New identities involving Cauchy numbers, harmonic numbers and zeta values

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Abstract In this article, we present several series identities linking together Cauchy numbers (also known as Bernoulli numbers of the second kind), special values of the Riemann zeta function and its derivative, and a generalization of the Roman harmonic numbers.

Keywords Cauchy numbers, Roman harmonic numbers, binomial identities, series with zeta values.

Mathematics Subject Classification (2020) 11B75, 11M06, 40G99.

1 Introduction

Several years ago, we proposed a method based on the Ramanujan summation of series which enabled us to generate a number of identities involving the Cauchy numbers together with the harmonic numbers and the values of the Riemann zeta function at positive integers [4]. Thanks to new formulas recently proved in our last paper [6], we are now in a position to complete these results by giving new closed form evaluations of the same kind (see Propositions 1 and 2). In order to do this, we make use of a quite natural generalization of Roman harmonic numbers (see Definition 2) which have been introduced in [5]. A notable novelty is the involvement in our formulas of certain alternating series with zeta values (see Definition 3) that also appear in [1, 6, 7]. In the aim to help the reader to find his way among these various formulas, a summary of the most noteworthy identities is given in the last section of the article.

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2 Preliminaries : reminder of the main definitions and results

We first recall various definitions and results that appeared in previous work, refering to the indicated references for the proof of these results.

2.1 Harmonic numbers and harmonic sums

Definition 1. The generalized harmonic numbers $H_n^{(r)}$ are defined for non-negative integers n and r by

$$H_0^{(r)} = 0 \quad \text{and} \quad H_n^{(r)} = \sum_{j=1}^{n} \frac{1}{j^r} \quad \text{for } n \geq 1\,. \tag{1}$$

For $r = 1$, they reduce to classical harmonic numbers $H_n = H_n^{(1)}$. The sums

$$\mathcal{S}_{r,p} = \sum_{n=1}^{\infty} \frac{H_n^{(r)}}{n^p}$$

for positive integers $p \geq 2$ are called linear Euler sums. We recall Euler's classical formula for $\mathcal{S}_{1,p}$ [9, Eq. (3.6)]:

$$2\mathcal{S}_{1,p} = (p+2)\zeta(p+1) - \sum_{j=1}^{p-2} \zeta(p-j)\zeta(j+1) \qquad (p \geq 2)\,.$$

Definition 2 ([5])**.** The harmonic sums $H_{n,k}^{(r)}$ are defined for non-negative integers n, r and k with $n \geq 1$, $r \geq 1$ and $k \geq 0$ by

$$H_{n,0}^{(r)} = \frac{1}{n^{r-1}} \quad \text{and} \quad H_{n,k}^{(r)} = \sum_{n \geq j_1 \geq \cdots \geq j_k \geq 1} \frac{1}{j_1\, j_2 \cdots j_k^r} \quad \text{for } k \geq 1\,. \tag{2}$$

In particular, for $k = 1$, they reduce to classical generalized harmonic numbers

$$H_{n,1}^{(r)} = H_n^{(r)}\,,$$

and for $k = 2$, they admit the expression

$$H_{n,2}^{(r)} = \sum_{j=1}^{n} \frac{H_j^{(r)}}{j}\,.$$

Remark 1. For $r = 1$, the harmonic sums $H_{n,k}^{(1)}$, that will be noted $H_{n,k}$ in the remainder of the article, are nothing else than the ordinary Roman harmonic numbers [8]. In particular, for the first ones, we have

$$H_{n,0} = 1\,, \quad H_{n,1} = H_n\,, \quad H_{n,2} = \sum_{j=1}^{n} \frac{H_j}{j}\,, \quad \text{etc.}$$

In the general case, it can be shown [4, Eq. (18)], [8, Eq. (29)] that

$$H_{n,k} = P_k\left(H_n, \ldots, H_n^{(k)}\right)\,,$$

where $P_k(x_1, \ldots, x_k)$ are the modified Bell polynomials [4, Definition 2].

Example 1.

$$H_{n,2} = \sum_{j=1}^{n} \frac{H_j}{j} = \frac{1}{2}(H_n)^2 + \frac{1}{2}H_n^{(2)}\,. \tag{3}$$

The harmonic sums $H_{n,k}^{(r)}$ verify the following identity [5, Eq. (4.7)]:

$$H_{n,k}^{(r)} = \sum_{j=1}^{n} (-1)^{j-1} \binom{n}{j} \frac{H_{j,r-1}}{j^k} \qquad (k \geq 1)\,. \tag{4}$$

Example 2. For $r = 1$, formula (4) reduces to

$$H_{n,k} = \sum_{j=1}^{n} (-1)^{j-1} \binom{n}{j} \frac{1}{j^k} \tag{5}$$

which is a classical property of Roman harmonic numbers [8, Eq. (20)]. For $r = 2$, formula (4) translates into

$$H_{n,k}^{(2)} = \sum_{j=1}^{n} (-1)^{j-1} \binom{n}{j} \frac{H_j}{j^k}\,. \tag{6}$$

By inverse binomial transform[1], formula (4) also admits a reciprocal:

$$\frac{H_{n,r-1}}{n^k} = \sum_{j=1}^{n} (-1)^{j-1} \binom{n}{j} H_{j,k}^{(r)} \qquad (k \geq 1)\,. \tag{7}$$

1. If $b(n) = \frac{1}{n}\sum_{j=1}^{n}(-1)^{j-1}\binom{n}{j} j\, a(j)$, then $a(n) = \frac{1}{n}\sum_{j=1}^{n}(-1)^{j-1}\binom{n}{j} j\, b(j)$ [5, Definition 5 and Corollary 1].

2.2 Alternating series with zeta values

Definition 3 ([1, 6, 7])**.** The numbers τ_p are defined for all positive integers p by the series representation

$$\tau_p = \sum_{k=1}^{\infty} (-1)^{k+p} \frac{\zeta(k+p)}{k} \,. \tag{8}$$

For $p \geq 2$, they verify the following identity [1, Proposition 7], [6, Lemma 1]:

$$\sum_{n=1}^{\infty} \frac{\ln(n+1)}{n^p} = -\zeta'(p) - (-1)^p \, \tau_p \,. \tag{9}$$

For $p = 1$, we have

$$\tau_1 = \int_0^1 \frac{\psi(x+1)+\gamma}{x} \, dx \,, \tag{10}$$

where ψ is the digamma function and $\gamma = -\psi(1)$ is Euler's constant.

2.3 Cauchy numbers

Definition 4 ([2, 4])**.** The *Cauchy numbers* c_n are defined for $n \geq 1$ by

$$c_n = \int_0^1 x(x-1)\cdots(x-n+1)\, dx \,.$$

The Cauchy numbers alternate in sign. As in [4], we introduce the sequence $\{\lambda_n\}_n$ of non-alternating Cauchy numbers defined by

$$\lambda_n = (-1)^{n-1} c_n \qquad (n \geq 1)\,.$$

The first terms of the sequence are the following:

$$\lambda_1 = \frac{1}{2}\,,\ \lambda_2 = \frac{1}{6}\,,\ \lambda_3 = \frac{1}{4}\,,\ \lambda_4 = \frac{19}{30}\,,\ \lambda_5 = \frac{9}{4}\,,\ \lambda_6 = \frac{863}{84}\,,\ \text{etc.}$$

We recall the transformation formula [3, Theorem 18] that links Cauchy numbers to the Ramanujan summation of series: if a is a function analytic in the half-plane $P = \{\mathrm{Re}(z) > 0\}$ such that there exists a constant $C > 0$ with

$$|a(z)| < C\, 2^{|z|} \quad \text{for all } z \in P \ ,$$

then

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\, n} \sum_{j=1}^{n} (-1)^{j-1} \binom{n}{j} j\, a(j) = \sum_{n \geq 1}^{\mathcal{R}} a(n) \tag{11}$$

where $\sum_{n\geq 1}^{\mathcal{R}}$ denotes the $\mathcal{R}$-sum of the series i.e. the sum of the series in the sense of Ramanujan's summation method [3, 4].

3 Series with Cauchy numbers and harmonic sums

A reinterpretation of [6, Propositions 1 and 2] via the transformation formula (11) now allows us to establish the following propositions:

Proposition 1. *For any integer $p \geq 1$, we have*

a)

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\,n} H_{n,p-1} = \begin{cases} \gamma\,, & \text{if } p = 1; \\ \zeta(p) - \frac{1}{p-1}\,, & \text{otherwise.} \end{cases} \tag{12}$$

b)

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\,n} H^{(2)}_{n,p-1} = \begin{cases} \frac{1}{2}\gamma^2 - \frac{1}{2}\zeta(2) + \gamma_1 + \tau_1\,, & \text{if } p = 1; \\ \mathcal{S}_{1,p} - \sum_{j=1}^{p-2} \frac{(-1)^{p-j}}{j}\,\zeta(p-j) + (-1)^p\zeta'(p) + \tau_p\,, & \text{otherwise} \end{cases} \tag{13}$$

with γ_1 the first Stieltjes constant.

Proof. It follows from formulas (4) and (11) that

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\,n} H^{(r)}_{n,k} = \sum_{n\geq 1}^{\mathcal{R}} \frac{H_{n,r-1}}{n^{k+1}} \qquad (k \geq 0,\, r \geq 1).$$

Specializing this identity with $k = p - 1$, we obtain

a) for $r = 1$,

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\,n} H_{n,p-1} = \sum_{n\geq 1}^{\mathcal{R}} \frac{1}{n^p}\,,$$

b) for $r = 2$,

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\,n} H^{(2)}_{n,p-1} = \sum_{n\geq 1}^{\mathcal{R}} \frac{H_n}{n^p}\,.$$

Hence, formula (12) results from [3, Eqs. (1.22) and (1.24)]) while formula (13) results from [6, Proposition 1 and Eq. (13)]. □

Example 3. Applied to small values of p, formulas (12) and (13) translate into the following identities:

1) For $p = 1$,

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\,n} = \gamma\,, \tag{14}$$

and

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\, n^2} = \frac{1}{2}\gamma^2 - \frac{1}{2}\zeta(2) + \gamma_1 + \tau_1 \,. \tag{15}$$

2) For $p = 2$,

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\, n} H_n = \zeta(2) - 1 \,, \tag{16}$$

and

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\, n} H_n^{(2)} = 2\zeta(3) + \zeta'(2) + \tau_2 \,. \tag{17}$$

3) For $p = 3$,

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\, n} \sum_{j=1}^{n} \frac{H_j}{j} = \zeta(3) - \frac{1}{2} \,, \tag{18}$$

and

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\, n} \sum_{j=1}^{n} \frac{H_j^{(2)}}{j} = \frac{5}{4}\zeta(4) - \zeta(2) - \zeta'(3) + \tau_3 \,. \tag{19}$$

Remark 2. a) Formulas (14) and (16) are well-known classical series representations of γ and $\zeta(2)$ respectively (cf. [2, 3, 4]). Formula (15) appears in [3, p. 133] by combining Eqs. (3.23) and (4.29).

b) Thanks to the relation (3), formula (18) can be rewritten under the following equivalent form:

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\, n} (H_n)^2 + \sum_{n=1}^{\infty} \frac{\lambda_n}{n!\, n} H_n^{(2)} = 2\zeta(3) - 1 \tag{20}$$

which coincides with [2, Eq. (9)]. Subtracting (17) from (20) enables to deduce yet another interesting identity:

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\, n} (H_n)^2 = -\zeta'(2) - \tau_2 - 1 \,. \tag{21}$$

c) By means of [6, Eq. (14)] we can also easily prove the following formula dual of (19):

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\, n} \sum_{j=1}^{n} \frac{H_j}{j^2} = \frac{7}{4}\zeta(4) + \zeta(2) + 2\zeta'(3) - 2\tau_3 - 1 \,. \tag{22}$$

The proposition below is a more convenient reformulation of [4, Theorem 10].

Proposition 2. *For any integer $p \geq 2$, we have*

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\, n^2} H_{n,p-1} = \gamma\zeta(p) + \zeta(p+1) - \mathcal{S}_{1,p} - \zeta'(p) - (-1)^p \tau_p - \sigma_p \tag{23}$$

with $\sigma_2 = 1$, and

$$\sigma_p = \frac{1+(-1)^p}{p} + \sum_{j=1}^{p-2} (-1)^j \zeta(p-j) \left[\frac{(j-1)!(p-1-j)!}{(p-1)!} - \frac{1}{j} \right] \quad \textit{for } p \geq 3.$$

Proof. It follows from formulas (7) and (11) that

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\, n^{k+1}} H_{n,r-1} = \sum_{n\geq 1}^{\mathcal{R}} \frac{H_{n,k}^{(r)}}{n} \qquad (k \geq 0,\ r \geq 1).$$

Specializing this identity with $k = 1$ and $r = p$, we get

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\, n^2} H_{n,p-1} = \sum_{n\geq 1}^{\mathcal{R}} \frac{H_n^{(p)}}{n}.$$

Formula (23) then results from [6, Proposition 2]. □

Example 4. Applied to small values of p, formula (23) translates into the following identities:

a) For $p = 2$,

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\, n^2} H_n = \gamma\zeta(2) - \zeta(3) - \zeta'(2) - \tau_2 - 1\,. \tag{24}$$

Subtracting (24) from (21) and replacing $H_n - \frac{1}{n}$ by H_{n-1} inside the expression, we obtain the surprisingly simple relation

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\, n} H_n\, H_{n-1} = \zeta(3) - \gamma\zeta(2)\,. \tag{25}$$

b) For $p = 3$,

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\, n^2} \sum_{j=1}^{n} \frac{H_j}{j} = \gamma\zeta(3) - \frac{1}{4}\zeta(4) - \frac{1}{2}\zeta(2) - \zeta'(3) + \tau_3\,. \tag{26}$$

Remark 3. Replacing n^2 by $n(n-1)$ in (24) leads to the simpler formula

$$\sum_{n=2}^{\infty} \frac{\lambda_n\, H_n}{n!\, n(n-1)} = \frac{1}{2}\ln(2\pi) - \frac{3}{2}\gamma - \frac{3}{2}\zeta(2) + \frac{5}{2}\,.$$

4 Summary of main formulas

The most noteworthy identities are listed below.

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\,n} = \gamma\,, \tag{A}$$

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\,n} H_n = \frac{\pi^2}{6} - 1\,, \tag{B}$$

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\,n} H_n\, H_{n-1} = \zeta(3) - \gamma\frac{\pi^2}{6}\,, \tag{C}$$

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\,n} \Big\{(H_n)^2 + H_n^{(2)}\Big\} = 2\zeta(3) - 1\,, \tag{D}$$

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\,n} H_n^{(2)} = 2\zeta(3) + \zeta'(2) + \tau_2\,, \tag{E}$$

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\,n} (H_n)^2 = -\zeta'(2) - \tau_2 - 1\,, \tag{F}$$

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\,n^2} = \frac{1}{2}\gamma^2 - \frac{\pi^2}{12} + \gamma_1 + \tau_1\,, \tag{G}$$

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\,n^2} H_n = \gamma\frac{\pi^2}{6} - \zeta(3) - \zeta'(2) - \tau_2 - 1\,, \tag{H}$$

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\,n^2} \Big\{(H_n)^2 + H_n^{(2)}\Big\} = 2\gamma\zeta(3) - \frac{\pi^4}{180} - \frac{\pi^2}{6} - 2\zeta'(3) + 2\tau_3\,, \tag{I}$$

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\,n} \sum_{j=1}^{n} \frac{H_j^{(2)}}{j} = \frac{\pi^4}{72} - \frac{\pi^2}{6} - \zeta'(3) + \tau_3\,, \tag{J}$$

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\,n} \sum_{j=1}^{n} \frac{H_j}{j^2} = \frac{7\pi^4}{360} + \frac{\pi^2}{6} + 2\zeta'(3) - 2\tau_3 - 1\,. \tag{K}$$

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