New identities involving Cauchy numbers, harmonic numbers and zeta values

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Abstract. In this article, we present a number of series identities involving Cauchy numbers (also known as Bernoulli numbers of the second kind) and certain harmonic sums which generalize Roman harmonic numbers.

Introduction

Several years ago, we proposed in [3] a method based on the Ramanujan summation of series which enabled us to generate a large number of identities involving Cauchy numbers, harmonic numbers and zeta values. By means of new formulas recently proved in our last paper [6], we complete these results by showing new identities of the same kind while, at the same time, reinterpreting those already known (see Propositions 1 and 2). A noteworthy novelty is the appearance in our closed form evaluations of certain interesting alternating series with zeta values (see Definition 3) which were recently introduced in [5] and thoroughly studied in [7]. We also introduce a rather natural generalization of Roman harmonic numbers (see Definition 2).

1 Preliminaries : reminder of the main definitions and results

We first recall a number of definitions and results that appear in previous work.

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1.1 Harmonic numbers and harmonic sums

Definition 1. The generalized harmonic numbers $H_n^{(r)}$ are defined for all natural numbers n and r by

$$H_0^{(r)} = 0 \quad \text{and} \quad H_n^{(r)} = \sum_{j=1}^{n} \frac{1}{j^r} \quad \text{for } n \geq 1\,. \tag{1}$$

For $r = 1$, they reduce to classical harmonic numbers $H_n = H_n^{(1)}$. The sums

$$\mathcal{S}_{p,q} = \sum_{n=1}^{\infty} \frac{H_n^{(p)}}{n^q}$$

for positive integers p and q with $q \geq 2$ are called linear Euler sums. We have the classical Euler's formula (cf. [6, 9]):

$$2\mathcal{S}_{1,p} = (p+2)\zeta(p+1) - \sum_{j=1}^{p-2} \zeta(p-j)\zeta(j+1) \qquad (p \geq 2)\,.$$

Definition 2 (cf. [4])**.** The harmonic sums $H_{n,k}^{(r)}$ are defined for all natural numbers n, r and k with $n \geq 1$, $r \geq 1$ and $k \geq 0$ by

$$H_{n,0}^{(r)} = \frac{1}{n^{r-1}} \quad \text{and} \quad H_{n,k}^{(r)} = \sum_{n \geq j_1 \geq \cdots \geq j_k \geq 1} \frac{1}{j_1\, j_2 \cdots j_k^r} \quad \text{for } k \geq 1\,. \tag{2}$$

In particular, for $k = 1$, they reduce to classical generalized harmonic numbers

$$H_{n,1}^{(r)} = H_n^{(r)}\,,$$

and for $k = 2$, we have

$$H_{n,2}^{(r)} = \sum_{j=1}^{n} \frac{H_j^{(r)}}{j}\,.$$

Remark 1. For $r = 1$, the harmonic sums $H_{n,k} := H_{n,k}^{(1)}$ are nothing else than the ordinary Roman harmonic numbers $c_n^{(k)}$ (cf. [8]). In particular, we have

$$H_{n,0} = 1\,, \quad H_{n,1} = H_n\,, \quad H_{n,2} = \sum_{j=1}^{n} \frac{H_j}{j}\,, \quad \text{etc.}$$

In the general case, it can be shown (cf. [3, Eq. (18)] and [8, Eq. (29)]) that

$$H_{n,k} = P_k\left(H_n, \ldots, H_n^{(k)}\right)$$

where $P_k(x_1, \ldots, x_k)$ are the modified Bell polynomials (cf. [3, Definition 2]).

The harmonic sums $H_{n,k}^{(r)}$ verify the following fundamental property (cf. [4, Corollary 8]):

$$H_{n,k}^{(r)} = \sum_{j=1}^{n} (-1)^{j-1} \binom{n}{j} \frac{H_{j,r-1}}{j^k} \quad (k \geq 1)\,. \tag{3}$$

For $r = 1$, formula (3) reduces to

$$H_{n,k} = \sum_{j=1}^{n} (-1)^{j-1} \binom{n}{j} \frac{1}{j^k} \tag{4}$$

which is a classical property of Roman harmonic numbers (cf. [8, Eq. (20)]), and for $r = 2$, formula (3) translates into

$$H_{n,k}^{(2)} = \sum_{j=1}^{n} (-1)^{j-1} \binom{n}{j} \frac{H_j}{j^k}\,. \tag{5}$$

Formula (3) also admits a kind of reciprocal:

$$\frac{H_{n,r-1}}{n^k} = \sum_{j=1}^{n} (-1)^{j-1} \binom{n}{j} H_{j,k}^{(r)} \quad (k \geq 1)\,. \tag{6}$$

1.2 Alternating series with zeta values

Definition 3 (cf. [5, 6, 7])**.** the conditionally convergent series τ_p are defined for any positive integer p by

$$\tau_p = \sum_{k=1}^{\infty} (-1)^{k+p} \frac{\zeta(k+p)}{k}\,. \tag{7}$$

For $p \geq 2$, they verify the relation

$$\sum_{n=1}^{\infty} \frac{\ln(n+1)}{n^p} = -\zeta'(p) - (-1)^p \,\tau_p\,, \tag{8}$$

and for $p = 1$, we have

$$\tau_1 = \int_0^1 \frac{\psi(x+1) + \gamma}{x}\, dx \tag{9}$$

where ψ is the digamma function.

Remark 2. The series τ_p defined above are linked to the series σ_p introduced and studied in [7] by the elementary relation

$$\tau_p = (-1)^{p-1} \sigma_p\,.$$

1.3 Cauchy numbers

Definition 4 (cf. [1, 2, 3])**.** The *non-alternating Cauchy numbers* λ_n are defined for all integers $n \geq 1$ by the recurrence relation

$$\sum_{k=1}^{n-1} \frac{\lambda_k}{k!\,(n-k)} = \frac{1}{n} \qquad \text{for } n \geq 2\,.$$

The first values are

$$\lambda_1 = \frac{1}{2}\,,\ \lambda_2 = \frac{1}{6}\,,\ \lambda_3 = \frac{1}{4}\,,\ \lambda_4 = \frac{19}{30}\,,\ \lambda_5 = \frac{9}{4}\,,\ \lambda_6 = \frac{863}{84}\,,\ \text{etc.}$$

Remark 3. The λ_n are linked to the classical Cauchy numbers c_n considered in [1] by the relation

$$\lambda_n = (-1)^{n-1} c_n \qquad \text{for } n \geq 1.$$

Similarly, they are linked by the same relation to the Bernoulli numbers of the second kind β_n considered in [2, § 4.2]:

$$\lambda_n = (-1)^{n-1}\beta_n = (-1)^{n-1}\int_0^1 x(x-1)\cdots(x-n+1)\,dx \qquad (n \geq 1)\,.$$

We recall the transformation formula that links Cauchy numbers to the Ramanujan summation of series (cf. [2, Theorem 18]) : if a is a function analytic in the half-plane $P = \{\mathrm{Re}(z) > 0\}$ such that there exists a constant $C > 0$ for which

$$|a(z)| < C\,2^{|z|} \qquad \text{for all } z \in P\ \ ,$$

then

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\,n} \sum_{j=1}^{n} (-1)^{j-1}\binom{n}{j}\, j\, a(j) = \sum_{n\geq 1}^{\mathcal{R}} a(n) \tag{10}$$

where $\sum_{n\geq 1}^{\mathcal{R}}$ denotes the $\mathcal{R}$-sum of the series (i.e. the sum of the series in the sense of Ramanujan's summation method).

2 Series with Cauchy numbers and harmonic sums

Proposition 1. Let p be an integer with $p \geq 1$, then

a)

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\,n} H_{n,p-1} = \begin{cases} \gamma & \text{for } p = 1 \\ \zeta(p) - \frac{1}{p-1} & \text{for } p > 1 \end{cases} \tag{11}$$

where $\gamma = -\psi(1)$ is Euler's constant.

b)

$$\sum_{n=1}^{\infty}\frac{\lambda_n}{n!\,n}H_{n,p-1}^{(2)}=\begin{cases}\frac{1}{2}\gamma^2-\frac{1}{2}\zeta(2)+\gamma_1+\tau_1 & \text{for } p=1\\ \mathcal{S}_{1,p}-\sum_{j=1}^{p-2}\frac{(-1)^{p-j}}{j}\,\zeta(p-j)+(-1)^p\zeta'(p)+\tau_p & \text{for } p>1\end{cases} \tag{12}$$

where γ_1 is the first Stieltjes constant.

Proof. It follows from formulas (3) and (10) that

$$\sum_{n=1}^{\infty}\frac{\lambda_n}{n!\,n}H_{n,k}^{(r)}=\sum_{n\geq 1}^{\mathcal{R}}\frac{H_{n,r-1}}{n^{k+1}}\qquad (k\geq 0,\ r\geq 1).$$

Applying the above formula with $k=p-1$, we obtain

a) for $r=1$,

$$\sum_{n=1}^{\infty}\frac{\lambda_n}{n!\,n}H_{n,p-1}=\sum_{n\geq 1}^{\mathcal{R}}\frac{1}{n^p}\,,$$

b) for $r=2$,

$$\sum_{n=1}^{\infty}\frac{\lambda_n}{n!\,n}H_{n,p-1}^{(2)}=\sum_{n\geq 1}^{\mathcal{R}}\frac{H_n}{n^p}\,.$$

Hence, formula (11) results from [2, Eqs. (1.22) and (1.24)]) while formula (12) results from [6, Proposition 1 and Eq. (13)]. $\square$

Example 1. a) For $p=1$, formulas (11)–(12) translate into

$$\sum_{n=1}^{\infty}\frac{\lambda_n}{n!\,n}=\gamma\,, \tag{13}$$

and

$$\sum_{n=1}^{\infty}\frac{\lambda_n}{n!\,n^2}=\frac{1}{2}\gamma^2-\frac{1}{2}\zeta(2)+\gamma_1+\tau_1\,. \tag{14}$$

b) For $p=2$, formulas (11)–(12) translate into

$$\sum_{n=1}^{\infty}\frac{\lambda_n}{n!\,n}H_n=\zeta(2)-1\,, \tag{15}$$

and

$$\sum_{n=1}^{\infty}\frac{\lambda_n}{n!\,n}H_n^{(2)}=2\zeta(3)+\zeta'(2)+\tau_2\,, \tag{16}$$

Remark 4. Formulas (13) and (15) are well-known classical series representations for γ and $\zeta(2)$ (cf. [1, 2, 3]).

Example 2. For $p = 3$, formulas (11)–(12) translate into

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\, n} \sum_{j=1}^{n} \frac{H_j}{j} = \zeta(3) - \frac{1}{2}\,, \tag{17}$$

and

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\, n} \sum_{j=1}^{n} \frac{H_j^{(2)}}{j} = \frac{5}{4}\zeta(4) - \zeta(2) - \zeta'(3) + \tau_3\,. \tag{18}$$

Remark 5. By means of [6, Eq. (14)], we can also easily prove the following formula dual of (18):

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\, n} \sum_{j=1}^{n} \frac{H_j}{j^2} = \frac{7}{4}\zeta(4) + \zeta(2) + 2\zeta'(3) - 2\tau_3 - 1\,. \tag{19}$$

Formula (14) admits the following generalization:

Proposition 2. Let p be an integer with $p \geq 2$, then

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\, n^2} H_{n,p-1} = \gamma\zeta(p) + \zeta(p+1) - \mathcal{S}_{1,p} - \zeta'(p) - (-1)^p\tau_p - \sigma_p \tag{20}$$

with $\sigma_2 = 1$ and

$$\sigma_p = \frac{1+(-1)^p}{p} + \sum_{j=1}^{p-2}(-1)^j\zeta(p-j)\left[\frac{(j-1)!(p-1-j)!}{(p-1)!} - \frac{1}{j}\right] \quad \text{for } p \geq 3.$$

Proof. It follows from formulas (6) and (10) that

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\, n^{k+1}} H_{n,r-1} = \sum_{n\geq 1}^{\mathcal{R}} \frac{H_{n,k}^{(r)}}{n} \qquad (k \geq 0,\ r \geq 1).$$

The above formula applied for $k = 1$ and $r = p$ gives

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\, n^2} H_{n,p-1} = \sum_{n\geq 1}^{\mathcal{R}} \frac{H_n^{(p)}}{n}\,.$$

Formula (20) then results from [6, Proposition 2]. $\square$

Example 3. Formula (20) applied with $p = 2$ leads to the following relation:

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\, n^2} H_n = \gamma\zeta(2) - \zeta(3) - \zeta'(2) - \tau_2 - 1\,. \tag{21}$$

Example 4. Since

$$H_{n,2} = \sum_{j=1}^{n} \frac{H_j}{j} = \frac{1}{2}(H_n)^2 + \frac{1}{2}H_n^{(2)}\,,$$

formula (17) can be rewritten under the following form:

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\, n}(H_n)^2 + \sum_{n=1}^{\infty} \frac{\lambda_n}{n!\, n} H_n^{(2)} = 2\zeta(3) - 1\,.$$

By means of formula (16), this enables to deduce yet another interesting identity:

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\, n}(H_n)^2 = -\zeta'(2) - \tau_2 - 1\,. \tag{22}$$

Subtracting (21) from (22) and writing $H_n - \frac{1}{n} = H_{n-1}$, we also obtain this nice relation:

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\, n} H_n\, H_{n-1} = \zeta(3) - \gamma\zeta(2)\,. \tag{23}$$

Example 5. Formula (20) applied with $p = 3$ gives the relation:

$$\frac{1}{2}\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\, n^2} \left\{(H_n)^2 + H_n^{(2)}\right\} = \gamma\zeta(3) - \frac{1}{4}\zeta(4) - \frac{1}{2}\zeta(2) - \zeta'(3) + \tau_3\,. \tag{24}$$

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