On the generalized Glaisher-Kinkelin constants and Blagouchine's integrals

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Abstract The main purpose of this article is to establish a close connection between a sequence of complex integrals introduced by Blagouchine and some important mathematical constants, namely the generalized Glaisher-Kinkelin constants (also known as the Bendersky constants) which occur quite naturally in analysis and number theory. At the end of this study, we also use a formula of Candelpergher to deduce an interesting expression of the alternating series $\sum_{n\geq 2}(-1)^n \frac{\zeta(n)}{n^s}$ for complex values of s.

Keywords Generalized Glaisher-Kinkelin constants, infinite series with zeta values, complex integration.

1 Introduction

The main purpose of this article is to highlight the link between the sequence of complex integrals $\{\mathcal{J}_k\}_{k\geq 0}$ defined by

$$\mathcal{J}_k = \int_{-\infty}^{+\infty} \frac{\zeta(\frac{1}{2}+ix)}{(2k+1+2ix)\cosh(\pi x)}\, dx\,,$$

and some important mathematical constants, namely the generalized Glaisher-Kinkelin constants (also known as the Bendersky constants) which occur quite naturally in analysis and number theory [1, 10, 11]. To establish these close connections, we make use of a relation (found by Blagouchine [2]) between the integral $\mathcal{J}_k$ and the alternating series

$$\nu_k := \sum_{n=2}^{\infty}(-1)^n \frac{\zeta(n)}{n+k}$$

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which have been thoroughly studied in [9]. This enable us to give a general expression of the integrals $\mathcal{J}_k$ for all non-negative integers k (see Theorem 1).

Recently, this deep relation between $\mathcal{J}_k$ and ν_k has been generalized by Candelpergher [6] (see Theorem 2), that allows us to provide, as a corollary, an interesting expression of the alternating series

$$\sum_{n=2}^{\infty}(-1)^n\frac{\zeta(n)}{n^s}$$

for any complex number s with $\mathrm{Re}(s) > \frac{1}{2}$ (see formula (10)).

2 Generalized Glaisher-Kinkelin constants

Definition 1 ([10, 11])**.** For any integer $k \geq 0$, the constant A_k are usually defined by

$$\begin{aligned}
\ln(A_0) &= \lim_{N\to\infty}\left\{\sum_{n=1}^{N}\ln n - \left(N+\frac{1}{2}\right)\ln N + N\right\}\,,\\
\ln(A_1) &= \lim_{N\to\infty}\left\{\sum_{n=1}^{N}n\ln n - \left(\frac{N^2}{2}+\frac{N}{2}+\frac{1}{12}\right)\ln N + \frac{N^2}{4}\right\}\,,\\
\ln(A_2) &= \lim_{N\to\infty}\left\{\sum_{n=1}^{N}n^2\ln n - \left(\frac{N^3}{3}+\frac{N^2}{2}+\frac{N}{6}\right)\ln N + \frac{N^3}{9} - \frac{N}{12}\right\}\,,
\end{aligned}$$

and more generally

$$\ln(A_k) = \lim_{N\to\infty}\left\{\sum_{n=1}^{N}n^k\ln n - P_k(N)\ln N + Q_k(N)\right\}\,,$$

where P_k and Q_k are polynomials of degree $k+1$ that can be explicitely computed (see e.g. [11, Eq. (1.1)]). The numbers A_k for $k = 0, 1, 2, \ldots$ are called the *generalized Glaisher-Kinkelin constants* (sometimes called the *Bendersky constants*). Adamchik [1, Proposition 4] has given an alternative expression of the constants A_k in terms of the derivatives of the Riemann zeta function. More precisely, we have

$$A_k = \exp\left\{\frac{H_k B_{k+1}}{k+1} - \zeta'(-k)\right\}\,, \tag{1}$$

where $H_k = \sum_{j=1}^{k}\frac{1}{j}$ is the k-th harmonic number with the usual convention $H_0 = 0$.

Example 1. The constant $A_0 = \exp(-\zeta'(0)) = \sqrt{2\pi}$ is the Stirling constant, and

$$A_1 = \exp\left(\frac{1}{12} - \zeta'(-1)\right)$$

is the classical Glaisher-Kinkelin constant.

The following relations are easily derived by differentiation of the Riemann functional equation for the zeta function:

$$\zeta'(-2k) = (-1)^k \frac{(2k)!}{2(2\pi)^{2k}} \zeta(2k+1) \qquad (k \geq 1)\,,$$

and

$$\zeta'(1-2k) = (-1)^{k+1} \frac{(2k)!}{k(2\pi)^{2k}} \zeta'(2k) + \frac{B_{2k}}{2k} \left(H_{2k-1} - \gamma - \ln 2\pi\right) \qquad (k \geq 1).$$

This enable to deduce from Adamchik's formula (1) the expressions

$$A_{2k-1} = \exp\left\{(-1)^k \frac{(2k)!}{k(2\pi)^{2k}} \zeta'(2k) + \frac{B_{2k}}{2k} \left(\gamma + \ln 2\pi\right)\right\} \qquad (k \geq 1)\,, \tag{2}$$

and

$$A_{2k} = \exp\left\{(-1)^{k+1} \frac{(2k)!}{2(2\pi)^{2k}} \zeta(2k+1)\right\} \qquad (k \geq 1)\,. \tag{3}$$

In particular, we can easily deduce from formulas (2) and (3) the following binomial identity which will be useful later:

Lemma 1. For $k \geq 1$,

$$\begin{aligned}\sum_{j=0}^{k-1} (-1)^j \binom{k}{j} \ln(A_j) = \frac{\ln(2\pi)}{k+1} - \frac{k-1}{k+1}\frac{\gamma}{2} \\ - \sum_{j=1}^{[\frac{k}{2}]} (-1)^j \binom{k}{2j-1} \frac{(2j)!}{j(2\pi)^{2j}} \zeta'(2j) \\ - \sum_{j=1}^{[\frac{k-1}{2}]} (-1)^j \binom{k}{2j} \frac{(2j)!}{2(2\pi)^{2j}} \zeta(2j+1)\,.\end{aligned} \tag{4}$$

Remark 1. Bendersky [3] introduced for the first time the sequence of numbers A_k without any consideration of their relation with the ζ-function. From the point of view of the summation of divergent series, the constants A_k can be interpreted as follows: let $\sum_{n\geq 1}^{\mathcal{R}} n^k \ln n$ denotes the $\mathcal{R}$-sum of the divergent series $\sum_{n\geq 1} n^k \ln n$ (i.e. the sum of the series in the sense of Ramanujan's summation method [5]), then, for any integer $k \geq 0$, we have (cf. [5, p. 68], [3, p. 280]):

$$\begin{aligned}\sum_{n\geq 1}^{\mathcal{R}} n^k \ln n &= -\zeta'(-k) - \frac{1}{(k+1)^2} \\ &= \ln(A_k) - \frac{H_k B_{k+1}}{k+1} - \frac{1}{(k+1)^2} \\ &= \int_0^1 \ln \Gamma_k(x+1)\, dx\,,\end{aligned}$$

where Γ_k is the Bendersky generalized gamma function [3, p. 279]. This function verifies in particular

$$\Gamma_k(n+1) = 1^{1^k} 2^{2^k} \cdots n^{n^k} \qquad \text{for any integer } n \geq 1\,.$$

Example 2. For the first values of k, we have

$$\begin{aligned}\sum_{n\geq 1}^{\mathcal{R}} \ln n &= \ln(\sqrt{2\pi}) - 1 = \int_0^1 \ln \Gamma(x+1)\, dx\,, \\ \sum_{n\geq 1}^{\mathcal{R}} n \ln n &= \ln(A_1) - \frac{1}{3} = \int_0^1 \ln K(x+1)\, dx\,,\end{aligned}$$

where $K = \Gamma_1$ is the classical hyperfactorial K-function [7].

3 Blagouchine's first integral

We give a direct proof of [2, Theorem 2] using Cauchy's residue theorem.

Proposition 1. For any integer $k \geq 0$, let $\mathcal{I}_k$ be the integral defined by

$$\mathcal{I}_k := \int_{-\infty}^{+\infty} \frac{\zeta(\frac{3}{2} + ix)}{(2k+1+2ix)\cosh(\pi x)}\, dx\,.$$

Then

$$\mathcal{I}_k = \nu_{k-1} \tag{5}$$

where ν_k is the conditionally convergent series defined by

$$\nu_k := \sum_{n=2}^{\infty} (-1)^n \frac{\zeta(n)}{n+k} \qquad (k \geq -1)\,.$$

Proof. For $k \geq 0$, let us consider the function

$$f_k(z) = \frac{\zeta(\frac{3}{2} + iz)}{(\frac{1}{2} + k + iz)\cosh(\pi z)} \,.$$

We have $\cosh(\pi z) = 0$ if and only if $z = i/2 + in$ with $n \in \mathbb{Z}$. For $n \geq 1$, the residue of f_k at $z = i/2 - in$ is

$$\frac{\zeta(1+n)}{(n+k)\pi \sinh(i\pi(\frac{1}{2} - n))} = \frac{\zeta(1+n)}{(n+k)i\pi \sin(\pi(\frac{1}{2} - n))} = \frac{(-1)^n \zeta(1+n)}{(n+k)i\pi} \,.$$

We integrate on a closed contour composed of the interval $D_R = [-R, R]$ and the lower semicircle C_R of radius R with center at 0. By the residue theorem, we can then write the following relation:

$$\frac{1}{2i\pi} \int_{C_R} f_k(z)\, dz + \frac{1}{2i\pi} \int_{D_R} f_k(z)\, dz = -\sum_{n=1}^{N_R} \mathrm{Res}(f_k; \frac{i}{2} - in) \,,$$

which, from the foregoing, translates into the identity

$$\int_{C_R} f_k(z)\, dz + \int_{D_R} f_k(z)\, dz = 2\sum_{n=1}^{N_R} (-1)^{n+1} \frac{\zeta(1+n)}{(n+k)} \,. \tag{6}$$

For $z \in C_R$, the parameterization $iz = Re^{it}$ with $-\pi/2 < t < \pi/2$, enables us to write

$$\begin{aligned}\left| \int_{C_R} f_k(z) dz \right| &= \left| \int_{-\pi/2}^{+\pi/2} \frac{\zeta(\frac{3}{2} + Re^{it})}{(\frac{1}{2} + k + Re^{it})\cosh(i\pi Re^{it})} Re^{it}\, dt \right| \\ &\leq \int_{-\pi/2}^{+\pi/2} \left| \frac{\zeta(\frac{3}{2} + Re^{it})}{(\frac{1}{2} + k + Re^{it})\cosh(i\pi Re^{it})} \right| R\, dt \,.\end{aligned}$$

Since $\frac{3}{2} + Re^{it}$ is in the half-plane $\mathrm{Re}(z) > 3/2$, its absolute value is bounded by $\zeta(\frac{3}{2})$, i.e.

$$\left| \zeta(\frac{3}{2} + Re^{it}) \right| \leq \zeta(\frac{3}{2}) \,.$$

Hence, when R increases towards infinity, we have the following limits:

$$\lim_{R\to\infty} \int_{C_R} f_k(z)\, dz = 0 \,,$$

$$\lim_{R\to\infty} \int_{D_R} f_k(z)\, dz = \int_{-\infty}^{+\infty} \frac{\zeta(\frac{3}{2} + ix)}{(\frac{1}{2} + k + ix)\cosh(\pi x)}\, dx = 2\,\mathcal{I}_k \,,$$

and

$$\lim_{R\to\infty} \sum_{n=1}^{N_R} (-1)^{n+1} \frac{\zeta(1+n)}{n+k} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\zeta(n+1)}{n+k} = \nu_{k-1} \,.$$

This allows us to deduce formula (5) by passing to the limit in (6). □

Example 3. For $k = 0$, formula (5) translates into

$$\int_{-\infty}^{+\infty} \frac{\zeta(\frac{3}{2} + ix)}{(1 + 2ix)\cosh(\pi x)}\,dx = \sum_{n=2}^{\infty} (-1)^n \frac{\zeta(n)}{n-1}\,,$$

and for $k = 1$

$$\int_{-\infty}^{+\infty} \frac{\zeta(\frac{3}{2} + ix)}{(3 + 2ix)\cosh(\pi x)}\,dx = \sum_{n=2}^{\infty} (-1)^n \frac{\zeta(n)}{n}\,.$$

The constant

$$\nu_{-1} = \sum_{n=2}^{\infty} (-1)^n \frac{\zeta(n)}{n-1} = \sum_{n=1}^{\infty} \frac{1}{n} \ln\left(1 + \frac{1}{n}\right) = 1.257746\ldots$$

has been thoroughly studied by Boyadzhiev [4][1]. Moreover, by a well-known series representation of Euler's constant γ, we also have

$$\nu_0 = \sum_{n=2}^{\infty} (-1)^n \frac{\zeta(n)}{n} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \ln\left(1 + \frac{1}{n}\right)\right) = \gamma = 0.577215\ldots\,.$$

4 Blagouchine's second integral

Proposition 2. For any integer $k \geq 0$, let $\mathcal{J}_k$ be the integral defined by

$$\mathcal{J}_k := \int_{-\infty}^{+\infty} \frac{\zeta(\frac{1}{2} + ix)}{(2k + 1 + 2ix)\cosh(\pi x)}\,dx\,.$$

Then we have the relation

$$\mathcal{J}_k = \frac{\gamma}{k+1} - \frac{1}{(k+1)^2} - \mathcal{I}_{k+1} \qquad (k \geq 0)\,. \tag{7}$$

Proof. For $\mathrm{Re}(z) > 0$, let us consider the function

$$F_k(z) := \frac{\zeta(z)}{2(k+z)\cos(\pi z - \frac{\pi}{2})}\,.$$

The residue theorem allows us to write the relation

$$\int_{\mathrm{Re}(z)=1/2} F_k(z)\,dz - \int_{\mathrm{Re}(z)=3/2} F_k(z)\,dz = \frac{\gamma}{k+1} - \frac{1}{(k+1)^2}$$

because the right-hand side of this equation is nothing but the residue of the function F_k at $z = 1$. This enable to deduce the relation (7). □

[1] This constant is noted M in [4] and K in [8, p. 142].

Theorem 1. We have $\mathcal{J}_0 = -1$,

$$\mathcal{J}_1 = \frac{1}{2}\ln(2\pi) - \frac{5}{4},$$
$$\mathcal{J}_2 = \frac{1}{3}\ln(2\pi) - \frac{1}{6}\gamma + \frac{\zeta'(2)}{\pi^2} - \frac{11}{19},$$

and for $k \geq 3$,

$$\mathcal{J}_k = \frac{1}{k+1}\ln(2\pi) - \frac{k-1}{2(k+1)}\gamma - \sum_{j=1}^{[\frac{k}{2}]}(-1)^j \binom{k}{2j-1}\frac{(2j)!}{j(2\pi)^{2j}}\zeta'(2j) - \sum_{j=1}^{[\frac{k-1}{2}]}(-1)^j \binom{k}{2j}\frac{(2j)!}{2(2\pi)^{2j}}\zeta(2j+1) - \frac{k^2+3k+1}{k(k+1)^2}. \quad (8)$$

Proof. By (5) we have $\mathcal{I}_{k+1} = \nu_k$ and, by [9, Proposition 1], we also have

$$\nu_k = \frac{\gamma}{k+1} + \sum_{j=0}^{k-1}(-1)^j\binom{k}{j}\zeta'(-j) + \frac{1}{k} + \sum_{j=0}^{k-1}\binom{k}{j}\frac{B_{j+1}H_j}{j+1} \qquad (k \geq 1).$$

Using Adamchik's formula (1), this expression may be rewritten as follows:

$$\mathcal{I}_{k+1} = \frac{\gamma}{k+1} + \frac{1}{k} - \sum_{j=0}^{k-1}(-1)^j\binom{k}{j}\ln(A_j) \qquad (k \geq 1).$$

Now, using the relation (7), we get $\mathcal{J}_0 = -1$ (since $\mathcal{I}_1 = \gamma$) and

$$\mathcal{J}_k = \sum_{j=0}^{k-1}(-1)^j\binom{k}{j}\ln(A_j) - \frac{1}{k} - \frac{1}{(k+1)^2} \qquad (k \geq 1).$$

Hence, formula (8) results from formula (4) (cf. Lemma 1). □

5 Further generalization

Using Fourier transform method, Candelpergher [6, Eq. (7)] recently proved the following beautiful relation which is the analogue of (7).

Theorem 2. for $k \geq 0$ and $\mathrm{Re}(s) > \frac{1}{2}$, we have

$$2^{s-1}\mathcal{J}_k(s) = \frac{\gamma}{(k+1)^s} - \frac{s}{(k+1)^{s+1}} - \nu_k(s), \quad (9)$$

with

$$\mathcal{J}_k(s) := \int_{-\infty}^{+\infty} \frac{\zeta(\frac{1}{2}+ix)}{(2k+1+2ix)^s \cosh(\pi x)}\, dx$$

and

$$\nu_k(s) := \sum_{n=2}^{\infty} (-1)^n \frac{\zeta(n)}{(n+k)^s}\,.$$

Applying (9) with $k = 0$ allows us to deduce the following identity:

Corollary 1.

$$\sum_{n=2}^{\infty} (-1)^n \frac{\zeta(n)}{n^s} = \gamma - s - \frac{1}{2} \int_{-\infty}^{+\infty} \frac{\zeta(\frac{1}{2}+ix)}{(\frac{1}{2}+ix)^s \cosh(\pi x)}\, dx \qquad (\mathrm{Re}(s) > \frac{1}{2})\,. \tag{10}$$

Example 4.

$$\sum_{n=2}^{\infty} (-1)^n \frac{\zeta(n)}{n^2} = \gamma - 2 - \frac{1}{2} \int_{-\infty}^{+\infty} \frac{\zeta(\frac{1}{2}+ix)}{(\frac{1}{2}+ix)^2 \cosh(\pi x)}\, dx\,.$$

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