New identities involving Cauchy numbers, harmonic numbers and zeta values

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Abstract In this article, we present a number of series identities linking together Cauchy numbers (also known as Bernoulli numbers of the second kind), the values of zeta and its derivative, and a generalization of Roman harmonic numbers.

Keywords: Cauchy numbers, Roman harmonic numbers, binomial identities, series with special numbers, zeta values.

Introduction

Several years ago, we proposed in [3] a method based on the Ramanujan summation of series which enabled us to generate a large number of identities involving Cauchy numbers, harmonic numbers and zeta values. By means of new formulas recently proved in our last paper [6], we are in a position to complete these results by showing new identities of the same kind (see Propositions 1 and 2). In order to do this, we make use of a rather natural generalization of Roman harmonic numbers introduced in [4] (see Definition 2). A notable novelty is the appearance in our formulas of certain alternating series involving the values of zeta that have recently been considered in [5] and thoroughly studied in [7] (see Definition 3).

1 Preliminaries : reminder of the main definitions and results

We first recall a number of definitions and results that appear in previous work.

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1.1 Harmonic numbers and harmonic sums

Definition 1. The generalized harmonic numbers $H_n^{(r)}$ are defined for all natural numbers n and r by

$$H_0^{(r)} = 0 \quad \text{and} \quad H_n^{(r)} = \sum_{j=1}^{n} \frac{1}{j^r} \quad \text{for } n \geq 1\,. \tag{1}$$

For $r = 1$, they reduce to classical harmonic numbers $H_n = H_n^{(1)}$. The sums

$$\mathcal{S}_{p,q} = \sum_{n=1}^{\infty} \frac{H_n^{(p)}}{n^q}$$

for positive integers p and q with $q \geq 2$ are called linear Euler sums. We have the classical Euler's formula (cf. [6, 9]):

$$2\mathcal{S}_{1,p} = (p+2)\zeta(p+1) - \sum_{j=1}^{p-2} \zeta(p-j)\zeta(j+1) \qquad (p \geq 2)\,.$$

Definition 2 (cf. [4])**.** The harmonic sums $H_{n,k}^{(r)}$ are defined for all natural numbers n, r and k with $n \geq 1$, $r \geq 1$ and $k \geq 0$ by

$$H_{n,0}^{(r)} = \frac{1}{n^{r-1}} \quad \text{and} \quad H_{n,k}^{(r)} = \sum_{n \geq j_1 \geq \cdots \geq j_k \geq 1} \frac{1}{j_1\, j_2 \cdots j_k^r} \quad \text{for } k \geq 1\,. \tag{2}$$

In particular, for $k = 1$, they reduce to classical generalized harmonic numbers

$$H_{n,1}^{(r)} = H_n^{(r)}\,,$$

and for $k = 2$, we have

$$H_{n,2}^{(r)} = \sum_{j=1}^{n} \frac{H_j^{(r)}}{j}\,.$$

Remark 1. For $r = 1$, the harmonic sums $H_{n,k} := H_{n,k}^{(1)}$ are nothing else than the ordinary Roman harmonic numbers $c_n^{(k)}$ (cf. [8]). In particular, we have

$$H_{n,0} = 1\,, \quad H_{n,1} = H_n\,, \quad H_{n,2} = \sum_{j=1}^{n} \frac{H_j}{j}\,, \quad \text{etc.}$$

In the general case, it can be shown (cf. [3, Eq. (18)] and [8, Eq. (29)]) that

$$H_{n,k} = P_k\left(H_n, \ldots, H_n^{(k)}\right)$$

where $P_k(x_1, \ldots, x_k)$ are the modified Bell polynomials (cf. [3, Definition 2]).

Example 1.

$$H_{n,2} = \sum_{j=1}^{n} \frac{H_j}{j} = \frac{1}{2}(H_n)^2 + \frac{1}{2} H_n^{(2)} \,. \tag{3}$$

The harmonic sums $H_{n,k}^{(r)}$ verify the following identity (cf. [4, Eq. (4.7)]):

$$H_{n,k}^{(r)} = \sum_{j=1}^{n} (-1)^{j-1} \binom{n}{j} \frac{H_{j,r-1}}{j^k} \quad (k \geq 1) \,. \tag{4}$$

Example 2. For $r = 1$, formula (4) reduces to

$$H_{n,k} = \sum_{j=1}^{n} (-1)^{j-1} \binom{n}{j} \frac{1}{j^k} \tag{5}$$

which is a classical property of Roman harmonic numbers (cf. [8, Eq. (20)]), and for $r = 2$, it is written as follows:

$$H_{n,k}^{(2)} = \sum_{j=1}^{n} (-1)^{j-1} \binom{n}{j} \frac{H_j}{j^k} \,. \tag{6}$$

By inverse binomial transform[1], the identity (4) also admits a kind of reciprocal:

$$\frac{H_{n,r-1}}{n^k} = \sum_{j=1}^{n} (-1)^{j-1} \binom{n}{j} H_{j,k}^{(r)} \quad (k \geq 1) \,. \tag{7}$$

1.2 Alternating series with zeta values

Definition 3 (cf. [5, 6, 7])**.** the conditionally convergent series τ_p are defined for any positive integer p by

$$\tau_p = \sum_{k=1}^{\infty} (-1)^{k+p} \frac{\zeta(k+p)}{k} \,. \tag{8}$$

For $p \geq 2$, they verify the relation

$$\sum_{n=1}^{\infty} \frac{\ln(n+1)}{n^p} = -\zeta'(p) - (-1)^p \, \tau_p \,, \tag{9}$$

and for $p = 1$, we have

$$\tau_1 = \int_0^1 \frac{\psi(x+1) + \gamma}{x} \, dx \tag{10}$$

where ψ is the digamma function.

1. If $b(n) = \frac{1}{n} \sum_{j=1}^{n} (-1)^{j-1} \binom{n}{j} j \, a(j)$, then $a(n) = \frac{1}{n} \sum_{j=1}^{n} (-1)^{j-1} \binom{n}{j} j \, b(j)$ (cf. [4, Definition 5 and Corollary 1]).

Remark 2. The series τ_p defined above are linked to the series σ_p introduced and studied in [7] by the elementary relation

$$\tau_p = (-1)^{p-1}\sigma_p \,.$$

1.3 Cauchy numbers

Definition 4 (cf. [1, 2, 3])**.** The *non-alternating Cauchy numbers* λ_n are defined for all integers $n \geq 1$ by the recurrence relation

$$\sum_{k=1}^{n-1} \frac{\lambda_k}{k!\,(n-k)} = \frac{1}{n} \quad \text{for } n \geq 2\,.$$

The first values are

$$\lambda_1 = \frac{1}{2}\,,\ \lambda_2 = \frac{1}{6}\,,\ \lambda_3 = \frac{1}{4}\,,\ \lambda_4 = \frac{19}{30}\,,\ \lambda_5 = \frac{9}{4}\,,\ \lambda_6 = \frac{863}{84}\,,\ \text{etc.}$$

Remark 3. The λ_n are linked to the classical Cauchy numbers denoted c_n in [1] by the relation

$$\lambda_n = (-1)^{n-1}c_n = (-1)^{n-1}\int_0^1 x(x-1)\cdots(x-n+1)\,dx \quad \text{for } n \geq 1.$$

We recall the transformation formula that links Cauchy numbers to the Ramanujan summation of series (cf. [2, Theorem 18]) : if a is a function analytic in the half-plane $P = \{\mathrm{Re}(z) > 0\}$ such that there exists a constant $C > 0$ with

$$|a(z)| < C\,2^{|z|} \quad \text{for all } z \in P\ ,$$

then

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\,n} \sum_{j=1}^{n} (-1)^{j-1}\binom{n}{j}\, j\, a(j) = \sum_{n\geq 1}^{\mathcal{R}} a(n) \tag{11}$$

where $\sum_{n\geq 1}^{\mathcal{R}}$ denotes the $\mathcal{R}$-sum of the series (i.e. the sum of the series in the sense of Ramanujan's summation method).

2 Series with Cauchy numbers and harmonic sums

Proposition 1. Let p be an integer with $p \geq 1$, then

a)

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\,n} H_{n,p-1} = \begin{cases} \gamma & \text{for } p = 1 \\ \zeta(p) - \frac{1}{p-1} & \text{for } p > 1 \end{cases} \tag{12}$$

where $\gamma = -\psi(1)$ is Euler's constant.

b)

$$\sum_{n=1}^{\infty}\frac{\lambda_n}{n!\,n}H_{n,p-1}^{(2)}=\begin{cases}\frac{1}{2}\gamma^2-\frac{1}{2}\zeta(2)+\gamma_1+\tau_1 & \text{for } p=1\\ \mathcal{S}_{1,p}-\sum_{j=1}^{p-2}\frac{(-1)^{p-j}}{j}\,\zeta(p-j)+(-1)^p\zeta'(p)+\tau_p & \text{for } p>1\end{cases} \tag{13}$$

where γ_1 is the first Stieltjes constant.

Proof. It follows from formulas (4) and (11) that

$$\sum_{n=1}^{\infty}\frac{\lambda_n}{n!\,n}H_{n,k}^{(r)}=\sum_{n\geq 1}^{\mathcal{R}}\frac{H_{n,r-1}}{n^{k+1}}\qquad (k\geq 0,\ r\geq 1).$$

Applying the above formula with $k=p-1$, we obtain

a) for $r=1$,

$$\sum_{n=1}^{\infty}\frac{\lambda_n}{n!\,n}H_{n,p-1}=\sum_{n\geq 1}^{\mathcal{R}}\frac{1}{n^p}\,,$$

b) for $r=2$,

$$\sum_{n=1}^{\infty}\frac{\lambda_n}{n!\,n}H_{n,p-1}^{(2)}=\sum_{n\geq 1}^{\mathcal{R}}\frac{H_n}{n^p}\,.$$

Hence, formula (12) results from [2, Eqs. (1.22) and (1.24)]) while formula (13) results from [6, Proposition 1 and Eq. (13)]. □

Example 3. a) For $p=1$, formulas (12)–(13) translate into

$$\sum_{n=1}^{\infty}\frac{\lambda_n}{n!\,n}=\gamma\,, \tag{14}$$

and

$$\sum_{n=1}^{\infty}\frac{\lambda_n}{n!\,n^2}=\frac{1}{2}\gamma^2-\frac{1}{2}\zeta(2)+\gamma_1+\tau_1\,. \tag{15}$$

b) For $p=2$, formulas (12)–(13) translate into

$$\sum_{n=1}^{\infty}\frac{\lambda_n}{n!\,n}H_n=\zeta(2)-1\,, \tag{16}$$

and

$$\sum_{n=1}^{\infty}\frac{\lambda_n}{n!\,n}H_n^{(2)}=2\zeta(3)+\zeta'(2)+\tau_2\,, \tag{17}$$

Remark 4. Formulas (14) and (16) are well-known classical series representations for γ and $\zeta(2)$ (cf. [1, 2, 3]).

Example 4. For $p = 3$, formulas (12)–(13) translate into

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\,n} \sum_{j=1}^{n} \frac{H_j}{j} = \zeta(3) - \frac{1}{2}\,, \tag{18}$$

and

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\,n} \sum_{j=1}^{n} \frac{H_j^{(2)}}{j} = \frac{5}{4}\zeta(4) - \zeta(2) - \zeta'(3) + \tau_3\,. \tag{19}$$

Remark 5. a) Thanks to the identity (3), formula (18) can be rewritten under the following form:

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\,n}(H_n)^2 + \sum_{n=1}^{\infty} \frac{\lambda_n}{n!\,n} H_n^{(2)} = 2\zeta(3) - 1\,. \tag{20}$$

Subtracting (17) from (20) we deduce yet another interesting identity:

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\,n}(H_n)^2 = -\zeta'(2) - \tau_2 - 1\,. \tag{21}$$

b) By means of [6, Eq. (14)] we can also easily prove the following formula dual of (19):

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\,n} \sum_{j=1}^{n} \frac{H_j}{j^2} = \frac{7}{4}\zeta(4) + \zeta(2) + 2\zeta'(3) - 2\tau_3 - 1\,. \tag{22}$$

Hence, by adding (19) and (22) we obtain

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\,n} \sum_{j=1}^{n} \left\{ \frac{H_j}{j^2} + \frac{H_j^{(2)}}{j} \right\} = 3\zeta(4) + \zeta'(3) - \tau_3 - 1\,. \tag{23}$$

Proposition 2. Let p be an integer with $p \geq 2$, then

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\,n^2} H_{n,p-1} = \gamma\zeta(p) + \zeta(p+1) - \mathcal{S}_{1,p} - \zeta'(p) - (-1)^p \tau_p - \sigma_p \tag{24}$$

with $\sigma_2 = 1$ and

$$\sigma_p = \frac{1 + (-1)^p}{p} + \sum_{j=1}^{p-2} (-1)^j \zeta(p-j) \left[\frac{(j-1)!(p-1-j)!}{(p-1)!} - \frac{1}{j} \right] \quad \text{for } p \geq 3.$$

Proof. It follows from formulas (7) and (11) that

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\, n^{k+1}} H_{n,r-1} = \sum_{n\geq 1}^{\mathcal{R}} \frac{H_{n,k}^{(r)}}{n} \qquad (k \geq 0,\ r \geq 1).$$

The above formula applied for $k=1$ and $r=p$ gives

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\, n^{2}} H_{n,p-1} = \sum_{n\geq 1}^{\mathcal{R}} \frac{H_{n}^{(p)}}{n}\,.$$

Formula (24) then results from [6, Proposition 2]. □

Example 5. Formula (24) applied with $p=2$ gives the following identity:

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\, n^{2}} H_n = \gamma\zeta(2) - \zeta(3) - \zeta'(2) - \tau_2 - 1\,. \tag{25}$$

Moreover, subtracting (25) from (21) and replacing $H_n - \frac{1}{n}$ by H_{n-1} leads to the nice expression

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\, n} H_n\, H_{n-1} = \zeta(3) - \gamma\zeta(2)\,. \tag{26}$$

Example 6. Formula (24) applied with $p=3$ gives the relation:

$$\frac{1}{2}\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\, n^{2}} \left\{(H_n)^2 + H_n^{(2)}\right\} = \gamma\zeta(3) - \frac{1}{4}\zeta(4) - \frac{1}{2}\zeta(2) - \zeta'(3) + \tau_3\,. \tag{27}$$

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