New identities involving Cauchy numbers, harmonic numbers and zeta values

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Abstract In this article, we present a number of series identities linking together Cauchy numbers (also known as Bernoulli numbers of the second kind), special values of the zeta function and its derivative, and a generalization of Roman harmonic numbers.

Keywords: Cauchy numbers, Roman harmonic numbers, binomial identities, series with special numbers, zeta values.

Introduction

Several years ago, we proposed in [4] a method based on the Ramanujan summation of series which enabled us to generate a substantial number of identities involving the Cauchy numbers together with the harmonic numbers and values of the Riemann zeta function at positive integers. Thanks to the formulas recently proved in our last paper [6], we are now in a position to complete these results by giving new closed form evaluations of the same kind (see the examples deduced from Propositions 1 and 2). In order to do this, we make use of a quite natural generalization of Roman harmonic numbers which have been introduced in [5] (see Definition 2). A notable novelty is the involvement in our formulas of certain alternating series with zeta values that appeared in [1] and [6] (see Definition 3).

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1 Preliminaries : reminder of the main definitions and results

We first recall a number of definitions and results that appear in previous work, refering to the indicated references for the proof of these results.

1.1 Harmonic numbers and harmonic sums

Definition 1. The generalized harmonic numbers $H_n^{(r)}$ are defined for non-negative integers n and r by

$$H_0^{(r)} = 0 \quad \text{and} \quad H_n^{(r)} = \sum_{j=1}^{n} \frac{1}{j^r} \quad \text{for } n \geq 1\,. \tag{1}$$

For $r = 1$, they reduce to classical harmonic numbers $H_n = H_n^{(1)}$. The sums

$$\mathcal{S}_{r,p} = \sum_{n=1}^{\infty} \frac{H_n^{(r)}}{n^p}$$

for positive integers $p \geq 2$ are called linear Euler sums. We recall the classical Euler's formula for $\mathcal{S}_{1,p}$ (cf. [6, 8]):

$$2\mathcal{S}_{1,p} = (p+2)\zeta(p+1) - \sum_{j=1}^{p-2} \zeta(p-j)\zeta(j+1) \qquad (p \geq 2)\,.$$

Definition 2 (cf. [5])**.** The harmonic sums $H_{n,k}^{(r)}$ are defined for non-negative integers n, r and k with $n \geq 1$, $r \geq 1$ and $k \geq 0$ by

$$H_{n,0}^{(r)} = \frac{1}{n^{r-1}} \quad \text{and} \quad H_{n,k}^{(r)} = \sum_{n \geq j_1 \geq \cdots \geq j_k \geq 1} \frac{1}{j_1\, j_2 \cdots j_k^r} \quad \text{for } k \geq 1\,. \tag{2}$$

In particular, for $k = 1$, they reduce to classical generalized harmonic numbers

$$H_{n,1}^{(r)} = H_n^{(r)}\,,$$

and for $k = 2$, we have

$$H_{n,2}^{(r)} = \sum_{j=1}^{n} \frac{H_j^{(r)}}{j}\,.$$

Remark 1. For $r = 1$, the harmonic sums $H_{n,k}^{(1)}$ that we shall note $H_{n,k}$ in the rest of the article are nothing else than the ordinary Roman harmonic numbers $c_n^{(k)}$ (cf. [7]). In particular, we have

$$H_{n,0} = 1 \,, \quad H_{n,1} = H_n \,, \quad H_{n,2} = \sum_{j=1}^{n} \frac{H_j}{j} \,, \quad \text{etc.}$$

In the general case, it can be shown (cf. [4, Eq. (18)] and [7, Eq. (29)]) that

$$H_{n,k} = P_k\left(H_n, \ldots, H_n^{(k)}\right)$$

where $P_k(x_1, \ldots, x_k)$ are the modified Bell polynomials (cf. [4, Definition 2]).

Example 1.

$$H_{n,2} = \sum_{j=1}^{n} \frac{H_j}{j} = \frac{1}{2}(H_n)^2 + \frac{1}{2}H_n^{(2)} \,. \tag{3}$$

The harmonic sums $H_{n,k}^{(r)}$ verify the following identity (cf. [5, Corollary 8]):

$$H_{n,k}^{(r)} = \sum_{j=1}^{n} (-1)^{j-1} \binom{n}{j} \frac{H_{j,r-1}}{j^k} \quad (k \geq 1) \,. \tag{4}$$

Example 2. For $r = 1$, formula (4) reduces to

$$H_{n,k} = \sum_{j=1}^{n} (-1)^{j-1} \binom{n}{j} \frac{1}{j^k} \tag{5}$$

which is a classical property of Roman harmonic numbers (cf. [7, Eq. (20)]), and for $r = 2$, it is written as follows:

$$H_{n,k}^{(2)} = \sum_{j=1}^{n} (-1)^{j-1} \binom{n}{j} \frac{H_j}{j^k} \,. \tag{6}$$

By inverse binomial transform[1], identity (4) also admits a reciprocal:

$$\frac{H_{n,r-1}}{n^k} = \sum_{j=1}^{n} (-1)^{j-1} \binom{n}{j} H_{j,k}^{(r)} \quad (k \geq 1) \,. \tag{7}$$

1. If $b(n) = \frac{1}{n} \sum_{j=1}^{n} (-1)^{j-1} \binom{n}{j} j \, a(j)$, then $a(n) = \frac{1}{n} \sum_{j=1}^{n} (-1)^{j-1} \binom{n}{j} j \, b(j)$ (cf. [5, Definition 5 and Corollary 1]).

1.2 Alternating series with zeta values

Definition 3 ([6])**.** For any positive integer p, let us define the infinite series τ_p by

$$\tau_p = \sum_{k=1}^{\infty} (-1)^{k+p} \frac{\zeta(k+p)}{k} \,. \tag{8}$$

For $p \geq 2$, these series verify the following identity (cf. [1, Proposition 7], [6, Lemma 1]):

$$\sum_{n=1}^{\infty} \frac{\ln(n+1)}{n^p} = -\zeta'(p) - (-1)^p \, \tau_p \,. \tag{9}$$

For $p = 1$, we have

$$\tau_1 = \int_0^1 \frac{\psi(x+1)+\gamma}{x} \, dx \,, \tag{10}$$

where ψ is the digamma function.

1.3 Cauchy numbers

Definition 4 (cf. [4])**.** The sequence of *non-alternating Cauchy numbers* $\{\lambda_n\}_n$ is defined for all positive integers n by the recurrence relation

$$\sum_{k=1}^{n} \frac{\lambda_k}{k!\,(n+1-k)} = \frac{1}{n+1} \,.$$

The first terms are

$$\lambda_1 = \frac{1}{2}\,,\ \lambda_2 = \frac{1}{6}\,,\ \lambda_3 = \frac{1}{4}\,,\ \lambda_4 = \frac{19}{30}\,,\ \lambda_5 = \frac{9}{4}\,,\ \lambda_6 = \frac{863}{84}\,,\ \text{etc.}$$

Remark 2. If c_n are the classical Cauchy numbers (cf. [2]) defined by

$$c_n = \int_0^1 x(x-1)\cdots(x-n+1)\,dx \quad \text{for } n \geq 1\,,$$

then λ_n and c_n are linked by the relation $\lambda_n = (-1)^{n-1} c_n$.

We recall the transformation formula that links Cauchy numbers to the Ramanujan summation of series (cf. [3, Theorem 18]) : if a is a function analytic in the half-plane $P = \{\mathrm{Re}(z) > 0\}$ such that there exists a constant $C > 0$ with

$$|a(z)| < C\,2^{|z|} \quad \text{for all } z \in P \ \ ,$$

then

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\,n} \sum_{j=1}^{n} (-1)^{j-1} \binom{n}{j} j\,a(j) = \sum_{n\geq 1}^{\mathcal{R}} a(n) \tag{11}$$

where $\sum_{n\geq 1}^{\mathcal{R}}$ denotes the $\mathcal{R}$-sum of the series (i.e. the sum of the series in the sense of Ramanujan's summation method).

2 Series with Cauchy numbers and harmonic sums

With the help of the results of [6], we can now state and prove the following propositions:

Proposition 1. For any integer $p \geq 1$, we have

a)

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\,n} H_{n,p-1} = \begin{cases} \gamma & \text{for } p = 1 \\ \zeta(p) - \frac{1}{p-1} & \text{for } p > 1 \end{cases} \tag{12}$$

where $\gamma = -\psi(1)$ is Euler's constant.

b)

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\,n} H^{(2)}_{n,p-1} = \begin{cases} \frac{1}{2}\gamma^2 - \frac{1}{2}\zeta(2) + \gamma_1 + \tau_1 & \text{for } p = 1 \\ \mathcal{S}_{1,p} - \sum_{j=1}^{p-2} \frac{(-1)^{p-j}}{j}\,\zeta(p-j) + (-1)^p \zeta'(p) + \tau_p & \text{for } p > 1 \end{cases} \tag{13}$$

where γ_1 is the first Stieltjes constant.

Proof. It follows from formulas (4) and (11) that

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\,n} H^{(r)}_{n,k} = \sum_{n\geq 1}^{\mathcal{R}} \frac{H_{n,r-1}}{n^{k+1}} \qquad (k \geq 0,\ r \geq 1).$$

Specializing this identity with $k = p-1$, we obtain

a) for $r = 1$,

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\,n} H_{n,p-1} = \sum_{n\geq 1}^{\mathcal{R}} \frac{1}{n^p}\,,$$

b) for $r = 2$,

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\,n} H^{(2)}_{n,p-1} = \sum_{n\geq 1}^{\mathcal{R}} \frac{H_n}{n^p}\,.$$

Hence, formula (12) results from [3, Eqs. (1.22) and (1.24)]) while formula (13) results from [6, Proposition 1 and Eq. (13)]. □

Example 3. Applied to small values of p, formulas (12) and (13) translate into the following identities:

1) For $p = 1$,

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\,n} = \gamma\,, \tag{14}$$

and

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\,n^2} = \frac{1}{2}\gamma^2 - \frac{1}{2}\zeta(2) + \gamma_1 + \tau_1\,. \tag{15}$$

2) For $p = 2$,

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\,n} H_n = \zeta(2) - 1\,, \tag{16}$$

and

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\,n} H_n^{(2)} = 2\zeta(3) + \zeta'(2) + \tau_2\,. \tag{17}$$

3) For $p = 3$,

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\,n} \sum_{j=1}^{n} \frac{H_j}{j} = \zeta(3) - \frac{1}{2}\,, \tag{18}$$

and

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\,n} \sum_{j=1}^{n} \frac{H_j^{(2)}}{j} = \frac{5}{4}\zeta(4) - \zeta(2) - \zeta'(3) + \tau_3\,. \tag{19}$$

Remark 3. a) Formulas (14) and (16) are well-known classical series representations for γ and $\zeta(2)$ (cf. [2, 3, 4]).

b) Thanks to the relation (3), formula (18) can be rewritten under the following equivalent form:

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\,n} (H_n)^2 + \sum_{n=1}^{\infty} \frac{\lambda_n}{n!\,n} H_n^{(2)} = 2\zeta(3) - 1\,. \tag{20}$$

Subtracting (17) from (20) we deduce yet another interesting identity:

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\,n} (H_n)^2 = -\zeta'(2) - \tau_2 - 1\,. \tag{21}$$

c) By means of [6, Eq. (14)] we can also easily prove the following formula dual of (19):

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\,n} \sum_{j=1}^{n} \frac{H_j}{j^2} = \frac{7}{4}\zeta(4) + \zeta(2) + 2\zeta'(3) - 2\tau_3 - 1\,. \tag{22}$$

Therefore, adding (19) and (22), we obtain

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\,n} \sum_{j=1}^{n} \left\{ \frac{H_j}{j^2} + \frac{H_j^{(2)}}{j} \right\} = 3\zeta(4) + \zeta'(3) - \tau_3 - 1\,. \tag{23}$$

The proposition below is a more usable reformulation of [4, Theorem 10].

Proposition 2. For any integer $p \geq 2$, we have

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\,n^2} H_{n,p-1} = \gamma\zeta(p) + \zeta(p+1) - \mathcal{S}_{1,p} - \zeta'(p) - (-1)^p \tau_p - \sigma_p \tag{24}$$

with $\sigma_2 = 1$ and

$$\sigma_p = \frac{1+(-1)^p}{p} + \sum_{j=1}^{p-2} (-1)^j \zeta(p-j) \left[\frac{(j-1)!(p-1-j)!}{(p-1)!} - \frac{1}{j} \right] \quad \text{for } p \geq 3.$$

Proof. It follows from formulas (7) and (11) that

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\,n^{k+1}} H_{n,r-1} = \sum_{n\geq 1}^{\mathcal{R}} \frac{H_{n,k}^{(r)}}{n} \qquad (k \geq 0,\ r \geq 1).$$

Specializing this identity with $k = 1$ and $r = p$, we get

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\,n^2} H_{n,p-1} = \sum_{n\geq 1}^{\mathcal{R}} \frac{H_n^{(p)}}{n}\,.$$

Formula (24) then results from [6, Proposition 2]. □

Example 4. Applied to small values of p, formula (24) translates into the following identities:

a) For $p = 2$,

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\,n^2} H_n = \gamma\zeta(2) - \zeta(3) - \zeta'(2) - \tau_2 - 1\,. \tag{25}$$

Moreover, subtracting (25) from (21) and replacing $H_n - \frac{1}{n}$ by H_{n-1} leads to the surprisingly simple relation

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\,n} H_n\, H_{n-1} = \zeta(3) - \gamma\zeta(2)\,. \tag{26}$$

b) For $p = 3$,

$$\frac{1}{2} \sum_{n=1}^{\infty} \frac{\lambda_n}{n!\,n^2} \left\{ (H_n)^2 + H_n^{(2)} \right\} = \gamma\zeta(3) - \frac{1}{4}\zeta(4) - \frac{1}{2}\zeta(2) - \zeta'(3) + \tau_3\,. \tag{27}$$

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