On some formulae related to Euler sums

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Abstract Using the Ramanujan summation method, we derive some unusual formulas for a class of Euler sums (including divergent Euler sums) similar to the classical relations due to Euler.

Keywords Euler sums, analytic continuation, Ramanujan summation of series, harmonic numbers, series with zeta values.

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Introduction

The study of Euler sums has a fairly long history dating back to the middle of the 18th century. In response to a letter from Goldbach dated from december 1742, Euler considered infinite sums of the form

$$\mathcal{S}_{p,q} = \sum_{n=1}^{\infty} \frac{H_n^{(p)}}{n^q} \, ,$$

where p and q are positive integers, and $H_n^{(p)} = \sum_{k=1}^{n} \frac{1}{k^p}$ are generalized harmonic numbers. For $p = 1$, the generalized harmonic numbers reduce to classical harmonic numbers $H_n = H_n^{(1)}$. The importance of harmonic numbers comes from the fact that they appear (sometimes quite unexpectedly) in different branches of number theory and combinatorics. In our times, the sums $\mathcal{S}_{p,q}$ are called the *linear Euler sums*. Euler discovered that for all pairs (p, q) with $p = 1$, or $p = q$, or $p + q$

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odd, these sums have expressions in terms of zeta values (i.e. the values of the Riemann zeta function $\zeta(s)=\sum_{n\geq 1} n^{-s}$ at positive integers), a remarkable result that will be also found and completed later by Nielsen [11, pp. 47–51]. Among the beautiful formulas discovered by Euler [10], the following two are particularly noteworthy:

- Euler's reciprocity formula:
$$\mathcal{S}_{p,q}+\mathcal{S}_{q,p}=\zeta(p)\zeta(q)+\zeta(p+q) \qquad (p,q\geq 2),$$
called "prima methodus", that allows to express $\mathcal{S}_{q,p}$ as a function of $\mathcal{S}_{p,q}$ and vice versa. In the particular case $p=q$, it results from this formula that
$$\mathcal{S}_{p,p}=\frac{1}{2}\left\{(\zeta(p))^2+\zeta(2p)\right\} ;$$
- Euler's formula: $\mathcal{S}_{1,2}=2\zeta(3)$, and
$$2\mathcal{S}_{1,p}=(p+2)\zeta(p+1)-\sum_{j=1}^{p-2}\zeta(p-j)\zeta(j+1) \qquad (p>2),$$
that Euler derives from his "secunda methodus", this famous formula will be several times rediscovered throughout the 20th century (see [13, Remark 3.1] for historical details).

Ramanujan's method of summation of series appears in Chapter VI of Ramanujan's second notebook [12]. Because of the ambiguities (observed by Hardy in Chapter XIII of his classical treatise on divergent series) contained in the definition of the "constant of a series" that made its use very tricky, Ramanujan's method, based on the Euler-Maclaurin summation formula, had fallen into neglect. This method has known a revival of interest at the end of the 20th century when a clear and rigorous definition of the sum of a series in the sense of Ramanujan summation was given by Candelpergher et al. [5] at the same time as the link with the usual summation was completely clarified. The reader will find in the recent monograph [2] a masterful synthesis of main definitions, fundamental properties, and scope of application of the Ramanujan summation.

Ramanujan's method is particulary well suited to linear Euler sums because it allows to easily handle both the convergence case and the divergence case; however this process of regularization is unusual and remains little known. In the remainder of this article, we consider the sum in the sense of Ramanujan summation corresponding to $\mathcal{S}_{p,q}$. To avoid confusion, this sum is noted $\mathcal{R}_{p,q}$, where the letter $\mathcal{R}$ is reminiscent of Ramanujan summation (see Definition 1). A complete evaluation of the sums $\mathcal{R}_{1,p}$, $\mathcal{R}_{p,1}$, and $\mathcal{R}_{p,p}$ is then given for each positive integer p. This allows us to provide a number of relations similar (though more complicated) to the classical relations mentioned above (see Propositions 1 to 4). Several interesting applications of these formulas are given in [4] and [8].

1 Ramanujan summation of Euler sums

Let us recall that the generalized harmonic numbers $H_n^{(p)}$ are defined for integers $n \geq 1$ and $p \geq 1$ by

$$H_n^{(p)} = \sum_{k=1}^{n} \frac{1}{k^p} .$$

When $p = 1$, they reduce to classical harmonic numbers denoted $H_n = H_n^{(1)}$. It is convenient to express them in the following form:

$$H_n^{(p)} = \frac{(-1)^{p-1}}{(p-1)!} \partial^{p-1} \psi(n+1) + \zeta(p) \qquad (p \geq 2) ,$$

and

$$H_n = \psi(n+1) + \gamma ,$$

where $\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}$ is the digamma function and $\gamma = -\psi(1)$ is the Euler constant [6, p. 95].

Definition 1. For any positive integer p, the function $s \mapsto \mathcal{Z}(p,s)$ is defined as the analytic continuation of the function defined in the half-plane $\text{Re}(s) > 1$ by

$$\mathcal{Z}(p,s) = \sum_{n=1}^{\infty} H_n^{(p)} \, n^{-s} - \int_1^{\infty} \psi_p(x) \, x^{-s} \, dx ,$$

where $\psi_1(x) = \psi(x+1) + \gamma$, and

$$\psi_p(x) = \frac{(-1)^{p-1}}{(p-1)!} \partial^{p-1} \psi(x+1) + \zeta(p) \qquad (p \geq 2) .$$

It follows from [2, Thm. 9] that this function can be analytically continued as an entire function in the whole $\mathbb{C}$. For each integer $q \in \mathbb{Z}$, $\mathcal{R}_{p,q}$ is defined by

$$\mathcal{R}_{p,q} := \mathcal{Z}(p,q) .$$

The value $\mathcal{R}_{p,q}$ is thus well-defined and may be interpreted as the $\mathcal{R}$-sum (i.e. the sum in the sense of Ramanujan summation) of the (possibly divergent) series $\sum_{n \geq 1} H_n^{(p)} \, n^{-q}$. Therefore, with the notations of [2], $\mathcal{R}_{p,q}$ is nothing else than $\sum_{n \geq 1}^{\mathcal{R}} \frac{H_n^{(p)}}{n^q}$.

Example 1 (values of $\mathcal{R}_{p,q}$ at $q = 0$)**.** It results from [2, p. 44] that

$$\mathcal{R}_{p,0} = \begin{cases} \frac{3}{2}\gamma - \frac{1}{2}\ln(2\pi) + \frac{1}{2} & \text{if } p = 1 , \\ \frac{3}{2}\zeta(2) - 2 & \text{if } p = 2 , \\ \frac{3}{2}\zeta(p) - \frac{p-2}{p-1}\zeta(p-1) - \frac{1}{p-1} & \text{if } p > 2 . \end{cases}$$

2 Evaluation of the sum $\mathcal{R}_{1,p}$

Let ζ_H be the harmonic zeta function [4, 7] defined for $\mathrm{Re}(s) > 1$ by

$$\zeta_H(s) = \sum_{n=1}^{\infty} H_n\, n^{-s}\,.$$

According to Definition 1, $\mathcal{Z}(1,s)$ is closely linked to $\zeta_H(s)$ through the relation

$$\mathcal{Z}(1,s) = \zeta_H(s) - \int_1^{\infty} x^{-s}\,(\psi(x+1)+\gamma)\;dx \qquad (\mathrm{Re}(s) > 1)\,.$$

In particular, since

$$\zeta_H(p) = \mathcal{S}_{1,p} \qquad (p \geq 2)\,,$$

it follows that

$$\mathcal{R}_{1,p} := \mathcal{Z}(1,p) = \mathcal{S}_{1,p} - \int_1^{\infty} \frac{\psi(x+1)+\gamma}{x^p}\,dx \qquad (p \geq 2)\,. \tag{1}$$

Definition 2. For any positive integer p, let τ_p be the real constant defined by the series representation

$$\tau_p := \sum_{k=1}^{\infty} (-1)^{k+p}\,\frac{\zeta(k+p)}{k}\,. \tag{2}$$

Remark 1. The sequence $\{\tau_p\}_p$ appears in [4] and [9]. The constant τ_1 has been thoroughly studied by Boyadzhiev [1] (see also [6, Ex. 92, p. 142]).

To give a better expression of the $\mathcal{R}$-sum $\mathcal{R}_{1,p}$, we first prove the following lemma:

Lemma 1. For $p > 2$, we have the relation

$$(-1)^p \int_1^{\infty} \frac{\psi(x+1)+\gamma}{x^p}\,dx = \sum_{j=1}^{p-2} \frac{(-1)^j}{j}\zeta(p-j) - \zeta'(p) - (-1)^p\tau_p\,. \tag{3}$$

For $p = 2$, this relation reduces to

$$\int_1^{\infty} \frac{\psi(x+1)+\gamma}{x^2}\,dx = -\zeta'(2) - \tau_2\,. \tag{4}$$

For $p = 1$, we have the identity

$$\int_0^{1} \frac{\psi(x+1)+\gamma}{x}\,dx = \tau_1.$$

Proof. For $p \geq 2$, the convergent series $\sum_{n\geq 1} \frac{\ln(n+1)}{n^p}$ may be splitted into the two series

$$\sum_{n=1}^{\infty} \frac{\ln(n+1)}{n^p} = \sum_{n=1}^{\infty} \frac{\ln(n)}{n^p} + \sum_{n=1}^{\infty} \frac{1}{n^p} \ln\left(1+\frac{1}{n}\right) .$$

The well-known expansion of $\ln(1+1/n)$ in power series leads to the identity

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \ln\left(1+\frac{1}{n}\right) = \sum_{n=1}^{\infty} \frac{1}{n^p}\left[\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}\left(\frac{1}{n}\right)^k\right] = (-1)^{p-1}\,\tau_p\,,$$

then it follows that

$$\sum_{n=1}^{\infty} \frac{\ln(n+1)}{n^p} = -\zeta'(p) - (-1)^p\,\tau_p\,. \tag{5}$$

On the other side, the partial fraction expansion

$$\frac{(-1)^p}{x^{p-1}(x+y)} = \sum_{j=0}^{p-2} \frac{(-1)^j}{x^{j+1}\,y^{p-1-j}} - \frac{1}{y^{p-1}(x+y)} \qquad (p \geq 2),$$

leads, after integration, to the formula

$$(-1)^p \int_1^{\infty} \frac{dx}{x^{p-1}(x+n)} = \sum_{j=1}^{p-2} \frac{(-1)^j}{j\,n^{p-1-j}} + \frac{\ln(n+1)}{n^{p-1}} \qquad (p > 2)\,,$$

valid for any positive integer n. After division by n, it can be rewritten as

$$\frac{\ln(n+1)}{n^p} = \sum_{j=1}^{p-2} \frac{(-1)^{j-1}}{j\,n^{p-j}} + (-1)^p \int_1^{\infty} \frac{x}{x^p\,n(x+n)}\,dx\,.$$

By summing this identity, we then obtain

$$\sum_{n=1}^{\infty} \frac{\ln(n+1)}{n^p} = \sum_{j=1}^{p-2} \frac{(-1)^{j-1}}{j}\zeta(p-j) + (-1)^p \int_1^{\infty} \frac{\psi(x+1)+\gamma}{x^p}\,dx \qquad (p > 2)\,. \tag{6}$$

Hence, formula (3) follows from (5) and (6) by substitution. For $p = 2$, it reduces to

$$\int_1^{\infty} \frac{\psi(x+1)+\gamma}{x^2}\,dx = \sum_{n=1}^{\infty} \frac{\ln(n+1)}{n^2} = -\zeta'(2) - \tau_2\,.$$

In the case $p = 1$, the well-known Taylor expansion

$$\psi(x+1) + \gamma = \sum_{n=1}^{\infty} (-1)^{n-1}\zeta(n+1)x^n \qquad (|x| < 1)$$

gives (after division of both sides by x and integration from 0 to 1)

$$\int_0^1 \frac{\psi(x+1)+\gamma}{x}\,dx = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{\zeta(n+1)}{n} = \tau_1.$$

□

Proposition 1. *For any positive integer $p > 2$, we have*

$$\mathcal{R}_{1,p} = \mathcal{S}_{1,p} - \sum_{j=1}^{p-2}\frac{(-1)^{p-j}}{j}\,\zeta(p-j) + (-1)^p\zeta'(p) + \tau_p\,, \tag{7}$$

where τ_p is defined by formula (2). For $p = 2$, it reduces to

$$\mathcal{R}_{1,2} = \mathcal{S}_{1,2} + \zeta'(2) + \tau_2 = 2\zeta(3) + \zeta'(2) + \tau_2\,.$$

Moreover, for $p \geq 2$, we have the formula

$$\mathcal{R}_{1,2p} = (p+1)\zeta(2p+1) - \sum_{j=1}^{p-1}\zeta(2p-j)\zeta(j+1) - \sum_{j=1}^{2p-2}\frac{(-1)^j}{j}\zeta(2p-j) + \zeta'(2p) + \tau_{2p}\,. \tag{8}$$

Proof. On one side, by formula (1), we have the relation

$$\mathcal{R}_{1,p} = \mathcal{S}_{1,p} - \int_1^{\infty}\frac{\psi(x+1)+\gamma}{x^p}\,dx\,,$$

and, on the other side, by Lemma 1, we have

$$\int_1^{\infty}\frac{\psi(x+1)+\gamma}{x^p}\,dx = -(-1)^p\zeta'(p) - \tau_p + \sum_{j=1}^{p-2}\frac{(-1)^{p-j}}{j}\,\zeta(p-j)\,.$$

Hence, formula (7) is immediately deduced from (1) and (3) by substitution, and (8) results from (7) and the following expression of $\mathcal{S}_{1,2p}$ given by Euler's formula [13, Thm. 3.1]:

$$\mathcal{S}_{1,2p} = (p+1)\zeta(2p+1) - \sum_{j=1}^{p-1}\zeta(2p-j)\zeta(j+1) \qquad (p > 1)\,. \tag{9}$$

□

Example 2.

$$\begin{aligned}
\mathcal{R}_{1,2} &= 2\zeta(3) + \zeta'(2) + \tau_2\,,\\
\mathcal{R}_{1,4} &= 3\zeta(5) - \zeta(3)\zeta(2) + \zeta(3) - \frac{1}{2}\zeta(2) + \zeta'(4) + \tau_4\,,\\
\mathcal{R}_{1,6} &= 4\zeta(7) - \zeta(3)\zeta(4) - \zeta(2)\zeta(5) + \zeta(5) - \frac{1}{2}\zeta(4) + \frac{1}{3}\zeta(3) - \frac{1}{4}\zeta(2)\\
&\quad + \zeta'(6) + \tau_6\,.
\end{aligned}$$

Remark 2. We point out here the analogy between our formula for $\mathcal{R}_{1,2p}$ and the "dual" formula

$$\mathcal{R}_{1,-2p} = \frac{1-2p}{2}\zeta(1-2p) + \zeta'(-2p) + \nu_{2p} \qquad (p \geq 1)$$

given in [7, Eq. (8)], where ν_p is defined by the series representation

$$\nu_p = \sum_{k=2}^{\infty} (-1)^k \frac{\zeta(k)}{k+p} \qquad (p \geq 1)\,.$$

3 Evaluation of the sum $\mathcal{R}_{p,1}$

We now give an evaluation of the "reciprocal" sum $\mathcal{R}_{p,1}$ corresponding to the divergent Euler sum $\mathcal{S}_{p,1}$.

Definition 3. Let σ_p defined by $\sigma_2 = 1$, and

$$\sigma_p = \frac{1+(-1)^p}{p} + \sum_{j=1}^{p-2} (-1)^j \zeta(p-j) \left[\frac{(j-1)!(p-1-j)!}{(p-1)!} - \frac{1}{j} \right] \qquad (p > 2)\,. \quad (10)$$

Example 3. The first values of σ_p are

$$\begin{aligned}
\sigma_2 &= 1\,,\\
\sigma_3 &= \frac{1}{2}\zeta(2)\,,\\
\sigma_4 &= \frac{2}{3}\zeta(3) - \frac{1}{3}\zeta(2) + \frac{1}{2}\,,\\
\sigma_5 &= \frac{3}{4}\zeta(4) - \frac{5}{12}\zeta(3) + \frac{1}{4}\zeta(2)\,,\\
\sigma_6 &= \frac{4}{5}\zeta(5) - \frac{9}{20}\zeta(4) + \frac{9}{10}\zeta(3) - \frac{1}{5}\zeta(2) + \frac{1}{3}\,.
\end{aligned}$$

Remark 3. Another interesting expression of σ_p is given by [3, Eq. (27)]. Let

$$Z(i,j) = \sum_{n=1}^{\infty} \frac{1}{n^i\,(n+1)^j} \qquad (i, j \geq 1)\,.$$

The partial fraction expansion of $\frac{1}{n^i\,(n+1)^j}$ shows that $Z(i,j)$ is a $\mathbb{Z}$-linear combination of zeta values and integers, and it results from [3, Eq. (27)] that

$$\sigma_p = \sum_{i+j=p} \frac{1}{j}\, Z(i,j) \qquad (p \geq 2)\,.$$

Proposition 2. *For any positive integer $p \geq 2$, we have*

$$\mathcal{R}_{p,1} = \gamma\zeta(p) + \zeta(p+1) - \mathcal{S}_{1,p} - \sigma_p - \zeta'(p) - (-1)^p \tau_p \,, \tag{11}$$

where τ_p and σ_p are respectively defined by formulas (2) *and* (10)*. Moreover, for $p \geq 2$, we have the formula*

$$\mathcal{R}_{2p,1} = \gamma\zeta(2p) - p\,\zeta(2p+1) + \sum_{j=1}^{p-1} \zeta(2p-j)\zeta(j+1) - \sigma_{2p} - \zeta'(2p) - \ \tau_{2p} \,. \tag{12}$$

Proof. By summing (in the sense of Ramanujan summation) the following equations :

$$\frac{H_n^{(p)}}{n} - \frac{1}{n}\zeta(p) = -\frac{1}{n}\sum_{m=n+1}^{\infty} \frac{1}{m^p} = \frac{1}{n}\frac{(-1)^{p-1}}{(p-1)!}\partial^{p-1}\psi(n+1) \,,$$

we get

$$\begin{aligned} \sum_{n\geq 1}^{\mathcal{R}} \left(\frac{H_n^{(p)}}{n} - \frac{\zeta(p)}{n} \right) &= -\sum_{n\geq 1}^{\mathcal{R}} \frac{1}{n} \sum_{m=n+1}^{\infty} \frac{1}{m^p} \\ &= \sum_{n\geq 1}^{\mathcal{R}} \frac{1}{n}\frac{(-1)^{p-1}}{(p-1)!}\partial^{p-1}\psi(n+1) \\ &= -\sum_{n=1}^{+\infty} \frac{1}{n} \sum_{m=n+1}^{+\infty} \frac{1}{m^p} + \frac{(-1)^p}{(p-1)!}\int_1^{\infty} \frac{\partial^{p-1}\psi(x+1)}{x}\, dx \,, \end{aligned}$$

where the symbol $\sum_{n\geq 1}^{\mathcal{R}}$ denotes the $\mathcal{R}$-sum of the series (see [2] for a precise definition). Since

$$\sum_{n=1}^{\infty} \frac{1}{n} \sum_{m=n+1}^{\infty} \frac{1}{m^p} = \sum_{n=1}^{\infty} \frac{H_n}{n^p} - \zeta(p+1) \,,$$

this can be rewritten

$$\sum_{n\geq 1}^{\mathcal{R}} \left(\frac{H_n^{(p)}}{n} - \frac{\zeta(p)}{n} \right) = \zeta(p+1) - \sum_{n=1}^{\infty} \frac{H_n}{n^p} + \frac{(-1)^p}{(p-1)!}\int_1^{\infty} \frac{\partial^{p-1}\psi(x+1)}{x}\, dx \,.$$

Thus we have

$$\sum_{n\geq 1}^{\mathcal{R}} \frac{H_n^{(p)}}{n} = \gamma\zeta(p) + \zeta(p+1) - \sum_{n=1}^{\infty} \frac{H_n}{n^p} + \frac{(-1)^p}{(p-1)!}\int_1^{\infty} \frac{\partial^{p-1}\psi(x+1)}{x}\, dx \,,$$

which, with our notations, translates into

$$\mathcal{R}_{p,1} = \gamma\zeta(p) + \zeta(p+1) - \mathcal{S}_{1,p} + \frac{(-1)^p}{(p-1)!}\int_1^{\infty} \frac{\partial^{p-1}\psi(x+1)}{x}\, dx \,. \tag{13}$$

The integral in the right member of (13) is evaluated by performing $p-1$ successive integrations by parts. When $p=2$, this is just

$$\int_1^\infty \frac{\partial\psi(x+1)}{x}\,dx = \int_1^\infty \frac{\psi(x+1)+\gamma}{x^2}\,dx \;-1\,,$$

which, by (4), is $-\zeta'(2)-\tau_2-1$. Hence, by (13), we have

$$\mathcal{R}_{2,1} = \gamma\zeta(2)+\zeta(3)-\mathcal{S}_{1,2}-\sigma_2-\zeta'(2)-\tau_2\,.$$

We now assume that $p>2$. Under this assumption, we have the identity

$$\partial^{p-k}\psi(2) = (-1)^{p-k}(p-k)! + (-1)^{p-k+1}(p-k)!\zeta(p-k+1) \qquad (p-k\geq 1)$$

[6, Prop. 9.6.41] from which results the relation

$$\frac{(-1)^p}{(p-1)!}\int_1^\infty \frac{\partial^{p-1}\psi(x+1)}{x}\,dx = (-1)^p\int_1^\infty \frac{\psi(x+1)+\gamma}{x^p}\,dx$$
$$+\frac{1}{(p-1)!}\sum_{k=0}^{p-3}(-1)^k k!(p-k-2)!\,\zeta(p-k-1) - \frac{1}{(p-1)!}\sum_{k=0}^{p-2}(-1)^k k!(p-k-2)!\,.$$

In this expression, the last term can be simplified by means of the formula

$$\frac{1}{(p-1)!}\sum_{k=0}^{p-2}(-1)^k k!(p-k-2)! = \frac{1}{p-1}\sum_{k=0}^{p-2}\frac{(-1)^k}{\binom{p-2}{k}} = \frac{1+(-1)^p}{p}$$

[14, Eq. (14)]. After reindexation, we can also write

$$\frac{1}{(p-1)!}\sum_{k=0}^{p-3}(-1)^k k!(p-k-2)!\,\zeta(p-k-1) = -\sum_{j=1}^{p-2}(-1)^j\frac{(j-1)!(p-j-1)!}{(p-1)!}\,\zeta(p-j)\,.$$

Moreover, by (3), we have

$$(-1)^p\int_1^\infty \frac{\psi(x+1)+\gamma}{x^p}\,dx = \sum_{j=1}^{p-2}\frac{(-1)^j}{j}\,\zeta(p-j)-\zeta'(p)-(-1)^p\tau_p\,.$$

Thanks to these simplifications, formula (13) can then be rewritten

$$\mathcal{R}_{p,1} = \gamma\zeta(p)+\zeta(p+1)-\mathcal{S}_{1,p}-\zeta'(p)-(-1)^p\tau_p-\sigma_p\,,$$

with

$$\sigma_p = \frac{1+(-1)^p}{p}+\sum_{j=1}^{p-2}(-1)^j\frac{(j-1)!(p-j-1)!}{(p-1)!}\,\zeta(p-j)-\sum_{j=1}^{p-2}\frac{(-1)^j}{j}\,\zeta(p-j)\,.$$

This completes the demonstration of the expected formula (11). Formula (12) is immediately deduced from (11) and Euler's formula (9). □

Example 4.

$$\begin{aligned}
\mathcal{R}_{2,1} &= \gamma\zeta(2) - \zeta(3) - 1 - \zeta'(2) - \tau_2\\
\mathcal{R}_{3,1} &= \gamma\zeta(3) - \frac{1}{4}\zeta(4) - \frac{1}{2}\zeta(2) - \zeta'(3) + \tau_3\,,\\
\mathcal{R}_{4,1} &= \gamma\zeta(4) - 2\zeta(5) + \zeta(3)\zeta(2) - \frac{2}{3}\zeta(3) + \frac{1}{3}\zeta(2) - \frac{1}{2} - \zeta'(4) - \tau_4\,,\\
\mathcal{R}_{5,1} &= \gamma\zeta(5) - \frac{3}{4}\zeta(6) - \frac{3}{4}\zeta(4) + \frac{1}{2}(\zeta(3))^2 + \frac{5}{12}\zeta(3) - \frac{1}{4}\zeta(2) - \zeta'(5) + \tau_5\,.
\end{aligned}$$

Remark 4. Formula (11) plays a crucial role in the proof of [4, Prop. 6].

4 Values of $\mathcal{R}_{p,p}$

4.1 The case $p = 1$

The following formula [2, Eq. (3.23)] allows us to extend formula (11) to the case $p = 1$. We have

$$\mathcal{R}_{1,1} = \frac{1}{2}\gamma^2 - \frac{1}{2}\zeta(2) + \gamma_1 + \tau_1\,, \tag{14}$$

where γ_1 is the first Stieltjes constant and τ_1 is the constant defined by (2). A new direct proof of this formula is given below.

Proof of formula (14). The relation

$$\mathcal{R}_{1,1} = \frac{1}{2}\gamma^2 + \frac{1}{2}\zeta(2) - \frac{1}{2} + \frac{1}{2}\int_0^1 \psi^2(x+1)\,dx$$

[2, Eq. (2.6)] is a direct consequence of [2, Thm. 3]. Since $\psi(x+1) = \psi(x) + 1/x$, this relation can be rewritten

$$\int_0^1 \left(\psi^2(x) + 2\frac{\psi(x)}{x} + \frac{1}{x^2}\right) dx = 2\mathcal{R}_{1,1} - \gamma^2 - \zeta(2) + 1\,.$$

Moreover, from [6, p. 145], we have

$$\int_0^1 \left(\psi^2(x) - \frac{2\gamma}{x} - \frac{1}{x^2}\right) dx = 2\gamma_1 - 2\zeta(2) + 1\,.$$

Subtracting these two expressions, we obtain the following

$$2\int_0^1 \left((\psi(x) + \gamma)\frac{1}{x} + \frac{1}{x^2}\right) dx = \mathcal{R}_{1,1} - \gamma^2 - \zeta(2) + 1 - (2\gamma_1 - 2\zeta(2) + 1)\,.$$

Since

$$(\psi(x)+\gamma)\frac{1}{x}+\frac{1}{x^2}=\frac{\psi(x+1)+\gamma}{x}\,,$$

we deduce the relation

$$2\int_0^1 \frac{\psi(x+1)+\gamma}{x}\,dx = 2\mathcal{R}_{1,1}+\zeta(2)-\gamma^2-2\gamma_1\,.$$

Applying Lemma 1 with $p=1$, we obtain formula (14) after division by 2. $\square$

4.2 The case $p>1$

For $p\geq 2$, the $\mathcal{R}$-sums $\mathcal{R}_{p,p}$ may be easily evaluated by means of the relation

$$\mathcal{R}_{p,p}=\mathcal{S}_{p,p}-\int_1^\infty \frac{\psi_p(x)}{x^p}\,dx\,,$$

with

$$\psi_p(x)=\frac{(-1)^{p-1}}{(p-1)!}\partial^{p-1}\psi(x+1)+\zeta(p)$$

which results from Definition 1, and the expression

$$\mathcal{S}_{p,p}=\frac{1}{2}\zeta(p)^2+\frac{1}{2}\zeta(2p)$$

which results directly from Euler's reciprocity formula. By performing $p-1$ successive integrations by parts, we obtain an expression of $\mathcal{R}_{p,p}$ in terms of zeta values $\zeta(2p),\zeta(2p-2),\cdots,\zeta(2)$, as well as $\zeta'(2p-1)$, τ_{2p-1} and a rational constant. In this way, we get

$$\mathcal{R}_{2,2}=\frac{7}{4}\zeta(4)+\zeta(2)+2\zeta'(3)-2\tau_3-1\,, \tag{15}$$

and the general formula is given by

$$\begin{aligned}\mathcal{R}_{p,p}&=\frac{1}{2}\zeta(p)^2+\frac{1}{2}\zeta(2p)-\frac{\zeta(p)}{p-1}\\&\quad+(-1)^p\binom{2p-2}{p-1}\left[\sum_{j=1}^{2p-3}\frac{(-1)^{j+1}}{j}\zeta(2p-1-j)+\zeta'(2p-1)-\tau_{2p-1}\right]\\&\qquad+\frac{1}{((p-1)!)^2}\sum_{k=2}^{p-1}(-1)^k(p-k)!(p+k-3)!\,\zeta(p+1-k)\\&\qquad\quad-\frac{1}{((p-1)!)^2}\sum_{k=2}^{p}(-1)^k(p-k)!(p+k-3)!\qquad(p\geq 3)\,.\end{aligned} \tag{16}$$

5 Reciprocity formulas

5.1 The even case

Proposition 3. *For any integer $p \geq 1$, we have*

$$\mathcal{R}_{1,2p} + \mathcal{R}_{2p,1} = \gamma\zeta(2p) + \zeta(2p+1) - \sum_{j=0}^{2p-2} (-1)^j A_{j,p}\, \zeta(2p-j) - \frac{1}{p} \tag{17}$$

with $A_{0,p} = 0$, and

$$A_{j,p} = \frac{(j-1)!(2p-1-j)!}{(2p-1)!} \qquad (j \geq 1)\,.$$

Proof. By adding identities (8) and (12), we get for $p \geq 2$,

$$\mathcal{R}_{1,2p} + \mathcal{R}_{2p,1} = \gamma\zeta(2p) + \zeta(2p+1) - \sum_{j=1}^{2p-2} \frac{(-1)^j}{j}\, \zeta(2p-j) - \sigma_{2p}\,.$$

Thus, for $p > 1$, formula (17) follows immediately by replacing σ_{2p} by its expression given by (10) and is extendable to the case $p = 1$ by setting $A_0 = 0$. □

Example 5. We have the following relations:

$$\begin{aligned}
\mathcal{R}_{1,2} + \mathcal{R}_{2,1} &= \gamma\zeta(2) + \zeta(3) - 1\,,\\
\mathcal{R}_{1,4} + \mathcal{R}_{4,1} &= \gamma\zeta(4) + \zeta(5) + \frac{1}{3}\zeta(3) - \frac{1}{6}\zeta(2) - \frac{1}{2}\,,\\
\mathcal{R}_{1,6} + \mathcal{R}_{6,1} &= \gamma\zeta(6) + \zeta(7) + \frac{1}{5}\zeta(5) - \frac{1}{20}\zeta(4) + \frac{1}{30}\zeta(3) - \frac{1}{20}\zeta(2) - \frac{1}{3}\,.
\end{aligned}$$

5.2 The odd case

Proposition 4. *For any integer $p \geq 2$, we have*

$$\begin{aligned}
\mathcal{R}_{1,2p-1} &+ \mathcal{R}_{2p-1,1}\\
&= \gamma\zeta(2p-1) + \zeta(2p) - \sum_{j=1}^{2p-3} (-1)^j C_{j,p}\, \zeta(2p-1-j)\\
&\qquad - 2\zeta'(2p-1) + 2\tau_{2p-1}
\end{aligned} \tag{18}$$

with

$$C_{j,p} = \frac{(j-1)!(2p-2-j)!}{(2p-2)!} - \frac{2}{j} \qquad (j \geq 1)\,.$$

Proof. By adding identities (7) and (11), we get

$$\mathcal{R}_{1,p} + \mathcal{R}_{p,1} = \gamma\zeta(p) + \zeta(p+1) - \sigma_p - (-1)^p \sum_{j=1}^{p-2} \frac{(-1)^j}{j}\,\zeta(p-j) \\ + (1-(-1)^p)\tau_p + ((-1)^p - 1)\zeta'(p)\,.$$

Hence we have the following relation

$$\mathcal{R}_{1,2p-1} + \mathcal{R}_{2p-1,1} = \gamma\zeta(2p-1) + \zeta(2p) - \sigma_{2p-1} + \sum_{j=1}^{2p-3} \frac{(-1)^j}{j}\zeta(2p-1-j) \\ - 2\zeta'(2p-1) + 2\tau_{2p-1}\,,$$

from which formula (18) is derived by replacing σ_{2p-1} by its expression given by (10). Note that, in the odd case, the constant term of σ_{2p-1} is null. □

Example 6. We have the following relations:

$$\begin{aligned} \mathcal{R}_{1,3} + \mathcal{R}_{3,1} &= \gamma\zeta(3) + \zeta(4) - \frac{3}{2}\zeta(2) - 2\zeta'(3) + 2\tau_3\,, \\ \mathcal{R}_{1,5} + \mathcal{R}_{5,1} &= \gamma\zeta(5) + \zeta(6) - \frac{7}{4}\zeta(4) + \frac{11}{12}\zeta(3) - \frac{7}{12}\zeta(2) - 2\zeta'(5) + 2\tau_5\,, \\ \mathcal{R}_{1,7} + \mathcal{R}_{7,1} &= \gamma\zeta(7) + \zeta(8) - \frac{11}{6}\zeta(6) + \frac{29}{30}\zeta(5) - \frac{17}{30}\zeta(4) + \frac{2}{5}\zeta(3) - \frac{11}{30}\zeta(2) \\ &\quad - 2\zeta'(7) + 2\tau_7\,. \end{aligned}$$

Remark 5. a) It should be noted that, in constrast to the even case, the reciprocity relation between $\mathcal{R}_{1,p}$ and $\mathcal{R}_{p,1}$ when p is odd involves the constants $\zeta'(p)$ and τ_p.

b) Another type of reciprocity formula linking $\mathcal{R}_{p,q}$ to $\mathcal{R}_{q,p}$ is given by [2, Eq. (2.10)]. If $q = p+1$, we have the following nice formula:

$$\mathcal{R}_{p,p+1} + \mathcal{R}_{p+1,p} = \zeta(2p+1) + \left(\zeta(p) - \frac{1}{p-1}\right)\zeta(p+1) - \frac{1}{p} \qquad (p \geq 2)\,.$$

Moreover, this relation extends to the case $p = 1$ for which the formula

$$\mathcal{R}_{1,2} + \mathcal{R}_{2,1} = \zeta(3) + \gamma\zeta(2) - 1$$

is regained. Unfortunately, no general "simple" reciprocity formula, valid for any p and q, is known.

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