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Abstract The main purpose of this article is to establish a close connection between two sequences of complex integrals introduced by Blagouchine and some important mathematical constants, namely the Euler-Mascheroni constant, the Cohen-Boyadzhiev constant, and the generalized Glaisher-Kinkelin constants (also known as the Bendersky constants) which occur quite naturally in analysis and number theory. At the end of this study, we also use a formula of Candelpergher to deduce an interesting expression of the alternating series $\sum_{n\geq 2}(-1)^n \frac{\zeta(n)}{n^s}$ for complex values of s.

Keywords Generalized Glaisher-Kinkelin constants, infinite series with zeta values, complex integration.

1 Introduction

The main purpose of this article is to highlight the link between the sequence of complex integrals $\{\mathcal{I}_k\}_{k\geq 0}$ and $\{\mathcal{J}_k\}_{k\geq 0}$ respectively defined by

$$\mathcal{I}_k = \int_{-\infty}^{+\infty} \frac{\zeta(\frac{3}{2}+ix)}{(2k+1+2ix)\cosh(\pi x)}\, dx$$

and

$$\mathcal{J}_k = \int_{-\infty}^{+\infty} \frac{\zeta(\frac{1}{2}+ix)}{(2k+1+2ix)\cosh(\pi x)}\, dx\,,$$

on one side, and some important mathematical constants, namely the Euler-Mascheroni constant, the Cohen-Boyadzhiev constant, and the generalized Glaisher-Kinkelin constants (also known as the Bendersky constants) which occur quite

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naturally in analysis and number theory [7, 10, 11]. To establish these close connections, we make use of a relation (found by Blagouchine [2, Theorem 1]) between the integral $\mathcal{J}_k$ and the alternating series

$$\nu_k := \sum_{n=2}^{\infty} (-1)^n \frac{\zeta(n)}{n+k}$$

studied in [8]. Recently, this deep relation has been thoroughly generalized by Candelpergher [6, Eq. (7)], that allows us to give an interesting expression of the alternating series

$$\sum_{n=2}^{\infty} (-1)^n \frac{\zeta(n)}{n^s}$$

for any complex number s with $\mathrm{Re}(s) > \frac{1}{2}$ (see formula (11)).

2 Generalized Glaisher-Kinkelin constants

Definition 1 ([1, 10, 11])**.** For any integer $k \geq 0$, the constant A_k is usually defined by

$$\ln(A_k) = \lim_{N\to\infty} \left\{ \sum_{n=1}^{N} n^k \ln n - P_k(N) \right\},$$

where $P_k(N)$ is given by $P_0(N) = \left(N + \frac{1}{2}\right) \ln N - N$, and

$$\begin{aligned} P_k(N) = & \left(\frac{N^{k+1}}{k+1} + \frac{N^k}{2} + k! \sum_{j=1}^{k} \frac{N^{k-j} B_{j+1}}{(j+1)!(k-j)!} \right) \ln N \\ & - \frac{N^{k+1}}{(k+1)^2} + k! \sum_{j=1}^{k} \frac{N^{k-j} B_{j+1}}{(j+1)!(k-j)!} \left\{ (1 - \delta_{k,j}) \sum_{i=1}^{j} \frac{1}{k-i+1} \right\} \qquad (k \geq 1), \end{aligned}$$

where B_j is the j-th Bernoulli number and $\delta_{k,j}$ is the Kronecker delta function. The numbers A_k (for $k = 0, 1, 2, \ldots$) are the *generalized Glaisher-Kinkelin constants* (sometimes called the *Bendersky constants*). In particular, it follows from this definition that

$$\ln(A_0) = \lim_{N\to\infty} \left\{ \sum_{n=1}^{N} \ln n - \left(N + \frac{1}{2}\right) \ln N + N \right\},$$

and

$$\ln(A_1) = \lim_{N\to\infty} \left\{ \sum_{n=1}^{N} n \ln n - \left(\frac{N^2}{2} + \frac{N}{2} + \frac{1}{12} \right) \ln N + \frac{N^2}{4} \right\}.$$

Remark 1. Adamchik [1, Proposition 4] has found a nice expression of theses constants in terms of the derivatives of the Riemann zeta function. More precisely, we have

$$A_k = \exp\left\{\frac{H_k B_{k+1}}{k+1} - \zeta'(-k)\right\} \qquad (k \geq 0)\,, \tag{1}$$

where $H_k = \sum_{j=1}^{k} \frac{1}{j}$ is the k-th harmonic number with the usual convention $H_0 = 0$. The following relations are easily deduced by differentiation of Riemann's functional equation for the zeta function:

$$\zeta'(-2k) = (-1)^k \frac{(2k)!}{2(2\pi)^{2k}} \zeta(2k+1) \qquad (k \geq 1)\,,$$

and

$$\zeta'(1-2k) = (-1)^{k+1} \frac{(2k)!}{k(2\pi)^{2k}} \zeta'(2k) + \frac{B_{2k}}{2k} \left(H_{2k-1} - \gamma - \ln 2\pi\right) \qquad (k \geq 1).$$

This enable to deduce from Adamchik's formula (1) the expressions

$$A_{2k-1} = \exp\left\{(-1)^k \frac{(2k)!}{k(2\pi)^{2k}} \zeta'(2k) + \frac{B_{2k}}{2k} \left(\gamma + \ln 2\pi\right)\right\} \qquad (k \geq 1)\,, \tag{2}$$

and

$$A_{2k} = \exp\left\{(-1)^{k+1} \frac{(2k)!}{2(2\pi)^{2k}} \zeta(2k+1)\right\} \qquad (k \geq 1)\,. \tag{3}$$

Example 1. The constant

$$A_0 = \exp(-\zeta'(0)) = \sqrt{2\pi}$$

is the Stirling constant,

$$A_1 = \exp\left(\frac{1}{12} - \zeta'(-1)\right) = \exp\left(-\frac{\zeta'(2)}{2\pi^2} + \frac{\ln(2\pi) + \gamma}{12}\right)$$

is the classical Glaisher-Kinkelin constant, and for $k = 2$, we have

$$A_2 = \exp\left(-\zeta'(-2)\right) = \exp\left(\frac{\zeta(3)}{4\pi^2}\right)\,.$$

Remark 2. Bendersky [3] introduced for the first time the sequence of numbers $L_k := \ln(A_k)$ without any consideration of their relation with the ζ-function. From the point of view of the summation of divergent series, the constants L_k can be interpreted as follows: if $\sum_{n\geq 1}^{\mathcal{R}} n^k \ln n$ denotes the $\mathcal{R}$-sum of the divergent series $\sum_{n\geq 1} n^k \ln n$ (i.e. the sum of the series in the sense of Ramanujan's summation method [5]), then, for any integer $k \geq 0$, we have

$$\sum_{n\geq 1}^{\mathcal{R}} n^k \ln n = L_k - \frac{H_k B_{k+1}}{k+1} - \frac{1}{(k+1)^2} = \int_0^1 \ln \Gamma_k(x+1)\, dx\,, \tag{4}$$

where Γ_k is the Bendersky generalized gamma function [3]. This function verifies in particular

$$\Gamma_k(n+1) = 1^{1^k} 2^{2^k} \cdots n^{n^k} \qquad \text{for any integer } n \geq 1\,.$$

Apparently unaware of Bendersky's work, Kurokawa and Ochiai [9, Theorem 2] have given a deep expression of the function Γ_k in terms of the derivative of the Hurwitz zeta function $\zeta(s,x)$ at $s=-k$. Precisely, they showed that

$$\Gamma_k(x) = \exp\{\zeta'(-k,x) - \zeta'(-k)\} \qquad \text{for } x > 0\,.$$

Example 2. For the first values of k, formula (4) translates into the identities

$$\sum_{n\geq 1}^{\mathcal{R}} \ln n = \ln(\sqrt{2\pi}) - 1 = \int_0^1 \ln \Gamma(x+1)\, dx\,,$$
$$\sum_{n\geq 1}^{\mathcal{R}} n \ln n = \ln(A_1) - \frac{1}{3} = \int_0^1 \ln K(x+1)\, dx\,,$$

where $K = \Gamma_1$ is the classical Kinkelin hyperfactorial function.

3 Blagouchine's first integral

We give a direct proof of [2, Theorem 2] using Cauchy's residue theorem.

Proposition 1. For any integer $k \geq 0$, let $\mathcal{I}_k$ be the integral defined by

$$\mathcal{I}_k := \int_{-\infty}^{+\infty} \frac{\zeta(\frac{3}{2}+ix)}{(2k+1+2ix)\cosh(\pi x)}\, dx\,.$$

Then

$$\mathcal{I}_k = \mu_k \tag{5}$$

where μ_k is the conditionally convergent series defined by

$$\mu_k := \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\zeta(n+1)}{n+k}\,.$$

Proof. For $k \geq 0$, let us consider the function

$$f_k(z) = \frac{\zeta(\frac{3}{2} + iz)}{(\frac{1}{2} + k + iz)\cosh(\pi z)}\,.$$

We have $\cosh(\pi z) = 0$ if and only if $z = i/2 + in$ with $n \in \mathbb{Z}$. For $n \geq 1$, the residue of f_k at $z = i/2 - in$ is

$$\frac{\zeta(1+n)}{(n+k)\pi \sinh(i\pi(\frac{1}{2} - n))} = \frac{\zeta(1+n)}{(n+k)i\pi \sin(\pi(\frac{1}{2} - n))} = \frac{(-1)^n \zeta(1+n)}{(n+k)i\pi}\,.$$

We integrate on a closed contour composed of the interval $D_R = [-R, R]$ and the lower semicircle C_R of radius R with center at 0. By the residue theorem, we can then write the following relation:

$$\frac{1}{2i\pi}\int_{C_R} f_k(z)\,dz + \frac{1}{2i\pi}\int_{D_R} f_k(z)\,dz = -\sum_{n=1}^{N_R} \mathrm{Res}(f_k; \frac{i}{2} - in)\,,$$

which, from the foregoing, translates into the identity

$$\int_{C_R} f_k(z)\,dz + \int_{D_R} f_k(z)\,dz = 2\sum_{n=1}^{N_R} (-1)^{n+1}\frac{\zeta(1+n)}{(n+k)}\,. \tag{6}$$

For $z \in C_R$, the parameterization $iz = Re^{it}$ with $-\pi/2 < t < \pi/2$, enables us to write

$$\begin{aligned}\left|\int_{C_R} f_k(z)dz\right| &= \left|\int_{-\pi/2}^{+\pi/2} \frac{\zeta(\frac{3}{2} + Re^{it})}{(\frac{1}{2} + k + Re^{it})\cosh(i\pi Re^{it})} Re^{it}\,dt\right| \\ &\leq \int_{-\pi/2}^{+\pi/2} \left|\frac{\zeta(\frac{3}{2} + Re^{it})}{(\frac{1}{2} + k + Re^{it})\cosh(i\pi Re^{it})}\right| R\,dt\,.\end{aligned}$$

Since $\frac{3}{2} + Re^{it}$ is in the half-plane $\mathrm{Re}(z) > 3/2$, its absolute value is bounded by $\zeta(\frac{3}{2})$, i.e.

$$\left|\zeta(\frac{3}{2} + Re^{it})\right| \leq \zeta(\frac{3}{2})\,.$$

Hence, when R increases towards infinity, we have the following limits:

$$\lim_{R\to\infty} \int_{C_R} f_k(z)\,dz = 0\,,$$

$$\lim_{R\to\infty} \int_{D_R} f_k(z)\,dz = \int_{-\infty}^{+\infty} \frac{\zeta(\frac{3}{2} + ix)}{(\frac{1}{2} + k + ix)\cosh(\pi x)}\,dx = 2\,\mathcal{I}_k\,,$$

and

$$\lim_{R\to\infty} \sum_{n=1}^{N_R} (-1)^{n+1}\frac{\zeta(1+n)}{n+k} = \sum_{n=1}^{\infty} (-1)^{n+1}\frac{\zeta(n+1)}{n+k} = \mu_k\,.$$

This allows us to deduce formula (5) by passing to the limit in (6). $\square$

Remark 3. The constant

$$\mu_0 = \sum_{n=2}^{\infty} (-1)^n \frac{\zeta(n)}{n-1} = \sum_{n=1}^{\infty} \frac{1}{n} \ln\left(1 + \frac{1}{n}\right) = 1.257746\ldots$$

has been thoroughly studied by Boyadzhiev [4] (see also [7, p. 142]). This constant is noted M in [4], K in [7], and also appears in [8] as ν_{-1}. By a well-known series representation of Euler's constant γ, we also have

$$\mu_1 = \sum_{n=2}^{\infty} (-1)^n \frac{\zeta(n)}{n} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \ln\left(1 + \frac{1}{n}\right)\right) = \gamma = 0.577215\ldots .$$

Example 3. For $k = 0$, formula (5) translates into

$$\mathcal{I}_0 = \int_{-\infty}^{+\infty} \frac{\zeta(\frac{3}{2} + ix)}{(1 + 2ix)\cosh(\pi x)}\, dx = \int_0^1 \frac{\psi(x+1) + \gamma}{x}\, dx\,,$$

where ψ is the digamma function, and for $k = 1$

$$\mathcal{I}_1 = \int_{-\infty}^{+\infty} \frac{\zeta(\frac{3}{2} + ix)}{(3 + 2ix)\cosh(\pi x)}\, dx = \gamma = -\psi(1)\,.$$

4 Blagouchine's second integral

Proposition 2. For any integer $k \geq 0$, let $\mathcal{J}_k$ be the integral defined by

$$\mathcal{J}_k := \int_{-\infty}^{+\infty} \frac{\zeta(\frac{1}{2} + ix)}{(2k + 1 + 2ix)\cosh(\pi x)}\, dx$$

Then

$$\mathcal{J}_k = \frac{\gamma}{k+1} - \frac{1}{(k+1)^2} - \mathcal{I}_{k+1} \qquad (k \geq 0)\,. \tag{7}$$

Since $\mathcal{I}_1 = \gamma$, it follows from this relation that $\mathcal{J}_0 = -1$; furthermore we have the identity

$$\mathcal{J}_k = \sum_{j=0}^{k-1} (-1)^j \binom{k}{j} \ln(A_j) - \frac{k^2 + 3k + 1}{k(k+1)^2} \qquad (k \geq 1)\,. \tag{8}$$

Proof. Formula (7) is a reformulation of [2, Theorem 1]. Now assume that $k \geq 1$ and let $\nu_k := \mu_{k+1}$. By [8, Proposition 1] and (5), we have

$$\mathcal{I}_{k+1} = \nu_k = \frac{\gamma}{k+1} + \sum_{j=0}^{k-1} (-1)^j \binom{k}{j} \zeta'(-j) + \frac{1}{k} + \sum_{j=0}^{k-1} \binom{k}{j} \frac{B_{j+1}\, H_j}{j+1}\,.$$

Now, using Adamchik's formula (1), this expression may be rewritten as follows:

$$\mathcal{I}_{k+1} = \frac{\gamma}{k+1} + \frac{1}{k} - \sum_{j=0}^{k-1} (-1)^j \binom{k}{j} \ln(A_j) \qquad (k \geq 1)\,.$$

Hence, formula (8) results from (7). □

Example 4. For $k = 1$, formula (8) simply reduces to

$$\mathcal{J}_1 = \int_{-\infty}^{+\infty} \frac{\zeta(\frac{1}{2} + ix)}{(3 + 2ix)\cosh(\pi x)}\, dx = \frac{1}{2}\ln(2\pi) - \frac{5}{4}\,,$$

and for $k = 2$,

$$\mathcal{J}_2 = \int_{-\infty}^{+\infty} \frac{\zeta(\frac{1}{2} + ix)}{(5 + 2ix)\cosh(\pi x)}\, dx = \frac{1}{3}\ln(2\pi) - \frac{1}{6}\gamma + \frac{\zeta'(2)}{\pi^2} - \frac{11}{19}\,.$$

Remark 4. The following general expression can be easily deduced from formulas (2) and (3):

$$\begin{aligned}\sum_{j=0}^{k-1} (-1)^j \binom{k}{j} \ln(A_j) &= \frac{\ln(2\pi)}{k+1} - \frac{k-1}{k+1}\frac{\gamma}{2} \\ &\quad - \sum_{j=1}^{[\frac{k}{2}]} (-1)^j \binom{k}{2j-1} \frac{(2j)!}{j(2\pi)^{2j}}\zeta'(2j) \\ &\quad - \sum_{j=1}^{[\frac{k-1}{2}]} (-1)^j \binom{k}{2j} \frac{(2j)!}{2(2\pi)^{2j}}\zeta(2j+1) \qquad (k \geq 3)\,. \qquad (9)\end{aligned}$$

5 Further generalization

Using Fourier transform method, Candelpergher [6, Eq. (7)] recently proved the following beautiful relation which generalizes (7):

Theorem 1. for $k \geq 0$ and $\mathrm{Re}(s) > \frac{1}{2}$, we have

$$2^{s-1}\mathcal{J}_k(s) = \frac{\gamma}{(k+1)^s} - \frac{s}{(k+1)^{s+1}} - \nu_k(s)\,, \qquad (10)$$

with

$$\mathcal{J}_k(s) := \int_{-\infty}^{+\infty} \frac{\zeta(\frac{1}{2} + ix)}{(2k + 1 + 2ix)^s \cosh(\pi x)}\, dx$$

and

$$\nu_k(s) := \sum_{n=2}^{\infty} (-1)^n \frac{\zeta(n)}{(n+k)^s}\,.$$

Applying (10) with $k=0$ allows us to deduce the following identity:

Corollary 1.

$$\sum_{n=2}^{\infty}(-1)^n\frac{\zeta(n)}{n^s}=\gamma-s-\frac{1}{2}\int_{-\infty}^{+\infty}\frac{\zeta(\frac{1}{2}+ix)}{(\frac{1}{2}+ix)^s\cosh(\pi x)}\,dx \qquad (\mathrm{Re}(s)>\frac{1}{2})\,. \tag{11}$$

Example 5.

$$\sum_{n=2}^{\infty}(-1)^n\frac{\zeta(n)}{n^2}=\gamma-2-\frac{1}{2}\int_{-\infty}^{+\infty}\frac{\zeta(\frac{1}{2}+ix)}{(\frac{1}{2}+ix)^2\cosh(\pi x)}\,dx\,.$$

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