left(\frac{\zeta(3)}{4\pi^2}\right).$$

The following relations are easily derived by differentiation of the Riemann functional equation for the zeta function:

$$\zeta'(-2k) = (-1)^k \frac{(2k)!}{2(2\pi)^{2k}}\zeta(2k+1) \qquad (k \geq 1),$$

and

$$\zeta'(1-2k) = (-1)^{k+1} \frac{(2k)!}{k(2\pi)^{2k}}\zeta'(2k) + \frac{B_{2k}}{2k}\left(H_{2k-1} - \gamma - \ln 2\pi\right) \qquad (k \geq 1).$$

This enable to deduce from Adamchik's formula (1) the expressions

$$A_{2k-1} = \exp\left\{(-1)^k \frac{(2k)!}{k(2\pi)^{2k}}\zeta'(2k) + \frac{B_{2k}}{2k}\left(\gamma + \ln 2\pi\right)\right\} \qquad (k \geq 1), \tag{2}$$

and

$$A_{2k} = \exp\left\{(-1)^{k+1} \frac{(2k)!}{2(2\pi)^{2k}}\zeta(2k+1)\right\} \qquad (k \geq 1). \tag{3}$$

In particular, we can easily deduce from formulas (2) and (3) the following binomial identity which will be useful in the proof of the forthcoming theorem 1.

Lemma 1. For $k \geq 1$,

$$\begin{aligned}\sum_{j=0}^{k-1}(-1)^j\binom{k}{j}\ln A_j = \frac{\ln 2\pi}{k+1} - \frac{k-1}{k+1}\frac{\gamma}{2} \\ - \sum_{j=1}^{[\frac{k}{2}]}(-1)^j\binom{k}{2j-1}\frac{(2j)!}{j(2\pi)^{2j}}\zeta'(2j) \\ - \sum_{j=1}^{[\frac{k-1}{2}]}(-1)^j\binom{k}{2j}\frac{(2j)!}{2(2\pi)^{2j}}\zeta(2j+1).\end{aligned} \tag{4}$$

Remark 1. Bendersky [2] introduced for the first time the sequence of numbers A_k without any consideration of their relation with the ζ-function. From the point of view of the summation of divergent series, the constants A_k can be interpreted as follows: let $\sum_{n\geq 1}^{\mathcal{R}} n^k \ln n$ denotes the $\mathcal{R}$-sum of the divergent series $\sum_{n\geq 1} n^k \ln n$ (i.e. the sum of the series in the sense of Ramanujan's summation method [4]), then, for any integer $k \geq 0$, we have (see [4, p. 68] and [2, p. 280]):

$$\begin{aligned}\sum_{n\geq 1}^{\mathcal{R}} n^k \ln n &= -\zeta'(-k) - \frac{1}{(k+1)^2} \\ &= \ln A_k - \frac{H_k B_{k+1}}{k+1} - \frac{1}{(k+1)^2} \\ &= \int_0^1 \ln \Gamma_k(x+1)\, dx\,,\end{aligned}$$

where Γ_k is the Bendersky generalized gamma function [2, p. 279]. This function verifies in particular

$$\Gamma_k(n+1) = 1^{1^k} 2^{2^k} \cdots n^{n^k} \qquad \text{for any integer } n \geq 1\,.$$

Example 2. For the first values of k, we have

$$\begin{aligned}\sum_{n\geq 1}^{\mathcal{R}} \ln n &= \frac{1}{2}\ln 2\pi - 1 = \int_0^1 \ln \Gamma(x+1)\, dx\,, \\ \sum_{n\geq 1}^{\mathcal{R}} n \ln n &= \ln A_1 - \frac{1}{3} = \int_0^1 \ln K(x+1)\, dx\,,\end{aligned}$$

where $K = \Gamma_1$ is the Kinkelin-Bendersky hyperfactorial function which can be defined by the relation (see [2, Eq. p. 302] and [9, Definition 3])

$$\ln K(x) = \frac{x^2 - x}{2} - \frac{x}{2}\ln 2\pi + \int_0^x \ln \Gamma(u)\, du \qquad (x \geq 0)\,.$$

Remark 2. Unaware of Bendersky's work and following an idea of Milnor, Kurokawa and Ochiai [7, Theorem 2] have given an expression of the function Γ_k in terms of the derivative of the Hurwitz zeta function $\zeta(s,x)$ at $s = -k$. Precisely, they showed that

$$\Gamma_k(x) = \exp\{\zeta'(-k,x) - \zeta'(-k)\} \qquad \text{for } x > 0\,,$$

a formula that can be seen as the analogue of Adamchik's formula for A_k.

3 Blagouchine's integrals and series with zeta values

Definition 2. For each non-negative integer k and each positive odd integer p, the integral $\mathcal{J}_{k,p}$ is defined by

$$\mathcal{J}_{k,p} = \int_{-\infty}^{+\infty} \frac{\zeta(\frac{p}{2} + ix)}{(2k + p + 2ix)\cosh(\pi x)}\, dx\,.$$

Proposition 1. We have the following relations:

$$\mathcal{J}_{k,1} = \frac{\gamma}{k+1} - \frac{1}{(k+1)^2} - \sum_{n=2}^{\infty} (-1)^n \frac{\zeta(n)}{n+k}\,, \tag{5}$$

and

$$\mathcal{J}_{k,p} = (-1)^{\frac{p+1}{2}} \sum_{n=\frac{p+1}{2}}^{\infty} (-1)^n \frac{\zeta(n)}{n+k} \quad \text{for } p = 3, 5, 7, \cdots \tag{6}$$

Proof. For $k \geq 0$, let us consider the function

$$f_k(z) = \frac{\zeta(z)}{(k+z)\sin(\pi z)}\,.$$

The function f_k has poles at integers $n \in \mathbb{Z}$. For $n \geq 2$, the residue of f_k at $z = n$ is

$$\mathrm{Res}(f_k; n) = \frac{(-1)^n \zeta(n)}{(n+k)\pi}\,.$$

For $n = 1$, f_k has a double pole and

$$\mathrm{Res}(f_k; 1) = -\frac{1}{\pi}\left(\frac{\gamma}{k+1} - \frac{1}{(k+1)^2}\right).$$

Applying the residue theorem, we get

$$-\frac{1}{2i\pi}\int_{\mathrm{Re}(z)=p/2} f_k(z)\,dz = \sum_{n>\frac{p}{2}} \mathrm{Res}(f_k; n)\,.$$

This leads to formulas (5) and (6). □

Theorem 1. We have

$$\mathcal{J}_{1,1} = \frac{1}{2}\ln 2\pi - \frac{5}{4}\,,$$
$$\mathcal{J}_{2,1} = \frac{1}{3}\ln 2\pi - \frac{1}{6}\gamma - \frac{11}{18} + \frac{\zeta'(2)}{\pi^2}\,,$$

and for $k \geq 3$,

$$\begin{aligned}\mathcal{J}_{k,1} = {} & \frac{1}{k+1}\ \ln 2\pi - \frac{k-1}{k+1}\,\frac{\gamma}{2} - \frac{k^2+3k+1}{k(k+1)^2} \\ & - \sum_{j=1}^{[\frac{k}{2}]} (-1)^j \binom{k}{2j-1} \frac{(2j)!}{j(2\pi)^{2j}}\zeta'(2j) - \sum_{j=1}^{[\frac{k-1}{2}]} (-1)^j \binom{k}{2j} \frac{(2j)!}{2(2\pi)^{2j}}\zeta(2j+1)\,. \end{aligned} \tag{7}$$

Proof. From [6, Proposition 1], we have

$$\sum_{n=2}^{\infty}(-1)^n\frac{\zeta(n)}{n+k}=\frac{\gamma}{k+1}+\sum_{j=0}^{k-1}(-1)^j\binom{k}{j}\zeta'(-j)+\frac{1}{k}+\sum_{j=0}^{k-1}\binom{k}{j}\frac{B_{j+1}\,H_j}{j+1}\qquad(k\geq 1)\,.$$

By Adamchik's formula (1), this expression may be rewritten as follows:

$$\sum_{n=2}^{\infty}(-1)^n\frac{\zeta(n)}{n+k}=\frac{\gamma}{k+1}+\frac{1}{k}-\sum_{j=0}^{k-1}(-1)^j\binom{k}{j}\ln A_j\qquad(k\geq 1)\,.$$

Then, using the relation (5), we get the following expression:

$$\mathcal{J}_{k,1}=\sum_{j=0}^{k-1}(-1)^j\binom{k}{j}\ln A_j-\frac{1}{k}-\frac{1}{(k+1)^2}\qquad(k\geq 1)\,. \tag{8}$$

Hence, formula (7) results from formula (8) and Lemma 1. □

Remark 3. Since $\sum_{n=2}^{\infty}(-1)^n\frac{\zeta(n)}{n}=\gamma$ (by a well-known series representation of Euler's constant γ), we also have $\mathcal{J}_{0,1}=-1$ by (5).

The following relation

$$\mathcal{J}_{k,3}=\frac{\gamma}{k+1}-\frac{1}{(k+1)^2}-\mathcal{J}_{k,1} \tag{9}$$

which results from (5) and (6) allows us to deduce the following corollary:

Corollary 1. We have

$$\begin{aligned}\mathcal{J}_{1,3}&=\frac{1}{2}\gamma-\frac{1}{2}\ln 2\pi+1\,,\\ \mathcal{J}_{2,3}&=\frac{1}{2}\gamma-\frac{1}{3}\ln 2\pi+\frac{1}{2}-\frac{\zeta'(2)}{\pi^2}\,,\end{aligned}$$

and for $k\geq 3$,

$$\begin{aligned}\mathcal{J}_{k,3}&=\frac{1}{2}\gamma-\frac{1}{k+1}\ln 2\pi+\frac{1}{k}\\ &\quad+\sum_{j=1}^{[\frac{k}{2}]}(-1)^j\binom{k}{2j-1}\frac{(2j)!}{j(2\pi)^{2j}}\zeta'(2j)+\sum_{j=1}^{[\frac{k-1}{2}]}(-1)^j\binom{k}{2j}\frac{(2j)!}{2(2\pi)^{2j}}\zeta(2j+1)\,.\end{aligned} \tag{10}$$

4 Further generalization

Using a Fourier transform method, Candelpergher [5, Eq. (7)] recently proved the following beautiful relation which is a natural generalization of (5).

Theorem 2. for $k \geq 0$ and $\mathrm{Re}(s) > \frac{1}{2}$, we have

$$2^{s-1}\mathcal{J}_{k,1}(s) = \frac{\gamma}{(k+1)^s} - \frac{s}{(k+1)^{s+1}} - \sum_{n=2}^{\infty}(-1)^n \frac{\zeta(n)}{(n+k)^s}\,. \tag{11}$$

with

$$\mathcal{J}_{k,1}(s) := \int_{-\infty}^{+\infty} \frac{\zeta(\frac{1}{2}+ix)}{(2k+1+2ix)^s \cosh(\pi x)}\,dx$$

Applying (11) with $k = 0$ allows us to deduce the following identity:

Corollary 2.

$$\sum_{n=2}^{\infty}(-1)^n \frac{\zeta(n)}{n^s} = \gamma - s - \frac{1}{2}\int_{-\infty}^{+\infty} \frac{\zeta(\frac{1}{2}+ix)}{(\frac{1}{2}+ix)^s \cosh(\pi x)}\,dx \qquad (\mathrm{Re}(s) > \frac{1}{2})\,. \tag{12}$$

Example 3. For $s = 1$, the representation $\sum_{n=2}^{\infty}(-1)^n \frac{\zeta(n)}{n} = \gamma$ is regained (since $\mathcal{J}_{0,1}(1) = -1$), and for $s = 2$, formula (12) gives the relation

$$\sum_{n=2}^{\infty}(-1)^n \frac{\zeta(n)}{n^2} = \gamma - 2 - \frac{1}{2}\int_{-\infty}^{+\infty} \frac{\zeta(\frac{1}{2}+ix)}{(\frac{1}{2}+ix)^2 \cosh(\pi x)}\,dx\,.$$

References

[1] V. Adamchik, Polygamma functions of negative order, *J. Comput. Appl. Math.* **100** (1998), 191–199.

[2] L. Bendersky, Sur la function gamma généralisée, *Acta Math.* **61** (1933), 263–322.

[3] I. V. Blagouchine, A complement to a recent paper on some infinite sums with the zeta values, preprint, 2020. Available at https://arxiv.org/abs/2001.00108

[4] B. Candelpergher, *Ramanujan Summation of Divergent Series*, Lecture Notes in Math. 2185, Springer, 2017.

[5] B. Candelpergher, An expansion of the Riemann Zeta function on the critical line, preprint, 2021. Available at https://hal.archives-ouvertes.fr/hal-03271709

[6] M-A. Coppo, A note on some alternating series involving zeta and multiple zeta values, *J. Math. Anal. Appl.* **475** (2019), 1831–1841.

[7] N. Kurokawa and H. Ochiai, Generalized Kinkelin's formulas *Kodai Math. J.* **30** (2007), 195–212.

[8] M. Perkins and R. A. Van Gorder, Closed-form calculation of infinite products of Glaisher-type related to Dirichlet series, *Ramanujan J.* **49** (2019), 371–389.

[9] J. Sondow and P. Hadjicostas, The generalized-Euler-constant function $\gamma(z)$ and a generalization of Somos's quadratic recurrence constant, *J. Math. Anal. Appl.* **332** (2007), 292–314.

[10] W. Wang, Some asymptotic expansions of hyperfactorial functions and generalized Glaisher-Kinkelin constants, *Ramanujan J.* **43** (2017), 513–533.