Remarks on a formula of Blagouchine

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Abstract We make some comments on an amazing formula recently discovered by Blagouchine.

Keywords Complex integration, generalized Glaisher-Kinkelin constants, infinite series with zeta values.

1 Introduction

The aim of this short note is to emphasize the link between the sequence of integrals $\{\mathcal{I}_k\}_{k\geq 0}$ defined by

$$\mathcal{I}_k = \int_{-\infty}^{+\infty} \frac{\zeta(\frac{3}{2}+ix)}{(2k+1+2ix)\cosh(\pi x)}\,dx$$

and some important mathematical constants, namely the Euler-Mascheroni constant, the Cohen-Boyadzhiev constant, and the generalized Glaisher-Kinkelin constants (also known as the Bendersky constants) which occur quite naturally in analysis and number theory [7, 10, 11]. In order to do this, we make use of an amazing formula recently discovered by Blagouchine [2, Theorem 2] that we combine with another nice formula given by the author in a previous article ([8, Proposition 1]). Let us note in passing that the following special case of Blagouchine's formula:

$$\mathcal{I}_0 = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{\zeta(n+1)}{n}$$

has already been mentioned (without proof) on page 1836 of [8].

2 Blagouchine's integral

First of all, we provide a complete simple proof of [2, Theorem 2] using Cauchy's residue theorem.

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Proposition 1. For any integer $k \geq 0$, let $\mathcal{I}_k$ be the complex valued integral

$$\mathcal{I}_k := \int_{-\infty}^{+\infty} \frac{\zeta(\frac{3}{2}+ix)}{(2k+1+2ix)\cosh(\pi x)}\,dx$$

and μ_k be the infinite sum

$$\mu_k := \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\zeta(n+1)}{n+k}\,.$$

Then we have the identity

$$\mathcal{I}_k = \mu_k\,. \tag{1}$$

Proof. For $k \geq 0$, let us consider the function

$$f_k(z) = \frac{\zeta(\frac{3}{2}+iz)}{(\frac{1}{2}+k+iz)\cosh(\pi z)}\,.$$

We have $\cosh(\pi z) = 0$ if and only if $z = i/2 + in$ with $n \in \mathbb{Z}$. For $n \geq 1$, the residue of f_k at $z = i/2 - in$ is

$$\frac{\zeta(1+n)}{(n+k)\pi \sinh(i\pi(\frac{1}{2}-n))} = \frac{\zeta(1+n)}{(n+k)i\pi \sin(\pi(\frac{1}{2}-n))} = \frac{(-1)^n \zeta(1+n)}{(n+k)i\pi}\,.$$

We integrate on a closed contour composed of the interval $D_R = [-R, R]$ and the lower semicircle C_R of radius R with center at 0. By the residue theorem, we can then write the following relation:

$$\frac{1}{2i\pi}\int_{C_R} f_k(z)\,dz + \frac{1}{2i\pi}\int_{D_R} f_k(z)\,dz = -\sum_{n=1}^{N_R} \mathrm{Res}(f_k; \frac{i}{2} - in)\,,$$

which, from the foregoing, translates into the identity

$$\int_{C_R} f_k(z)\,dz + \int_{D_R} f_k(z)\,dz = 2\sum_{n=1}^{N_R} (-1)^{n+1} \frac{\zeta(1+n)}{(n+k)}\,. \tag{2}$$

For $z \in C_R$, the parameterization $iz = Re^{it}$ with $-\pi/2 < t < \pi/2$, enables us to write

$$\left|\int_{C_R} f_k(z)dz\right| = \left|\int_{-\pi/2}^{+\pi/2} \frac{\zeta(\frac{3}{2}+Re^{it})}{(\frac{1}{2}+k+Re^{it})\cosh(i\pi Re^{it})} Re^{it}\,dt\right|$$
$$\leq \int_{-\pi/2}^{+\pi/2} \left|\frac{\zeta(\frac{3}{2}+Re^{it})}{(\frac{1}{2}+k+Re^{it})\cosh(i\pi Re^{it})}\right| R\,dt\,.$$

Since $\frac{3}{2} + Re^{it}$ is in the half-plane $\mathrm{Re}(z) > 3/2$, its absolute value is bounded by $\zeta(\frac{3}{2})$, i.e.

$$\left|\zeta(\frac{3}{2} + Re^{it})\right| \le \zeta(\frac{3}{2})\,.$$

Hence, when R increases towards infinity, we have the following limits:

$$\lim_{R\to\infty} \int_{C_R} f_k(z)\,dz = 0\,,$$

$$\lim_{R\to\infty} \int_{D_R} f_k(z)\,dz = \int_{-\infty}^{+\infty} \frac{\zeta(\frac{3}{2}+ix)}{(\frac{1}{2}+k+ix)\cosh(\pi x)}\,dx = 2\,\mathcal{I}_k\,,$$

and

$$\lim_{R\to\infty} \sum_{n=1}^{N_R} (-1)^{n+1} \frac{\zeta(1+n)}{(n+k)} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\zeta(1+n)}{(n+k)} = \mu_k\,.$$

This allows us to deduce formula (1) by passing to the limit in (2). □

Remark 1. The constant

$$\mu_0 = \sum_{n=2}^{\infty} (-1)^n \frac{\zeta(n)}{n-1} = \sum_{n=1}^{\infty} \frac{1}{n} \ln\left(1 + \frac{1}{n}\right) = 1.257746\ldots$$

has been thoroughly studied by Boyadzhiev [4] (see also [7, p. 142]). This constant is noted M in [4], K in [7], and also appears as ν_{-1} in [8]. By a well-known series representation of Euler's constant γ, we also have

$$\mu_1 = \sum_{n=2}^{\infty} (-1)^n \frac{\zeta(n)}{n} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \ln\left(1 + \frac{1}{n}\right)\right) = \gamma = 0.577215\ldots\,.$$

Example 1. For $k = 0$ and $k = 1$ respectively, formula (1) translates into

$$\mathcal{I}_0 = \int_{-\infty}^{+\infty} \frac{\zeta(\frac{3}{2}+ix)}{(1+2ix)\cosh(\pi x)}\,dx = \int_0^1 \frac{\psi(x+1)+\gamma}{x}\,dx\,, \tag{3}$$

where ψ is the digamma function, and

$$\mathcal{I}_1 = \int_{-\infty}^{+\infty} \frac{\zeta(\frac{3}{2}+ix)}{(3+2ix)\cosh(\pi x)}\,dx = \gamma = -\psi(1)\,. \tag{4}$$

3 Link with the generalized Glaisher-Kinkelin constants

Definition 1 ([1, 10, 11])**.** For any integer $k \geq 0$, the constant A_k is usually defined by

$$\ln(A_k) = \lim_{N\to\infty} \left\{ \sum_{n=1}^{N} n^k \ln n - P_k(N) \right\} ,$$

where $P_k(N)$ is given by $P_0(N) = \left(N + \frac{1}{2}\right) \ln N - N$, and

$$\begin{aligned} P_k(N) = &\left(\frac{N^{k+1}}{k+1} + \frac{N^k}{2} + k! \sum_{j=1}^{k} \frac{N^{k-j} B_{j+1}}{(j+1)!(k-j)!} \right) \ln N \\ &- \frac{N^{k+1}}{(k+1)^2} + k! \sum_{j=1}^{k} \frac{N^{k-j} B_{j+1}}{(j+1)!(k-j)!} \left\{ (1-\delta_{k,j}) \sum_{i=1}^{j} \frac{1}{k-i+1} \right\} \qquad (k \geq 1)\,, \end{aligned}$$

where B_j is the j-th Bernoulli number and $\delta_{k,j}$ is the Kronecker delta function. The numbers A_k (for $k = 0, 1, 2, \ldots$) are the *generalized Glaisher-Kinkelin constants* (sometimes called the *Bendersky constants*). In particular, it follows from this definition that

$$\ln(A_0) = \lim_{N\to\infty} \left\{ \sum_{n=1}^{N} \ln n - \left(N + \frac{1}{2}\right) \ln N + N \right\} ,$$

and

$$\ln(A_1) = \lim_{N\to\infty} \left\{ \sum_{n=1}^{N} n \ln n - \left(\frac{N^2}{2} + \frac{N}{2} + \frac{1}{12} \right) \ln N + \frac{N^2}{4} \right\} .$$

Remark 2. Adamchik [1, Proposition 4] has given a nice expression of theses constants in terms of the derivatives of the Riemann zeta function. More precisely, he showed that

$$A_k = \exp\left\{ \frac{H_k B_{k+1}}{k+1} - \zeta'(-k) \right\} \qquad (k \geq 0)\,, \tag{5}$$

where $H_k = \sum_{j=1}^{k} \frac{1}{j}$ is the k-th harmonic number with the usual convention $H_0 = 0$. The following relations are easily deduced by differentiation of Riemann's functional equation for the zeta function:

$$\zeta'(-2k) = (-1)^k \frac{(2k)!}{2(2\pi)^{2k}} \zeta(2k+1) \qquad (k \geq 1)\,,$$

and

$$\zeta'(1-2k) = (-1)^{k+1}\frac{(2k)!}{k(2\pi)^{2k}}\zeta'(2k) + \frac{B_{2k}}{2k}\left(H_{2k-1} - \gamma - \ln 2\pi\right) \qquad (k \geq 1).$$

This enable to deduce from Adamchik's formula (5) the expressions

$$A_{2k-1} = \exp\left\{(-1)^k\frac{(2k)!}{k(2\pi)^{2k}}\zeta'(2k) + \frac{B_{2k}}{2k}\left(\gamma + \ln 2\pi\right)\right\} \qquad (k \geq 1)\,, \tag{6}$$

and

$$A_{2k} = \exp\left\{(-1)^{k+1}\frac{(2k)!}{2(2\pi)^{2k}}\zeta(2k+1)\right\} \qquad (k \geq 1)\,. \tag{7}$$

Example 2. The constant $A_0 = \exp(-\zeta'(0)) = \sqrt{2\pi}$ is the Stirling constant,

$$A_1 = \exp\left(\frac{1}{12} - \zeta'(-1)\right) = \exp\left(-\frac{\zeta'(2)}{2\pi^2} + \frac{\ln(2\pi)+\gamma}{12}\right)$$

is the Glaisher-Kinkelin constant, and

$$A_2 = \exp\left(-\zeta'(-2)\right) = \exp\left(\frac{\zeta(3)}{4\pi^2}\right)\,.$$

Remark 3. Bendersky [3] introduced for the first time the sequence of numbers $L_k := \ln(A_k)$ without any consideration of their relation with the ζ-function. From the point of view of the summation of divergent series, the constants L_k should be interpreted as follows: if $\sum_{n\geq 1}^{\mathcal{R}} n^k \ln n$ denotes the $\mathcal{R}$-sum (i.e. the sum in the sense of Ramanujan's summation method [5]) of the divergent series $\sum_{n\geq 1} n^k \ln n$, then, for any integer $k \geq 0$, we have

$$\sum_{n\geq 1}^{\mathcal{R}} n^k \ln n = L_k - \frac{H_k B_{k+1}}{k+1} - \frac{1}{(k+1)^2} = \int_0^1 \ln\Gamma_k(x+1)\,dx\,, \tag{8}$$

where Γ_k is the Bendersky generalized gamma function [3]. In particular, this function verifies

$$\Gamma_k(n+1) = 1^{1^k}2^{2^k}\cdots n^{n^k} \qquad \text{for any integer } n \geq 1\,.$$

Unaware of Bendersky's work, Kurokawa and Ochiai [9, Theorem 2] have given a deep expression of the function Γ_k in terms of the derivative of the Hurwitz zeta function $\zeta(s,x)$ at $s=-k$. Precisely, they showed that

$$\Gamma_k(x) = \exp\left\{\zeta'(-k,x) - \zeta'(-k)\right\} \qquad \text{for } x > 0\,.$$

For the first values of k, formula (8) translates into the identities

$$\sum_{n\geq 1}^{\mathcal{R}} \ln n = \ln(\sqrt{2\pi}) - 1 = \int_0^1 \ln\Gamma(x+1)\,dx\,,$$
$$\sum_{n\geq 1}^{\mathcal{R}} n\ln n = \ln(A_1) - \frac{1}{3} = \int_0^1 \ln K(x+1)\,dx\,,$$

where $K = \Gamma_1$ is the Kinkelin hyperfactorial function.

Proposition 2. For any integer $k \geq 1$, we have the following identity:

$$\mathcal{I}_{k+1} = \frac{\gamma}{k+1} + \frac{1}{k} - \sum_{j=0}^{k-1}(-1)^j\binom{k}{j}\ln(A_j)\,. \tag{9}$$

Proof. We have shown [8, Proposition 1] that

$$\mu_{k+1} = \frac{\gamma}{k+1} + \sum_{j=0}^{k-1}(-1)^j\binom{k}{j}\zeta'(-j) + \frac{1}{k} + \sum_{j=0}^{k-1}\binom{k}{j}\frac{B_{j+1}\,H_j}{j+1} \qquad (k\geq 1)\,.$$

By means of (5), this expression may be rewritten as follows:

$$\mu_{k+1} = \frac{\gamma}{k+1} + \frac{1}{k} - \sum_{j=0}^{k-1}(-1)^j\binom{k}{j}\ln(A_j) \qquad (k\geq 1)\,.$$

Hence, formula (9) results from (1). □

From (9) and formulas (6)–(7), we deduce the following identities:

Corollary 1. For $k = 2, 3$, we have

$$\mathcal{I}_2 = \int_{-\infty}^{+\infty}\frac{\zeta(\frac{3}{2}+ix)}{(5+2ix)\cosh(\pi x)}\,dx = \frac{1}{2}\gamma + 1 - \frac{1}{2}\ln(2\pi)\,, \tag{10}$$
$$\mathcal{I}_3 = \int_{-\infty}^{+\infty}\frac{\zeta(\frac{3}{2}+ix)}{(7+2ix)\cosh(\pi x)}\,dx = \frac{1}{2}\gamma + \frac{1}{2} - \frac{1}{3}\ln(2\pi) - \frac{\zeta'(2)}{\pi^2}\,, \tag{11}$$

and for any integer $k \geq 4$, we have the general formula:

$$\mathcal{I}_k = \int_{-\infty}^{+\infty}\frac{\zeta(\frac{3}{2}+ix)}{(2k+1+2ix)\cosh(\pi x)}\,dx = \frac{1}{2}\gamma + \frac{1}{k-1} - \frac{1}{k}\ln(2\pi)$$
$$+ \sum_{j=1}^{[\frac{k-1}{2}]}(-1)^j\binom{k-1}{2j-1}\frac{(2j)!}{j(2\pi)^{2j}}\zeta'(2j) + \sum_{j=1}^{[\frac{k}{2}]-1}(-1)^j\binom{k-1}{2j}\frac{(2j)!}{2(2\pi)^{2j}}\zeta(2j+1)\,. \tag{12}$$

Example 3. In particular, for $k = 4$, formula (12) translates into

$$\mathcal{I}_4 = \int_{-\infty}^{+\infty} \frac{\zeta(\frac{3}{2}+ix)}{(9+2ix)\cosh(\pi x)}\,dx = \frac{1}{2}\gamma + \frac{1}{3} - \frac{1}{4}\ln(2\pi) - \frac{3\zeta'(2)}{2\pi^2} - \frac{3\zeta(3)}{4\pi^2}\,.$$

Remark 4. Using a Fourier transform, Candelpergher [6, Eq. (7)] recently proved the following beautiful relation which generalizes [2, Theorem 1]: for $\mathrm{Re}(s) > \frac{1}{2}$ and $k \geq 0$, we have

$$2^{s-1}\int_{-\infty}^{+\infty} \frac{\zeta(\frac{1}{2}+ix)}{(2k+1+2ix)^s\cosh(\pi x)}\,dx = \frac{\gamma}{(k+1)^s} - \frac{s}{(k+1)^{s+1}} - \mu_{k+1}(s)\,, \quad (13)$$

with

$$\mu_k(s) := \sum_{n=1}^{\infty}(-1)^{n-1}\frac{\zeta(n+1)}{(n+k)^s}\,.$$

In particular, since $\mu_{k+1}(1) = \mathcal{I}_{k+1}$, it follows from (13) and (9) that

$$\int_{-\infty}^{+\infty} \frac{\zeta(\frac{1}{2}+ix)}{(2k+1+2ix)\cosh(\pi x)}\,dx = \sum_{j=0}^{k-1}(-1)^j\binom{k}{j}\ln(A_j) - \frac{k^2+3k+1}{k(k+1)^2} \qquad (k \geq 1). \quad (14)$$

In the case $k = 1$, this formula simply reduces to

$$\int_{-\infty}^{+\infty} \frac{\zeta(\frac{1}{2}+ix)}{(3+2ix)\cosh(\pi x)}\,dx = \ln(\sqrt{2\pi}) - \frac{5}{4}\,.$$

Example 4. By applying (13) with $k = 0$, we obtain the following identity:

$$\sum_{n=2}^{\infty}(-1)^n\frac{\zeta(n)}{n^s} = \gamma - s - \frac{1}{2}\int_{-\infty}^{+\infty} \frac{\zeta(\frac{1}{2}+ix)}{(\frac{1}{2}+ix)^s\cosh(\pi x)}\,dx \qquad (\mathrm{Re}(s) > \frac{1}{2})\,. \quad (15)$$

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