Generalized Glaisher-Kinkelin constants and Blagouchine’s integrals

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Abstract The main purpose of this article is to establish a close connection between two sequences of complex integrals introduced by Blagouchine and some important mathematical constants, namely the Euler-Mascheroni constant, the Cohen-Boyadzhiev constant, and the generalized Glaisher-Kinkelin constants (also known as the Bendersky constants) which occur quite naturally in analysis and number theory. At the end of this study, we also use a formula of Candelpergher to deduce an interesting expression of the alternating series \( \sum_{n \geq 2} (-1)^n \frac{\zeta(n)}{n^s} \) for complex values of \( s \).

Keywords Generalized Glaisher-Kinkelin constants, infinite series with zeta values, complex integration.

1 Introduction

The main purpose of this article is to highlight the link between the sequence of complex integrals \( \{I_k\}_{k \geq 0} \) and \( \{J_k\}_{k \geq 0} \) respectively defined by

\[
I_k = \int_{-\infty}^{+\infty} \frac{\zeta\left(\frac{3}{2} + ix\right)}{(2k + 1 + 2ix) \cosh(\pi x)} \, dx
\]

and

\[
J_k = \int_{-\infty}^{+\infty} \frac{\zeta\left(\frac{1}{2} + ix\right)}{(2k + 1 + 2ix) \cosh(\pi x)} \, dx,
\]

on one side, and some important mathematical constants, namely the Euler-Mascheroni constant, the Cohen-Boyadzhiev constant, and the generalized Glaisher-Kinkelin constants (also known as the Bendersky constants) which occur quite
naturally in analysis and number theory [7, 10, 11]. To establish these close connections, we make use of a relation (found by Blagouchine [2, Theorem 1]) between the integral \( J_k \) and the alternating series

\[
\nu_k := \sum_{n=2}^{\infty} \frac{(-1)^n \zeta(n)}{n+k}
\]

studied in [8]. Recently, this deep relation has been thoroughly generalized by Candelpergher [6, Eq. (7)], that allows us to give an interesting expression of the alternating series

\[
\sum_{n=2}^{\infty} \frac{(-1)^n \zeta(n)}{n^s}
\]

for any complex number \( s \) with \( \text{Re}(s) > \frac{1}{2} \) (see formula (11)).

2 Generalized Glaisher-Kinkelin constants

Definition 1 ([1, 10, 11]). For any integer \( k \geq 0 \), the constant \( A_k \) is usually defined by

\[
\ln(A_k) = \lim_{N \to \infty} \left\{ \sum_{n=1}^{N} n^k \ln n - P_k(N) \right\},
\]

where \( P_k(N) \) is given by \( P_0(N) = \left(N + \frac{1}{2}\right) \ln N - N \), and

\[
P_k(N) = \left( \frac{N^{k+1}}{k+1} + \frac{N^k}{2} + k! \sum_{j=1}^{k} \frac{N^{k-j} B_{j+1}}{(j+1)! (k-j)!} \right) \ln N
\]

\[ - \frac{N^{k+1}}{(k+1)^2} + k! \sum_{j=1}^{k} \frac{N^{k-j} B_{j+1}}{(j+1)! (k-j)!} \left( 1 - \delta_{k,j} \right) \sum_{i=1}^{j} \frac{1}{k - i + 1} \]  \quad (k \geq 1),

where \( B_j \) is the \( j \)-th Bernoulli number and \( \delta_{k,j} \) is the Kronecker delta function. The numbers \( A_k \) (for \( k = 0, 1, 2, \ldots \)) are the generalized Glaisher-Kinkelin constants (sometimes called the Bendersky constants). In particular, it follows from this definition that

\[
\ln(A_0) = \lim_{N \to \infty} \left\{ \sum_{n=1}^{N} \ln n - \left( N + \frac{1}{2} \right) \ln N + N \right\},
\]

and

\[
\ln(A_1) = \lim_{N \to \infty} \left\{ \sum_{n=1}^{N} n \ln n - \left( \frac{N^2}{2} + \frac{N}{2} + \frac{1}{12} \right) \ln N + \frac{N^2}{4} \right\}.\]
Remark 1. Adamchik [1, Proposition 4] has found a nice expression of these constants in terms of the derivatives of the Riemann zeta function. More precisely, we have

\[ A_k = \exp \left\{ \frac{H_k B_{k+1}}{k+1} - \zeta'(-k) \right\} \quad (k \geq 0), \quad (1) \]

where \( H_k = \sum_{j=1}^{k} \frac{1}{j} \) is the \( k \)-th harmonic number with the usual convention \( H_0 = 0 \).

The following relations are easily deduced by differentiation of Riemann’s functional equation for the zeta function:

\[ \zeta'(-2k) = (-1)^k \frac{(2k)!}{2(2\pi)^{2k}} \zeta(2k+1) \quad (k \geq 1), \]

and

\[ \zeta'(1 - 2k) = (-1)^{k+1} \frac{(2k)!}{k(2\pi)^{2k}} \zeta'(2k) + \frac{B_{2k}}{2k} (H_{2k-1} - \gamma - \ln 2\pi) \quad (k \geq 1). \]

This enables to deduce from Adamchik’s formula (1) the expressions

\[ A_{2k-1} = \exp \left\{ (-1)^k \frac{(2k)!}{k(2\pi)^{2k}} \zeta'(2k) + \frac{B_{2k}}{2k} (\gamma + \ln 2\pi) \right\} \quad (k \geq 1), \quad (2) \]

and

\[ A_{2k} = \exp \left\{ (-1)^{k+1} \frac{(2k)!}{2(2\pi)^{2k}} \zeta(2k+1) \right\} \quad (k \geq 1). \quad (3) \]

Example 1. The constant

\[ A_0 = \exp(-\zeta'(0)) = \sqrt{2\pi} \]

is the Stirling constant,

\[ A_1 = \exp \left( \frac{1}{12} - \zeta'(-1) \right) = \exp \left( -\frac{\zeta'(2)}{2\pi^2} + \frac{\ln(2\pi) + \gamma}{12} \right) \]

is the classical Glaisher-Kinkelin constant, and for \( k = 2 \), we have

\[ A_2 = \exp(-\zeta'(-2)) = \exp \left( \frac{\zeta(3)}{4\pi^2} \right). \]
Remark 2. Bendersky [3] introduced for the first time the sequence of numbers $L_k := \ln(A_k)$ without any consideration of their relation with the $\zeta$-function. From the point of view of the summation of divergent series, the constants $L_k$ can be interpreted as follows: if $\sum_{n \geq 1}^R n^k \ln n$ denotes the $R$-sum of the divergent series $\sum_{n \geq 1} n^k \ln n$ (i.e. the sum of the series in the sense of Ramanujan’s summation method [5]), then, for any integer $k \geq 0$, we have

$$\sum_{n \geq 1}^R n^k \ln n = L_k - \frac{H_k B_{k+1}}{k+1} - \frac{1}{(k+1)^2} = \int_0^1 \ln \Gamma_k(x + 1) \, dx,$$

where $\Gamma_k$ is the Bendersky generalized gamma function [3]. This function verifies in particular

$$\Gamma_k(n + 1) = 1^{1^k} 2^{2^k} \cdots n^{n^k} \quad \text{for any integer } n \geq 1.$$

Apparently unaware of Bendersky’s work, Kurokawa and Ochiai [9, Theorem 2] have given a deep expression of the function $\Gamma_k$ in terms of the derivative of the Hurwitz zeta function $\zeta(s, x)$ at $s = -k$. Precisely, they showed that

$$\Gamma_k(x) = \exp \{ \zeta'(-k, x) - \zeta'(-k) \} \quad \text{for } x > 0.$$

Example 2. For the first values of $k$, formula (4) translates into the identities

$$\sum_{n \geq 1}^R \ln n = \ln(\sqrt{2\pi}) - 1 = \int_0^1 \ln \Gamma(x + 1) \, dx,$$

$$\sum_{n \geq 1}^R n \ln n = \ln(A_1) - \frac{1}{3} = \int_0^1 \ln K(x + 1) \, dx,$$

where $K = \Gamma_1$ is the classical Kinkelin hyperfactorial function.

## 3 Blagouchine’s first integral

We give a direct proof of [2, Theorem 2] using Cauchy’s residue theorem.

Proposition 1. For any integer $k \geq 0$, let $I_k$ be the integral defined by

$$I_k := \int_{-\infty}^{+\infty} \frac{\zeta\left(\frac{3}{2} + ix\right)}{(2k + 1 + 2ix) \cosh(\pi x)} \, dx.$$

Then

$$I_k = \mu_k$$

where $\mu_k$ is the conditionally convergent series defined by

$$\mu_k := \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\zeta(n + 1)}{n + k}.$$
Proof. For \( k \geq 0 \), let us consider the function

\[
f_k(z) = \frac{\zeta\left(\frac{3}{2} + iz\right)}{\left(\frac{1}{2} + k + iz\right) \cosh(\pi z)}.
\]

We have \( \cosh(\pi z) = 0 \) if and only if \( z = i/2 + in \) with \( n \in \mathbb{Z} \). For \( n \geq 1 \), the residue of \( f_k \) at \( z = i/2 - in \) is

\[
\frac{\zeta(1 + n)}{(n + k)\pi \sinh(i\pi(\frac{1}{2} - n))} = \frac{\zeta(1 + n)}{(n + k)i\pi}.
\]

We integrate on a closed contour composed of the interval \( D_R = [-R, R] \) and the lower semicircle \( C_R \) of radius \( R \) with center at 0. By the residue theorem, we can then write the following relation:

\[
\frac{1}{2i\pi} \int_{C_R} f_k(z)\,dz + \frac{1}{2i\pi} \int_{D_R} f_k(z)\,dz = -\sum_{n=1}^{N_R} \text{Res}(f_k; i/2 - in),
\]

which, from the foregoing, translates into the identity

\[
\int_{C_R} f_k(z)\,dz + \int_{D_R} f_k(z)\,dz = 2\sum_{n=1}^{N_R} (-1)^{n+1} \frac{\zeta(1 + n)}{(n + k)}.
\] (6)

For \( z \in C_R \), the parameterization \( iz = Re^{it} \) with \( -\pi/2 < t < \pi/2 \), enables us to write

\[
\left|\int_{C_R} f_k(z)\,dz\right| = \left| \int_{-\pi/2}^{\pi/2} \frac{\zeta\left(\frac{3}{2} + Re^{it}\right)}{\left(\frac{1}{2} + k + Re^{it}\right) \cosh(i\pi Re^{it})} Re^{it} \,dt \right|
\leq \int_{-\pi/2}^{\pi/2} \left| \frac{\zeta\left(\frac{3}{2} + Re^{it}\right)}{\left(\frac{1}{2} + k + Re^{it}\right) \cosh(i\pi Re^{it})} \right| R \,dt.
\]

Since \( \frac{3}{2} + Re^{it} \) is in the half-plane \( \text{Re}(z) > 3/2 \), its absolute value is bounded by \( \zeta(\frac{3}{2}) \), i.e.

\[
\left| \zeta\left(\frac{3}{2} + Re^{it}\right) \right| \leq \zeta\left(\frac{3}{2}\right).
\]

Hence, when \( R \) increases towards infinity, we have the following limits:

\[
\lim_{R \to \infty} \int_{C_R} f_k(z)\,dz = 0,
\]

\[
\lim_{R \to \infty} \int_{D_R} f_k(z)\,dz = \int_{-\infty}^{+\infty} \frac{\zeta\left(\frac{3}{2} + ix\right)}{\left(\frac{1}{2} + k + ix\right) \cosh(\pi x)} \,dx = 2\mathcal{I}_k,
\]

and

\[
\lim_{R \to \infty} \sum_{n=1}^{N_R} (-1)^{n+1} \frac{\zeta(1 + n)}{n + k} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\zeta(n + 1)}{n + k} = \mu_k.
\]

This allows us to deduce formula (5) by passing to the limit in (6). \( \square \)
Remark 3. The constant
\[ \mu_0 = \sum_{n=2}^{\infty} (-1)^n \frac{\zeta(n)}{n-1} = \sum_{n=1}^{\infty} \frac{1}{n} \ln \left( 1 + \frac{1}{n} \right) = 1.257746 \ldots \]
has been thoroughly studied by Boyadzhiev [4] (see also [7, p. 142]). This constant is noted \( M \) in [4], \( K \) in [7], and also appears in [8] as \( \nu_{-1} \). By a well-known series representation of Euler’s constant \( \gamma \), we also have
\[ \mu_1 = \sum_{n=2}^{\infty} (-1)^n \frac{\zeta(n)}{n} = \sum_{n=1}^{\infty} \left( \frac{1}{n} \ln \left( 1 + \frac{1}{n} \right) \right) = \gamma = 0.577215 \ldots . \]

Example 3. For \( k = 0 \), formula (5) translates into
\[ \mathcal{I}_0 = \int_{-\infty}^{+\infty} \frac{\zeta\left( \frac{3}{2} + ix \right)}{(1 + 2ix) \cosh(\pi x)} \, dx = \int_{0}^{1} \frac{\psi(x+1) + \gamma}{x} \, dx, \]
where \( \psi \) is the digamma function, and for \( k = 1 \)
\[ \mathcal{I}_1 = \int_{-\infty}^{+\infty} \frac{\zeta\left( \frac{3}{2} + ix \right)}{(3 + 2ix) \cosh(\pi x)} \, dx = \gamma = -\psi(1). \]

4 Blagouchine’s second integral

Proposition 2. For any integer \( k \geq 0 \), let \( \mathcal{J}_k \) be the integral defined by
\[ \mathcal{J}_k := \int_{-\infty}^{+\infty} \frac{\zeta\left( \frac{3}{2} + ix \right)}{(2k + 1 + 2ix) \cosh(\pi x)} \, dx \]
Then
\[ \mathcal{J}_k = \gamma \frac{k}{k+1} - \frac{1}{(k+1)^2} - \mathcal{I}_{k+1} \quad (k \geq 0). \quad (7) \]
Since \( \mathcal{I}_1 = \gamma \), it follows from this relation that \( \mathcal{J}_0 = -1 \); furthermore we have the identity
\[ \mathcal{J}_k = \sum_{j=0}^{k-1} (-1)^j \binom{k}{j} \ln(A_j) - \frac{k^2 + 3k + 1}{k(k+1)^2} \quad (k \geq 1). \quad (8) \]

Proof. Formula (7) is a reformulation of [2, Theorem 1]. Now assume that \( k \geq 1 \) and let \( \nu_k := \mu_{k+1} \). By [8, Proposition 1] and (5), we have
\[ \mathcal{I}_{k+1} = \nu_k = \gamma \frac{k}{k+1} + \sum_{j=0}^{k-1} (-1)^j \binom{k}{j} \zeta'(-j) + \frac{1}{k} + \sum_{j=0}^{k-1} \binom{k}{j} B_{j+1} H_j \frac{1}{j+1}. \]
Now, using Adamchik’s formula (1), this expression may be rewritten as follows:

\[ I_{k+1} = \frac{\gamma}{k+1} + \frac{1}{k} - \sum_{j=0}^{k-1} (-1)^j \binom{k}{j} \ln(A_j) \]  

\( k \geq 1 \).

Hence, formula (8) results from (7).

**Example 4.** For \( k = 1 \), formula (8) simply reduces to

\[ J_1 = \int_{-\infty}^{+\infty} \frac{\zeta\left(\frac{1}{2} + ix\right)}{(3 + 2ix) \cosh(\pi x)} \, dx = \frac{1}{2} \ln(2\pi) - \frac{5}{4}, \]

and for \( k = 2 \),

\[ J_2 = \int_{-\infty}^{+\infty} \frac{\zeta\left(\frac{1}{2} + ix\right)}{(5 + 2ix) \cosh(\pi x)} \, dx = \frac{1}{3} \ln(2\pi) - \frac{1}{6} \gamma + \frac{\zeta'(2)}{\pi^2} - \frac{11}{19}. \]

**Remark 4.** The following general expression can be easily deduced from formulas (2) and (3):

\[ \sum_{j=0}^{k-1} (-1)^j \binom{k}{j} \ln(A_j) = \frac{\ln(2\pi)}{k+1} - \frac{k-1}{k+1} \gamma 
- \sum_{j=1}^{\left\lfloor \frac{k}{2} \right\rfloor} (-1)^j \binom{k}{2j-1} \frac{(2j)!}{j(2\pi)^{2j}} \zeta(2j) 
- \sum_{j=1}^{\left\lfloor \frac{k-1}{2} \right\rfloor} (-1)^j \binom{k}{2j} \frac{(2j)!}{2(2\pi)^{2j}} \zeta(2j+1) \quad (k \geq 3). \quad (9) \]

### 5 Further generalization

Using Fourier transform method, Candelpergher [6, Eq. (7)] recently proved the following beautiful relation which generalizes (7):

**Theorem 1.** for \( k \geq 0 \) and \( \text{Re}(s) > \frac{1}{2} \), we have

\[ 2^{s-1} J_k(s) = \frac{\gamma}{(k+1)^s} - \frac{s}{(k+1)^{s+1}} - \nu_k(s), \quad (10) \]

with

\[ J_k(s) := \int_{-\infty}^{+\infty} \frac{\zeta\left(\frac{1}{2} + ix\right)}{(2k + 1 + 2ix)^s \cosh(\pi x)} \, dx \]

and

\[ \nu_k(s) := \sum_{n=2}^{\infty} (-1)^n \frac{\zeta(n)}{(n+k)^s}. \]
Applying (10) with \( k = 0 \) allows us to deduce the following identity:

**Corollary 1.**

\[
\sum_{n=2}^{\infty} \frac{(-1)^n \zeta(n)}{n^s} = \gamma - s - \frac{1}{2} \int_{-\infty}^{+\infty} \frac{\zeta\left(\frac{1}{2} + ix\right)}{(\frac{1}{2} + ix)^s \cosh(\pi x)} \, dx \quad \text{(Re}(s) > \frac{1}{2}). \tag{11}
\]

**Example 5.**

\[
\sum_{n=2}^{\infty} \frac{(-1)^n \zeta(n)}{n^2} = \gamma - 2 - \frac{1}{2} \int_{-\infty}^{+\infty} \frac{\zeta\left(\frac{1}{2} + ix\right)}{(\frac{1}{2} + ix)^2 \cosh(\pi x)} \, dx.
\]

**References**


