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# On the generalized Glaisher-Kinkelin constants and Blagouchine's integrals

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**Abstract** The main purpose of this article is to establish a close connection between a sequence of complex integrals introduced by Blagouchine and some important mathematical constants, namely the generalized Glaisher-Kinkelin constants (also known as the Bendersky constants) which occur quite naturally in analysis and number theory.

**Keywords** Generalized Glaisher-Kinkelin constants, infinite series with zeta values, complex integration.

## 1 Introduction

The main purpose of this article is to highlight the link between the sequence of complex integrals  $\{\mathcal{J}_{k,p}\}$  (for integers  $k \geq 0$  and  $p \geq 1$  with  $p$  odd) defined by

$$\mathcal{J}_{k,p} = \int_{-\infty}^{+\infty} \frac{\zeta(\frac{p}{2} + ix)}{(2k + p + 2ix) \cosh(\pi x)} dx,$$

and some important mathematical constants, namely the generalized Glaisher-Kinkelin constants (also known as the Bendersky constants) which occur quite naturally in analysis and number theory [1, 8, 10]. Blagouchine [3] introduced these integrals in the cases  $p = 1$  and  $p = 3$ . To establish this close connection, we make use of a relation between the integral  $\mathcal{J}_{k,p}$  and the alternating series

$$\sum_{n=N_p}^{\infty} (-1)^n \frac{\zeta(n)}{n+k} \quad \text{with } N_p = \max(2, \frac{p+1}{2})$$

that we deduce from the residue theorem (see Proposition 1). Previously, these series were studied in detail in [6]. In particular, this enables us to give a general

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expression of the integrals  $\mathcal{J}_{k,1}$  and  $\mathcal{J}_{k,3}$  for all positive integers  $k$  (see Theorem 1 and Corollary 1).

Recently, the relation between  $\mathcal{J}_{k,1}$  and  $\mathcal{J}_{k,3}$  has been generalized by Candelpergher [5] (see Theorem 2). This allows us to give, as a corollary, an expression of the Dirichlet series

$$\sum_{n=2}^{\infty} (-1)^n \frac{\zeta(n)}{n^s}$$

which is valid for any complex number  $s$  with  $\operatorname{Re}(s) > \frac{1}{2}$  (see Corollary 2).

## 2 Generalized Glaisher-Kinkelin constants

**Definition 1** ([8, 10]). For any integer  $k \geq 0$ , the constant  $A_k$  are usually defined by

$$\begin{aligned} \ln A_0 &= \lim_{N \rightarrow \infty} \left\{ \sum_{n=1}^N \ln n - \left(N + \frac{1}{2}\right) \ln N + N \right\}, \\ \ln A_1 &= \lim_{N \rightarrow \infty} \left\{ \sum_{n=1}^N n \ln n - \left(\frac{N^2}{2} + \frac{N}{2} + \frac{1}{12}\right) \ln N + \frac{N^2}{4} \right\}, \\ \ln A_2 &= \lim_{N \rightarrow \infty} \left\{ \sum_{n=1}^N n^2 \ln n - \left(\frac{N^3}{3} + \frac{N^2}{2} + \frac{N}{6}\right) \ln N + \frac{N^3}{9} - \frac{N}{12} \right\}, \end{aligned}$$

and more generally

$$\ln A_k = \lim_{N \rightarrow \infty} \left\{ \sum_{n=1}^N n^k \ln n - P_k(N) \ln N + Q_k(N) \right\},$$

where  $P_k$  and  $Q_k$  are polynomials of degree  $k+1$  that can be explicitly computed (see e.g. [10, Eq. (1.1)]). The numbers  $A_k$  for  $k = 0, 1, 2, \dots$  are called the *generalized Glaisher-Kinkelin constants* (sometimes called the *Bendersky constants*). Adamchik [1, Proposition 4] has given an alternative expression of the constants  $A_k$  in terms of the derivatives of the Riemann zeta function. More precisely, we have

$$A_k = \exp \left\{ \frac{H_k B_{k+1}}{k+1} - \zeta'(-k) \right\}, \quad (1)$$

where  $H_k = \sum_{j=1}^k \frac{1}{j}$  is the  $k$ -th harmonic number with the usual convention  $H_0 = 0$ .

**Example 1.** The constant  $A_0 = \exp(-\zeta'(0)) = \sqrt{2\pi}$  is the Stirling constant,

$$A_1 = \exp \left( \frac{1}{12} - \zeta'(-1) \right)$$

is the classical Glaisher-Kinkelin constant, and for  $k = 2$ , we have

$$A_2 = \exp(-\zeta'(-2)) = \exp\left(\frac{\zeta(3)}{4\pi^2}\right).$$

The following relations are easily derived by differentiation of the Riemann functional equation for the zeta function:

$$\zeta'(-2k) = (-1)^k \frac{(2k)!}{2(2\pi)^{2k}} \zeta(2k+1) \quad (k \geq 1),$$

and

$$\zeta'(1-2k) = (-1)^{k+1} \frac{(2k)!}{k(2\pi)^{2k}} \zeta'(2k) + \frac{B_{2k}}{2k} (H_{2k-1} - \gamma - \ln 2\pi) \quad (k \geq 1).$$

This enable to deduce from Adamchik's formula (1) the expressions

$$A_{2k-1} = \exp\left\{(-1)^k \frac{(2k)!}{k(2\pi)^{2k}} \zeta'(2k) + \frac{B_{2k}}{2k} (\gamma + \ln 2\pi)\right\} \quad (k \geq 1), \quad (2)$$

and

$$A_{2k} = \exp\left\{(-1)^{k+1} \frac{(2k)!}{2(2\pi)^{2k}} \zeta(2k+1)\right\} \quad (k \geq 1). \quad (3)$$

In particular, we can easily deduce from formulas (2) and (3) the following binomial identity which will be useful in the proof of the forthcoming theorem 1.

**Lemma 1.** For  $k \geq 1$ ,

$$\begin{aligned} \sum_{j=0}^{k-1} (-1)^j \binom{k}{j} \ln A_j &= \frac{\ln 2\pi}{k+1} - \frac{k-1}{k+1} \frac{\gamma}{2} \\ &\quad - \sum_{j=1}^{\lfloor \frac{k}{2} \rfloor} (-1)^j \binom{k}{2j-1} \frac{(2j)!}{j(2\pi)^{2j}} \zeta'(2j) \\ &\quad - \sum_{j=1}^{\lfloor \frac{k-1}{2} \rfloor} (-1)^j \binom{k}{2j} \frac{(2j)!}{2(2\pi)^{2j}} \zeta(2j+1). \quad (4) \end{aligned}$$

*Remark 1.* Bendersky [2] introduced for the first time the sequence of numbers  $A_k$  without any consideration of their relation with the  $\zeta$ -function. From the point of view of the summation of divergent series, the constants  $A_k$  can be interpreted

as follows: let  $\sum_{n \geq 1}^{\mathcal{R}} n^k \ln n$  denotes the  $\mathcal{R}$ -sum of the divergent series  $\sum_{n \geq 1} n^k \ln n$  (i.e. the sum of the series in the sense of Ramanujan's summation method [4]), then, for any integer  $k \geq 0$ , we have (see [4, p. 68] and [2, p. 280]):

$$\begin{aligned} \sum_{n \geq 1}^{\mathcal{R}} n^k \ln n &= -\zeta'(-k) - \frac{1}{(k+1)^2} \\ &= \ln A_k - \frac{H_k B_{k+1}}{k+1} - \frac{1}{(k+1)^2} \\ &= \int_0^1 \ln \Gamma_k(x+1) dx, \end{aligned}$$

where  $\Gamma_k$  is the Bendersky generalized gamma function [2, p. 279]. This function verifies in particular

$$\Gamma_k(n+1) = 1^{1^k} 2^{2^k} \dots n^{n^k} \quad \text{for any integer } n \geq 1.$$

**Example 2.** For the first values of  $k$ , we have

$$\begin{aligned} \sum_{n \geq 1}^{\mathcal{R}} \ln n &= \frac{1}{2} \ln 2\pi - 1 = \int_0^1 \ln \Gamma(x+1) dx, \\ \sum_{n \geq 1}^{\mathcal{R}} n \ln n &= \ln A_1 - \frac{1}{3} = \int_0^1 \ln K(x+1) dx, \end{aligned}$$

where  $K = \Gamma_1$  is the Kinkelin-Bendersky hyperfactorial function which can be defined by the relation (see [2, Eq. p. 302] and [9, Definition 3])

$$\ln K(x) = \frac{x^2 - x}{2} - \frac{x}{2} \ln 2\pi + \int_0^x \ln \Gamma(u) du \quad (x \geq 0).$$

*Remark 2.* Unaware of Bendersky's work and following an idea of Milnor, Kurokawa and Ochiai [7, Theorem 2] have given an expression of the function  $\Gamma_k$  in terms of the derivative of the Hurwitz zeta function  $\zeta(s, x)$  at  $s = -k$ . Precisely, they showed that

$$\Gamma_k(x) = \exp \{ \zeta'(-k, x) - \zeta'(-k) \} \quad \text{for } x > 0,$$

a formula that can be seen as the analogue of Adamchik's formula for  $A_k$ .

### 3 Blagouchine's integrals and series with zeta values

**Definition 2.** For each non-negative integer  $k$  and each positive odd integer  $p$ , the integral  $\mathcal{J}_{k,p}$  is defined by

$$\mathcal{J}_{k,p} = \int_{-\infty}^{+\infty} \frac{\zeta(\frac{p}{2} + ix)}{(2k + p + 2ix) \cosh(\pi x)} dx.$$

**Proposition 1.** We have the following relations:

$$\mathcal{J}_{k,1} = \frac{\gamma}{k+1} - \frac{1}{(k+1)^2} - \sum_{n=2}^{\infty} (-1)^n \frac{\zeta(n)}{n+k}, \quad (5)$$

and

$$\mathcal{J}_{k,p} = (-1)^{\frac{p+1}{2}} \sum_{n=\frac{p+1}{2}}^{\infty} (-1)^n \frac{\zeta(n)}{n+k} \quad \text{for } p = 3, 5, 7, \dots \quad (6)$$

*Proof.* For  $k \geq 0$ , let us consider the function

$$f_k(z) = \frac{\zeta(z)}{(k+z) \sin(\pi z)}.$$

The function  $f_k$  has poles at integers  $n \in \mathbb{Z}$ . For  $n \geq 2$ , the residue of  $f_k$  at  $z = n$  is

$$\text{Res}(f_k; n) = \frac{(-1)^n \zeta(n)}{(n+k)\pi}.$$

For  $n = 1$ ,  $f_k$  has a double pole and

$$\text{Res}(f_k; 1) = -\frac{1}{\pi} \left( \frac{\gamma}{k+1} - \frac{1}{(k+1)^2} \right).$$

Applying the residue theorem, we get

$$-\frac{1}{2i\pi} \int_{\text{Re}(z)=p/2} f_k(z) dz = \sum_{n > \frac{p}{2}} \text{Res}(f_k; n).$$

This leads to formulas (5) and (6). □

**Theorem 1.** We have

$$\begin{aligned} \mathcal{J}_{1,1} &= \frac{1}{2} \ln 2\pi - \frac{5}{4}, \\ \mathcal{J}_{2,1} &= \frac{1}{3} \ln 2\pi - \frac{1}{6} \gamma - \frac{11}{18} + \frac{\zeta'(2)}{\pi^2}, \end{aligned}$$

and for  $k \geq 3$ ,

$$\begin{aligned} \mathcal{J}_{k,1} &= \frac{1}{k+1} \ln 2\pi - \frac{k-1}{k+1} \frac{\gamma}{2} - \frac{k^2+3k+1}{k(k+1)^2} \\ &\quad - \sum_{j=1}^{\lfloor \frac{k}{2} \rfloor} (-1)^j \binom{k}{2j-1} \frac{(2j)!}{j(2\pi)^{2j}} \zeta'(2j) - \sum_{j=1}^{\lfloor \frac{k-1}{2} \rfloor} (-1)^j \binom{k}{2j} \frac{(2j)!}{2(2\pi)^{2j}} \zeta(2j+1). \quad (7) \end{aligned}$$

*Proof.* From [6, Proposition 1], we have

$$\sum_{n=2}^{\infty} (-1)^n \frac{\zeta(n)}{n+k} = \frac{\gamma}{k+1} + \sum_{j=0}^{k-1} (-1)^j \binom{k}{j} \zeta'(-j) + \frac{1}{k} + \sum_{j=0}^{k-1} \binom{k}{j} \frac{B_{j+1} H_j}{j+1} \quad (k \geq 1).$$

By Adamchik's formula (1), this expression may be rewritten as follows:

$$\sum_{n=2}^{\infty} (-1)^n \frac{\zeta(n)}{n+k} = \frac{\gamma}{k+1} + \frac{1}{k} - \sum_{j=0}^{k-1} (-1)^j \binom{k}{j} \ln A_j \quad (k \geq 1).$$

Then, using the relation (5), we get the following expression:

$$\mathcal{J}_{k,1} = \sum_{j=0}^{k-1} (-1)^j \binom{k}{j} \ln A_j - \frac{1}{k} - \frac{1}{(k+1)^2} \quad (k \geq 1). \quad (8)$$

Hence, formula (7) results from formula (8) and Lemma 1.  $\square$

*Remark 3.* Since  $\sum_{n=2}^{\infty} (-1)^n \frac{\zeta(n)}{n} = \gamma$  (by a well-known series representation of Euler's constant  $\gamma$ ), we also have  $\mathcal{J}_{0,1} = -1$  by (5).

The following relation

$$\mathcal{J}_{k,3} = \frac{\gamma}{k+1} - \frac{1}{(k+1)^2} - \mathcal{J}_{k,1} \quad (9)$$

which results from (5) and (6) allows us to deduce the following corollary:

**Corollary 1.** We have

$$\begin{aligned} \mathcal{J}_{1,3} &= \frac{1}{2} \gamma - \frac{1}{2} \ln 2\pi + 1, \\ \mathcal{J}_{2,3} &= \frac{1}{2} \gamma - \frac{1}{3} \ln 2\pi + \frac{1}{2} - \frac{\zeta'(2)}{\pi^2}, \end{aligned}$$

and for  $k \geq 3$ ,

$$\begin{aligned} \mathcal{J}_{k,3} &= \frac{1}{2} \gamma - \frac{1}{k+1} \ln 2\pi + \frac{1}{k} \\ &+ \sum_{j=1}^{\lfloor \frac{k}{2} \rfloor} (-1)^j \binom{k}{2j-1} \frac{(2j)!}{j(2\pi)^{2j}} \zeta'(2j) + \sum_{j=1}^{\lfloor \frac{k-1}{2} \rfloor} (-1)^j \binom{k}{2j} \frac{(2j)!}{2(2\pi)^{2j}} \zeta(2j+1). \quad (10) \end{aligned}$$

## 4 Further generalization

Using a Fourier transform method, Candelpergher [5, Eq. (7)] recently proved the following beautiful relation which is a natural generalization of (5).

**Theorem 2.** for  $k \geq 0$  and  $\operatorname{Re}(s) > \frac{1}{2}$ , we have

$$2^{s-1} \mathcal{J}_{k,1}(s) = \frac{\gamma}{(k+1)^s} - \frac{s}{(k+1)^{s+1}} - \sum_{n=2}^{\infty} (-1)^n \frac{\zeta(n)}{(n+k)^s}. \quad (11)$$

with

$$\mathcal{J}_{k,1}(s) := \int_{-\infty}^{+\infty} \frac{\zeta(\frac{1}{2} + ix)}{(2k+1+2ix)^s \cosh(\pi x)} dx$$

Applying (11) with  $k = 0$  allows us to deduce the following identity:

**Corollary 2.**

$$\sum_{n=2}^{\infty} (-1)^n \frac{\zeta(n)}{n^s} = \gamma - s - \frac{1}{2} \int_{-\infty}^{+\infty} \frac{\zeta(\frac{1}{2} + ix)}{(\frac{1}{2} + ix)^s \cosh(\pi x)} dx \quad (\operatorname{Re}(s) > \frac{1}{2}). \quad (12)$$

**Example 3.** For  $s = 1$ , the representation  $\sum_{n=2}^{\infty} (-1)^n \frac{\zeta(n)}{n} = \gamma$  is regained (since  $\mathcal{J}_{0,1}(1) = -1$ ), and for  $s = 2$ , formula (12) gives the relation

$$\sum_{n=2}^{\infty} (-1)^n \frac{\zeta(n)}{n^2} = \gamma - 2 - \frac{1}{2} \int_{-\infty}^{+\infty} \frac{\zeta(\frac{1}{2} + ix)}{(\frac{1}{2} + ix)^2 \cosh(\pi x)} dx.$$

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