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New identities involving Cauchy numbers, harmonic numbers and zeta values

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Abstract In this article, we present a class of identities linking together Cauchy numbers, the special values of the Riemann zeta function and its derivative, and a generalization of the Roman harmonic numbers, which represents a significant refinement and improvement of our earlier work on the subject.

Keywords Cauchy numbers; Roman harmonic numbers; binomial identities; series with zeta values; Ramanujan summation of series.

Mathematics Subject Classification (2020) 05A19, 11B75, 11M06, 40G99.

1 Introduction

Several years ago, we introduced a method based on the Ramanujan summation of series which enabled us to generate a number of interesting identities linking together Cauchy numbers (also known as Bernoulli numbers of the second kind), harmonic numbers and values of the Riemann zeta function at positive integers [4]. Thanks to new formulas recently proved in our last paper [6], we can refine and improve most of these earlier results by providing new closed form evaluations of the same kind (see Proposition 1 and Corollary 1). In order to do this, we consider a natural generalization of the Roman harmonic numbers that were first introduced in [5] (see Definition 2), and use a general transformation formula that links the Cauchy numbers to the Ramanujan summation of series [3, Theorem 18]. A noteworthy fact is the presence in our formulas of certain alternating series with zeta values (see Definition 3) that recently appeared in different contexts [1, 6, 7]. In the aim to help the reader to find his way among our various formulas, a summary of the most noteworthy identities, ranked in ascending order of complexity, is given in the penultimate section of the article.

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2 Preliminaries : reminder of the main definitions and results

We first recall various definitions and results that appeared in our previous work, referring to the indicated references for the proof of these results.

2.1 Harmonic numbers

Definition 1. The generalized harmonic numbers $H_n^{(r)}$ are defined for non-negative integers n and r by

$$H_0^{(r)} = 0 \quad \text{and} \quad H_n^{(r)} = \sum_{j=1}^n \frac{1}{j^r} \quad \text{for } n \geq 1. \quad (1)$$

For $r = 1$, they reduce to classical harmonic numbers $H_n = H_n^{(1)}$. The sums

$$\mathcal{S}_{r,p} = \sum_{n=1}^{\infty} \frac{H_n^{(r)}}{n^p}$$

for positive integers $p \geq 2$ are called linear Euler sums. We recall Euler's celebrated formula for $\mathcal{S}_{1,p}$ [10, Eq. (3.6)]:

$$2\mathcal{S}_{1,p} = (p+2)\zeta(p+1) - \sum_{j=1}^{p-2} \zeta(p-j)\zeta(j+1) \quad (p \geq 2).$$

Definition 2 ([5]). The generalized Roman harmonic numbers $H_{n,k}^{(r)}$ are defined for non-negative integers n, r and k with $n \geq 1$, $r \geq 1$ and $k \geq 0$ by

$$H_{n,0}^{(r)} = \frac{1}{n^{r-1}} \quad \text{and} \quad H_{n,k}^{(r)} = \sum_{n \geq j_1 \geq \dots \geq j_k \geq 1} \frac{1}{j_1 j_2 \dots j_k^r} \quad \text{for } k \geq 1. \quad (2)$$

In particular, for $k = 1$, they reduce to classical generalized harmonic numbers

$$H_{n,1}^{(r)} = H_n^{(r)},$$

and for $k = 2$, they admit the expression

$$H_{n,2}^{(r)} = \sum_{j=1}^n \frac{H_j^{(r)}}{j}.$$

For $r = 1$, the harmonic numbers $H_{n,k}^{(1)}$ will be noted $H_{n,k}$ in the remainder of the article.

Remark 1. The harmonic numbers $H_{n,k}$ are nothing else than the ordinary Roman harmonic numbers [9]. In particular, for the first ones, we have

$$H_{n,0} = 1, \quad H_{n,1} = H_n, \quad H_{n,2} = \sum_{j=1}^n \frac{H_j}{j}, \quad \text{etc.}$$

It is well-known (cf. [4, Eq. (18)], [9, Eq. (29)]) that the Roman harmonic number $H_{n,k} = H_{n,k}^{(1)}$ can be written as a sum of products of generalized harmonic numbers $H_n^{(1)}, \dots, H_n^{(k)}$. In particular, for $k = 2$, this property translates into the formula

$$H_{n,2} = \sum_{j=1}^n \frac{H_j}{j} = \frac{1}{2}(H_n)^2 + \frac{1}{2}H_n^{(2)}. \quad (3)$$

The harmonic sums $H_{n,k}^{(r)}$ verify the following binomial identity [5, Eq. (4.7)]:

$$H_{n,k}^{(r)} = \sum_{j=1}^n (-1)^{j-1} \binom{n}{j} \frac{H_{j,r-1}}{j^k} \quad (k \geq 1). \quad (4)$$

By inverse binomial transform¹, formula (4) also admits a reciprocal:

$$\frac{H_{n,r-1}}{n^k} = \sum_{j=1}^n (-1)^{j-1} \binom{n}{j} H_{j,k}^{(r)} \quad (k \geq 1). \quad (5)$$

Example 1. For $r = 1$, formula (4) reduces to

$$H_{n,k} = \sum_{j=1}^n (-1)^{j-1} \binom{n}{j} \frac{1}{j^k}$$

which is a classical property of Roman harmonic numbers [9, Eq. (20)], and for $r = 2$, it may be written

$$H_{n,k}^{(2)} = \sum_{j=1}^n (-1)^{j-1} \binom{n}{j} \frac{H_j}{j^k}.$$

In particular, we have

$$H_n^{(2)} = \sum_{j=1}^n (-1)^{j-1} \binom{n}{j} \frac{H_j}{j}$$

and

$$\sum_{j=1}^n \frac{H_j^{(2)}}{j} = \sum_{j=1}^n (-1)^{j-1} \binom{n}{j} \frac{H_j}{j^2}.$$

1. If $b(n) = \frac{1}{n} \sum_{j=1}^n (-1)^{j-1} \binom{n}{j} j a(j)$, then $a(n) = \frac{1}{n} \sum_{j=1}^n (-1)^{j-1} \binom{n}{j} j b(j)$ (cf. [5, Definition 5 and Corollary 1]).

2.2 Alternating series with zeta values

Definition 3 ([1, 6, 7]). The numbers τ_p are defined for all positive integers p by the series representation

$$\tau_p = \sum_{k=1}^{\infty} (-1)^{k+p} \frac{\zeta(k+p)}{k}. \quad (6)$$

For $p \geq 2$, they verify the following identity [1, Proposition 7], [6, Lemma 1]:

$$\sum_{n=1}^{\infty} \frac{\ln(n+1)}{n^p} = -\zeta'(p) - (-1)^p \tau_p. \quad (7)$$

For $p = 1$, we have [1, Theorem 2 (a)]

$$\tau_1 = \int_0^1 \frac{\psi(x+1) + \gamma}{x} dx,$$

where ψ is the digamma function and $\gamma = -\psi(1)$ is Euler's constant.

2.3 Cauchy numbers

Definition 4 ([2, 4, 8]). The classical Cauchy numbers c_n are defined for $n \geq 1$ by

$$c_n = \int_0^1 x(x-1) \cdots (x-n+1) dx.$$

The Cauchy numbers alternate in sign. Using the notations introduced in [4], we consider the sequence $\{\lambda_n\}_n$ of non-alternating Cauchy numbers defined by

$$\lambda_n = (-1)^{n-1} c_n \quad (n \geq 1).$$

The first terms of the sequence are the following:

$$\lambda_1 = \frac{1}{2}, \lambda_2 = \frac{1}{6}, \lambda_3 = \frac{1}{4}, \lambda_4 = \frac{19}{30}, \lambda_5 = \frac{9}{4}, \lambda_6 = \frac{863}{84}, \text{ etc.}$$

3 Series with Cauchy numbers and Roman harmonic numbers

We first recall the transformation formula [3, Theorem 18] that links the Cauchy numbers to the Ramanujan summation of series: if a is a function analytic in the half-plane $P = \{\operatorname{Re}(z) > 0\}$ such that there exists a constant $C > 0$ with

$$|a(z)| < C 2^{|z|} \quad \text{for all } z \in P,$$

then

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n! n} \sum_{j=1}^n (-1)^{j-1} \binom{n}{j} j a(j) = \sum_{n \geq 1}^{\mathcal{R}} a(n) \quad (8)$$

where $\sum_{n \geq 1}^{\mathcal{R}}$ denotes the \mathcal{R} -sum of the series i.e. the sum of the series in the sense of Ramanujan's summation method [3, 4]. In particular, using formula (4) and its reciprocal (5), we deduce from (8) the following reciprocal identities:

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n! n} H_{n,k}^{(r)} = \sum_{n \geq 1}^{\mathcal{R}} \frac{H_{n,r-1}}{n^{k+1}} \quad (k \geq 0, r \geq 1). \quad (9)$$

and

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n! n^{k+1}} H_{n,r-1} = \sum_{n \geq 1}^{\mathcal{R}} \frac{H_{n,k}^{(r)}}{n} \quad (k \geq 0, r \geq 1). \quad (10)$$

We can now state the following proposition:

Proposition 1. *For any integer $p \geq 2$, we have*

a)

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n! n} H_{n,p-1} = \zeta(p) - \frac{1}{p-1}. \quad (11)$$

b)

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n! n^2} H_{n,p-1} = \gamma \zeta(p) + \zeta(p+1) - \mathcal{S}_{1,p} - \zeta'(p) - (-1)^p \tau_p - \sigma_p \quad (12)$$

with

$$\sigma_p = \frac{1 + (-1)^p}{p} + \sum_{j=1}^{p-2} (-1)^j \zeta(p-j) \left\{ \frac{(j-1)!(p-1-j)!}{(p-1)!} - \frac{1}{j} \right\}.$$

c)

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n! n} H_{n,p-1}^{(2)} = \mathcal{S}_{1,p} - \sum_{j=1}^{p-2} \frac{(-1)^{p-j}}{j} \zeta(p-j) + (-1)^p \zeta'(p) + \tau_p. \quad (13)$$

Summing (12) and (13), we obtain the following corollary:

Corollary 1. *For an even integer $p \geq 2$, we have*

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n! n^2} \left\{ H_{n,p-1} + n H_{n,p-1}^{(2)} \right\} = \gamma \zeta(p) + \zeta(p+1) - \sum_{j=1}^{p-2} \frac{(-1)^j}{j} \zeta(p-j) - \sigma_p. \quad (14)$$

Proof. Applying (10) with $r = p$ gives for $k = 0$

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n! n} H_{n,p-1} = \sum_{n \geq 1} \frac{H_{n,0}^{(p)}}{n} = \sum_{n \geq 1} \frac{1}{n^p} = \zeta(p) - \frac{1}{p-1}$$

(cf. [3, Eq. (1.22)]), and for $k = 1$

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n! n^2} H_{n,p-1} = \sum_{n \geq 1} \frac{H_n^{(p)}}{n}.$$

Hence formula (12) results from [6, Eq. (10)]. Specializing(9) with $k = p - 1$ gives for $r = 2$

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n! n} H_{n,p-1}^{(2)} = \sum_{n \geq 1} \frac{H_n}{n^p}.$$

Hence formula (13) results from [6, Eq. (6)]. \square

Remark 2.

- a) Since $H_{n,0} = 1$ and $H_{n,0}^{(2)} = \frac{1}{n}$, formulae (11) to (13) extend to the case $p = 1$ through the identities

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n! n} = \gamma$$

(cf. [2, 4, 8]), and

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n! n^2} = \frac{1}{2} \gamma^2 - \frac{1}{2} \zeta(2) + \gamma_1 + \tau_1$$

where γ_1 is the first Stieltjes constant (cf. [6, Eq. (13)]).

- b) Taking into account the identity (7), formula (12) can be seen as a refinement of [4, Eq. (27)].

Example 2. For the first values of p , formulas (11), (12) and (13) translate respectively into the following identities:

- 1) For $p = 2$,

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n! n} H_n = \zeta(2) - 1, \quad (15)$$

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n! n^2} H_n = \gamma \zeta(2) - \zeta(3) - \zeta'(2) - \tau_2 - 1, \quad (16)$$

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n! n} H_n^{(2)} = 2\zeta(3) + \zeta'(2) + \tau_2. \quad (17)$$

2) For $p = 3$,

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n! n} \sum_{j=1}^n \frac{H_j}{j} = \zeta(3) - \frac{1}{2}, \quad (18)$$

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n! n^2} \sum_{j=1}^n \frac{H_j}{j} = \gamma \zeta(3) - \frac{1}{4} \zeta(4) - \frac{1}{2} \zeta(2) - \zeta'(3) + \tau_3, \quad (19)$$

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n! n} \sum_{j=1}^n \frac{H_j^{(2)}}{j} = \frac{5}{4} \zeta(4) - \zeta(2) - \zeta'(3) + \tau_3. \quad (20)$$

Remark 3.

a) Formula (15) is a well-known series representation of $\zeta(2)$ (cf. [2, 4, 8]).

b) Replacing n^2 by $n(n-1)$ in (16) leads to the simpler formula

$$\sum_{n=2}^{\infty} \frac{\lambda_n}{n! n(n-1)} H_n = \frac{1}{2} \ln(2\pi) - \frac{3}{2} \gamma - \frac{3}{2} \zeta(2) + \frac{5}{2}. \quad (21)$$

c) By means of (3), formula (18) can be rewritten under the following equivalent form:

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n! n} (H_n)^2 + \sum_{n=1}^{\infty} \frac{\lambda_n}{n! n} H_n^{(2)} = 2\zeta(3) - 1 \quad (22)$$

which coincides with [2, Eq. (9)]. Subtracting (17) from (22) enables to deduce another interesting identity:

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n! n} (H_n)^2 = -\zeta'(2) - \tau_2 - 1. \quad (23)$$

d) Subtracting (16) from (23) and replacing $H_n - \frac{1}{n}$ by H_{n-1} inside the expression gives the surprisingly simple relation

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n! n} H_n H_{n-1} = \zeta(3) - \gamma \zeta(2). \quad (24)$$

e) By means of [6, Eq. (14)] we can also easily prove the following dual formula of (20):

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n! n} \sum_{j=1}^n \frac{H_j}{j^2} = \frac{7}{4} \zeta(4) + \zeta(2) + 2\zeta'(3) - 2\tau_3 - 1. \quad (25)$$

4 Summary of main formulas

The most noteworthy identities are listed below in increasing order of complexity.

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!n} H_n = \frac{\pi^2}{6} - 1 \quad (\text{A})$$

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!n} \{(H_n)^2 + H_n^{(2)}\} = 2\zeta(3) - 1 \quad (\text{B})$$

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!n} H_n H_{n-1} = \zeta(3) - \gamma \frac{\pi^2}{6} \quad (\text{C})$$

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!n^2} \{H_n + nH_n^{(2)}\} = \gamma \frac{\pi^2}{6} + \zeta(3) - 1 \quad (\text{D})$$

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!n} (H_n)^2 = -\zeta'(2) - \tau_2 - 1 \quad (\text{E})$$

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!n} H_n^{(2)} = 2\zeta(3) + \zeta'(2) + \tau_2 \quad (\text{F})$$

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!n^2} H_n = \gamma \frac{\pi^2}{6} - \zeta(3) - \zeta'(2) - \tau_2 - 1 \quad (\text{G})$$

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!n^2} \{(H_n)^2 + H_n^{(2)}\} = 2\gamma\zeta(3) - \frac{\pi^4}{180} - \frac{\pi^2}{6} - 2\zeta'(3) + 2\tau_3 \quad (\text{H})$$

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!n} \sum_{j=1}^n \frac{H_j^{(2)}}{j} = \frac{\pi^4}{72} - \frac{\pi^2}{6} - \zeta'(3) + \tau_3 \quad (\text{I})$$

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!n} \sum_{j=1}^n \frac{H_j}{j^2} = \frac{7\pi^4}{360} + \frac{\pi^2}{6} + 2\zeta'(3) - 2\tau_3 - 1 \quad (\text{J})$$

5 Conclusion

Theoretically, the reciprocal identities (9) and (10) should allow us to compute a rather large class of infinite sums involving Cauchy numbers and (generalized) Roman harmonic numbers. Unfortunately, finding an explicit evaluation of the \mathcal{R} -sums in the right-hand side of these identities remains a difficult task in most of cases. This highlights both the interest and the practical limits of our method.

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