Adjoint $(1, 1)$-classes on threefolds

Andreas Höring

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ABSTRACT. We answer a question of Filip and Tosatti concerning a basepoint-free theorem for transcendental $(1,1)$-classes on threefolds.

1. INTRODUCTION

In a recent preprint Filip and Tosatti proposed the following transcendental version of the basepoint-free theorem:

1.1. Conjecture. [FT17, Conj.1.2] *Let X be a compact Kähler manifold, and let α be a nef $(1,1)$-class on X such that $\alpha - K_X$ is nef and big. Then the class α is semiample: there exists a morphism with connected fibres $\psi : X \to Z$ onto a normal compact Kähler space and a Kähler class α_Z on Z such that $\alpha = \psi^* \alpha_Z$.*

If α is the class of a $\mathbb{Q}$-divisor (and hence X is projective), this statement is equivalent to the basepoint-free theorem (cf. [BCHM10, Thm.3.9.1] for $\mathbb{R}$-divisors). Filip and Tosatti proved this conjecture for surfaces [FT17, Thm.1.3]. The aim of this note is to clarify the situation in dimension three:

1.2. Theorem. *Conjecture 1.1 holds if $\dim X = 3$ and $\alpha - K_X$ is a Kähler class.*

The assumption is slightly stronger than in the conjecture, but should be enough for potential applications to the Kähler-Ricci flow. Theorem 1.2 was shown by Tosatti and Zhang [TZ18, Thm.1.4] when the class α is nef, but not big, so we only have to deal with the case where the morphism ψ is bimeromorphic. More precisely we prove the following:

1.3. Theorem. *Let X be a normal $\mathbb{Q}$-factorial compact Kähler threefold with terminal singularities. Let ω be a Kähler class on X such that $\alpha := K_X + \omega$ is nef and big. Then there exists a bimeromorphic morphism $\psi : X \to Z$ onto a normal compact Kähler space with isolated rational singularities and a Kähler class α_Z on Z such that $\alpha = \psi^* \alpha_Z$.*

If the curves contracted by ψ define an extremal *ray* in the (generalised) Mori cone of X, this is the contraction theorem for Kähler threefolds (see Remark 3.5). In the setting of Theorem 1.3 we contract, more generally, an extremal *face* of the Mori cone. The obvious idea is to reduce to the case of extremal rays by running a MMP $X \dashrightarrow Y$ where we contract only $\alpha_\bullet$-trivial $K_\bullet$-negative extremal rays. However Y is typically *not* the space we are looking for. Indeed if one of the steps of the MMP is a flip, the space Y can contain α_Y-trivial curves that are K_Y-positive. These curves can not contracted by the MMP, so we need some additional argument.

At the moment we do not know how to prove Conjecture 1.1 if the class $\alpha - K_X$ is merely nef and big. The main reason is that for crepant birational contractions the geometry of the exceptional locus is more complicated, so the strategy used for K_X-negative extremal rays in [HP16, CHP16] will not work. However it is easy to settle a special case:

1.4. Proposition. *Let X be a normal $\mathbb{Q}$-factorial compact Kähler threefold with terminal singularities such that $K_X \equiv 0$. Let α be a nef and big $(1,1)$-class on X. Then there exists a bimeromorphic morphism $\psi : X \to Z$ onto a normal compact Kähler space with rational singularities and a Kähler class α_Z on Z such that $\alpha = \psi^* \alpha_Z$.*

The proof is based on the observation that the decomposition theorem [CHP16, Dru18] essentially reduces the problem to the case of surfaces proven by Filip and Tosatti. In a similar spirit the Beauville-Bogomolov decomposition [Bea83b] reduces Conjecture 1.1 for manifolds with trivial canonical class to the case of Hyperkähler manifolds. This case is of independent interest (cf. [AV18]), we hope to come back to it in the future.

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2. DEFINITIONS AND TERMINOLOGY

We will use standard terminology of the minimal model program (MMP) as explained in [KM98] or [Deb01], see also [HP16, Sect.2-3] for minor modifications in the Kähler setting. For notions from analytic geometry we refer to [DP04], [Dem12].

Let X be a normal compact complex space with at most rational singularities. Suppose that X is in the Fujiki class, i.e. X is bimeromorphic to a compact Kähler manifold. A $(1,1)$-class on X is an element of $H^{1,1}_{\mathrm{BC}}(X)$, the Bott-Chern group of $(1,1)$-currents that are locally $\partial\bar{\partial}$-exact (cf. [HP16, Sect.2] for details). Even if X is projective, the inclusion

$$\mathrm{NS}(X) \otimes \mathbb{R} \subset H^{1,1}_{\mathrm{BC}}(X)$$

is typically not an equality. However if $H^2(X, \mathcal{O}_X) = 0$, we can apply Kodaira's criterion to a desingularisation to see that X is projective and every $(1,1)$-class is an $\mathbb{R}$-divisor class.

Let $\alpha \in H^{1,1}_{\mathrm{BC}}(X)$ be a $(1,1)$-class. An irreducible curve $C \subset X$ is α-trivial if $\alpha \cdot C = 0$. The class α is big if it contains a Kähler current, it is modified Kähler if there exists a modification $\mu : X' \to X$ and a Kähler form ω on X' such that $[\mu_* \omega] = \alpha$. A modified Kähler class is big, moreover for every hypersurface $S \subset X$, the restriction $\alpha|_S$ is big: since the hypersurface S is not in the image of the μ-exceptional locus, the restriction of the current $\mu_* \omega$ to S is well-defined and a Kähler current.

Following [Pau98, Defn.3] one can define when a $(1,1)$-class is nef. A nef $(1,1)$-class is always nef in the algebraic sense, i.e. for every subvariety $Z \subset X$ we have

$$Z \cdot \alpha^{\dim Z} \geq 0.$$

For a nef and big class α on X one defines the Null locus

$$\mathrm{Null}(\alpha) = \cup_{Z \cdot \alpha^{\dim Z} = 0} Z,$$

where the union runs over all positive-dimensional subvarieties. A priori the Null locus is a countable union of subvarieties, but by a theorem of Collins and Tosatti [CT15] (applied to the pull-back of the class α to a desingularisation) we know that the Null locus has only finitely many irreducible components. Moreover we have the following :

2.1. Lemma. *Let X be a normal compact complex space with isolated rational singularities. Let $\alpha \in H^{1,1}_{\mathrm{BC}}(X)$ be a nef and big class such that $Z \cdot \alpha^{\dim Z} > 0$ for every positive-dimensional subvariety. Then α is a Kähler class, i.e. there exists a Kähler form ω on X such that $[\omega] = \alpha$.*

Proof. Let $\mu : X' \to X$ be a resolution of singularities. Then $\mu^* \alpha$ is a nef and big class such that the Null locus coincides with the μ-exceptional locus. By [CT15, Thm.1.1] this implies that the non-Kähler locus of $\mu^* \alpha$ coincides with the μ-exceptional locus. Combining Demailly's regularisation [Dem92] and [Bou04, Thm.3.17(ii)] this implies that there exists a Kähler current T in the class $\mu^* \alpha$ which is singular exactly along the exceptional locus. The push-forward $\mu_* T$ is thus a Kähler current in the class α that is singular at most in the finitely many singular points of X. Using a regularized maximum, we obtain the smooth form ω (cf. the proof of [FT17, Thm.1.3] for details). $\qquad\square$

3. Bimeromorphic geometry

The following lemma is a consequence of Araujo's description of the mobile cone [Ara10, Thm.1.3], the proof is part of the proof of [HP16, Prop.7.11].

3.1. Lemma. *Let F be a projective surface such that $H^2(F, \mathcal{O}_F) = 0$. Let α_F be a nef (1,1)-class such that*

$$\alpha_F^2 = 0, \quad K_F \cdot \alpha_F < 0.$$

Then F is covered by α_F-trivial curves.

We will need a slightly stronger version of [HP16, Prop.7.11]:

3.2. Lemma. *Let Y be a normal $\mathbb{Q}$-factorial compact Kähler threefold with terminal singularities. Let ω_Y be a modified Kähler class on Y such that $\alpha_Y := K_Y + \omega_Y$ is nef and big. Let $S \subset Y$ be an integral surface such that $\alpha_Y^2 \cdot S = 0$. Then S is Moishezon and covered by α_Y-trivial curves.*

Proof. Let $\pi : S' \to S$ be the composition of normalisation and minimal resolution of S, then we have

(1) $$K_{S'} = \pi^* K_S - E$$

with E an effective $\mathbb{Q}$-divisor on S' (see [HP16, Sect.4.1]). Since S is a Kähler space, the surface S' is Kähler.

1st case. The pull-back $\pi^* \alpha_Y|_S$ is numerically trivial. Then we have $-\pi^* K_Y|_S = \pi^* \omega_Y|_S$. Since ω_Y is modified Kähler the restriction $\omega_Y|_S$ is a big (1,1)-class. Thus $-\pi^* K_Y|_S$ is a big line bundle on S' and S' is Moishezon. In particular S' (and hence S) is covered by curves, they are all $\alpha_Y|_S$-trivial.

2nd case. The pull-back $\pi^* \alpha_Y|_S$ is not numerically trivial. The nef and big class α_Y defines an intersection form on $N^1(Y)$ which, by the Hodge index theorem, has signature $(1, d)$. Since $\alpha_Y^2 \cdot S = 0$ this implies that either $\alpha_Y \cdot S = 0$ or

$\alpha_Y \cdot S^2 < 0$. The former case is excluded since $\pi^* \alpha_Y|_S$ is not numerically trivial. By the adjunction formula and using $K_Y = \alpha_Y - \omega_Y$ we obtain

$$K_S \cdot \alpha_Y|_S = (K_Y + S) \cdot S \cdot \alpha_Y = \alpha_Y^2 \cdot S - \alpha_Y \cdot \omega_Y \cdot S + \alpha_Y \cdot S^2.$$

The first term is zero, the other terms are negative, so $K_S \cdot \alpha_Y|_S < 0$. By (1) this implies

$$K_{S'} \cdot \pi^* \alpha_Y|_S \le K_S \cdot \alpha_Y|_S < 0.$$

Since $\pi^* \alpha_Y|_S$ is a nef class, this shows that $K_{S'}$ is not pseudoeffective. In particular we have $H^2(S', \mathcal{O}_{S'}) = H^0(S', K_{S'}) = 0$. By Kodaira's criterion S' is projective and we conclude with Lemma 3.1. $\qquad\square$

3.3. Proposition. *Let Y be a normal compact Kähler threefold with isolated rational singularities. Let α_Y be a (1,1)-class on Y that is nef and big. Suppose that every irreducible component of $\mathrm{Null}(\alpha_Y)$ has dimension one. Then there exists a bimeromorphic morphism $\mu_Y : Y \to Z$ onto a normal compact complex space Z such that every connected component of $\mathrm{Null}(\alpha_Y)$ is contracted onto a point, and*

$$Y \setminus \mathrm{Null}(\alpha_Y) \simeq Z \setminus \mu_Y(\mathrm{Null}(\alpha_Y))$$

is an isomorphism.

3.4. Remark. We do not claim that α_Y is a pull-back of a $(1,1)$-class from Z. In fact this already fails for nef and big divisors on projective surfaces.

Proof. The proof of [HP16, Thm.7.12] applies without changes. Indeed the first part of the proof consists in verifying that the nef supporting class is nef and big and has one-dimensional non-Kähler locus (which is exactly our assumption). The rest of the proof uses only this property. $\qquad\square$

3.5. Remark. We need a technical remark concerning the use of MMP in our setting: let X be normal $\mathbb{Q}$-factorial compact Kähler threefold with terminal singularities. Suppose either that K_X is pseudoeffective (or equivalently, that X is not uniruled), or that the MRC fibration is an almost holomorphic map $X \dashrightarrow B$ onto a compact Kähler surface. Let α be a nef and big class on X and suppose that there exists a curve $C \subset X$ such that

$$\alpha \cdot C = 0, \ K_X \cdot C < 0.$$

Then there exists a contraction $\mu : X \to X'$ of a K_X-negative extremal ray Γ such that $\alpha \cdot \Gamma = 0$.

If K_X is pseudoeffective, this is completely standard: by the cone theorem [HP16, Thm.1.2] we can decompose the class

$$C = \eta + \sum \lambda_i \Gamma_i$$

where $K_X \cdot \eta \ge 0$, the coefficients $\lambda_i \ge 0$ and Γ_i are rational curves generating extremal rays in the cone $\overline{\mathrm{NA}}(X)$. Since $K_X \cdot C < 0$ there exists at least one coefficient $\lambda_{i_0} > 0$. Since α is nef and $\alpha \cdot C = 0$ we have $\alpha \cdot \Gamma_{i_0} = 0$. By the contraction theorem [HP16, Thm.1.3] we can contract the extremal ray $\mathbb{R}^+ \Gamma_{i_0}$.

If the base of the MRC fibration has dimension two, we only have weaker forms of the cone and contraction theorem [HP15]. Let F be a general fibre of the MRC-fibration. Then $\alpha \cdot F > 0$, since α is nef and big. Let ω_X be a Kähler class on X.

Then for all $\varepsilon > 0$ the class

$$\omega_\varepsilon := 2\frac{\alpha + \varepsilon(\alpha \cdot F)\omega_X}{(\alpha + \varepsilon(\alpha \cdot F)\omega_X) \cdot F}$$

is a normalised Kähler class in the sense of [HP15, Defn.1.2.]. Moreover, since $\alpha \cdot C = 0$, we know that for $0 < \varepsilon \ll 1$ the intersection number

$$(K_X + \omega_\varepsilon) \cdot C < 0.$$

Thus we can use [HP15, Thm.3.13, Thm.3.15] to conclude as above.

Proof of Theorem 1.3. If $H^2(X, \mathcal{O}_X) = 0$ the nef and big class α is an $\mathbb{R}$-divisor class, so we can conclude with the basepoint-free theorem for $\mathbb{R}$-divisors. Suppose from now on that $H^2(X, \mathcal{O}_X) \neq 0$. Then X is either not uniruled or the MRC fibration is an almost holomorphic map $X \dashrightarrow B$ onto a compact Kähler surface (cf. [HP15, Sect.1]). Note that these properties are invariant under the minimal model program.

Step 1. Running a MMP. We start by running an $\alpha_\bullet$-trivial $K_\bullet$-MMP. More precisely set

$$X_0 := X, \qquad \omega_0 := \omega, \qquad \alpha_0 := \alpha.$$

We define inductively a sequence of bimeromorphic maps as follows: for $i \in \mathbb{N}$, suppose that there exists a curve $C \subset X$ such that $K_{X_{i-1}} \cdot C < 0$ and $\alpha \cdot C = 0$. By Remark 3.5 there exists a contraction $\varphi_i : X_{i-1} \to X_i$ of a $K_{X_{i-1}}$-negative extremal ray Γ_{i-1} such that $\alpha_{i-1} \cdot \Gamma_{i-1} = 0$. The contraction is not of fibre type, since otherwise X_{i-1} is covered by α_{i-1}-trivial curves, contradicting the hypothesis that α_{i-1} is nef and big. Thus the contraction is bimeromorphic, and we denote by $\mu_i : X_{i-1} \dashrightarrow X_i$ the divisorial contraction or, for a small ray, its flip. Since the contraction is α_{i-1}-trivial, the push-forward $(\mu_i)_*\alpha_{i-1} =: \alpha_i$ is a nef and big $(1,1)$-class α_i on X_i. We also set $(\mu_i)_*\omega_{i-1} =: \omega_i$, so $K_{X_i} + \omega_i = \alpha_i$. The class ω_i is not necessarily Kähler, but since $(\mu_i)^{-1}$ does not contract any divisors, it is a modified Kähler class.

By Mori's termination of flips for terminal threefolds [Mor88], the MMP terminates after finitely many steps, so composing the μ_i we obtain a bimeromorphic morphism

$$\mu : X \dashrightarrow Y$$

such that Y is a normal $\mathbb{Q}$-factorial compact Kähler threefold with terminal singularities, the class $\mu_*\alpha =: \alpha_Y$ is nef and big and $\alpha_Y = K_Y + \omega_Y$ with ω_Y a modified Kähler class. Since Y is the outcome of the $\alpha_\bullet$-trivial $K_\bullet$-MMP, we have $\alpha_Y \cdot C > 0$ for every curve C such that $K_Y \cdot C < 0$.

We claim that $\text{Null}(\alpha_Y)$ does not contain an irreducible surface S: otherwise we know by Lemma 3.2 that S is covered α_Y-trivial negative curves $(C_t)_{t \in T}$. Since ω_Y is modified Kähler, the restriction $\omega_Y|_S$ is a big $(1,1)$-class. Thus for a general curve C_t we have $\omega_Y \cdot C_t > 0$. Thus we obtain that $K_Y \cdot C_t < 0$, in contradiction to the preceding paragraph.

Step 2. Construction of Z. By the claim the Null locus of α_Y has pure dimension one. By Proposition 3.3 there exists thus a bimeromorphic map $\mu_Y : Y \to Z$ contracting the connected components of $\text{Null}(\alpha_Y)$ onto points. We claim that the composition $\psi := \mu_Y \circ \mu : X \dashrightarrow Z$ is holomorphic: denote by $\Gamma_\mu \subset X \times Y$ the graph of μ and by $p : \Gamma_\mu \to X$ the projection onto X. Denote by $\Gamma_\psi \subset X \times Z$

the graph of ψ. Then ψ is a morphism if and only if the projection $\Gamma_\psi \to X$ is an isomorphism.

Since ψ is the composition of the meromorphic map μ with the holomorphic map μ_Y this is equivalent to showing that the p-fibres are contained in $\mathrm{Null}(\alpha_Y)^1$. Yet for a threefold-MMP, the p-fibres are the (strict transforms) of flipped curves. Since we ran an $\alpha_\bullet$-trivial MMP, the class $\alpha_\bullet$ is trivial on both the contracted and the flipped curves. Thus the p-fibres are α_Y-trivial.

Thus we have constructed a bimeromorphic morphism $\psi : X \to Z$ such that the restriction of α to every fibre is numerically trivial and the exceptional locus coincides with $\mathrm{Null}(\alpha)$. Since $\alpha = K_X + \omega$ and ω is Kähler, this morphism is projective with relatively ample line bundle $-K_X$. In particular by relative Kodaira vanishing [Anc87] we have $R^j\psi_*\mathcal{O}_X = 0$ for all $j \geq 0$. Since X has terminal, hence rational singularities, this implies that Z rational singularities. By [HP16, Lemma 3.3.] there exists thus a (1,1)-class α_Z on Z such that $\alpha = \psi^*\alpha_Z$. By Lemma 2.1 the class α_Z is Kähler. $\qquad\square$

Remark. In general the space Z is not $\mathbb{Q}$-factorial, since μ_Y is a small contraction. Thus Z does not necessarily have terminal singularities.

4. Calabi-Yau case

Proof of Proposition 1.4. By the decomposition theorem [Dru18, Thm.1.2] [CHP16, Thm.9.2] there exists a finite cover $\mu : \tilde{X} \to X$ that is étale over the nonsingular locus such that $\tilde{X}$ is either a torus, a Calabi-Yau threefold (in particular $H^2(\tilde{X}, \mathcal{O}_{\tilde{X}}) = 0$) or a product of an elliptic curve E with a K3 surface S.

If $\tilde{X}$ is a torus, the pull-back $\mu^*\alpha$ is Kähler. Thus α itself is Kähler and we are done. If $\tilde{X}$ is a Calabi-Yau, the (1,1)-class α is an $\mathbb{R}$-divisor class and we conclude by the basepoint-free theorem.

Thus we are left to deal with the case $\tilde{X} \simeq E \times S$. Set $\tilde{\alpha} := \mu^*\alpha$. For a K3 surface we have $H^0(S, \Omega_S) = 0$, which immediately implies

$$H^{1,1}_{\mathrm{BC}}(X) = p_E^* H^{1,1}_{\mathrm{BC}}(E) \times p_S^* H^{1,1}_{\mathrm{BC}}(S),$$

where $p_E : \tilde{X} \to E$ and $p_S : \tilde{X} \to S$ are the projections on the factors. Thus we can write

$$\tilde{\alpha} = \lambda F + p_S^*\alpha_S$$

where F is a p_E-fibre, $\lambda \in \mathbb{R}$, and α_S is a (1,1)-class on S. The restriction of the nef and big class $\tilde{\alpha}$ to a general p_E-fibre is nef and big, so α_S is nef and big. The restriction of the nef and big class $\tilde{\alpha}$ to a general p_S-fibre is nef and big, so $\lambda > 0$.

Let now Z be an irreducible component on the null locus of $\tilde{\alpha}$. If Z is a curve, we have

$$0 = \tilde{\alpha} \cdot Z = \lambda F \cdot Z + \alpha_S \cdot (p_S)_* Z.$$

By what precedes we obtain $F \cdot Z = 0$, so Z is contained in a fibre $p_E^{-1}(t_0)$. Yet then Z trivially deforms in a family $Z_t = (t \times Z)_{t \in E}$, and $\alpha \cdot Z_t = 0$ for all the curves in this family. Thus Z is not an irreducible component of the null locus.

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[1]Since $\Gamma_\mu \subset X \times Y$, the p-fibres are naturally embedded in Y.

This shows that any irreducible component of the null locus is a surface Z. If the map $p_S|_Z : Z \to S$ is generically finite, the pull-back $p_S|_Z^* \alpha_S$ is big. Hence $\tilde\alpha|_Z$ is nef and big, but this contradicts the property of being in the null locus. Thus $p_S(Z)$ is an irreducible curve, and since

$$Z \subset p_S^{-1}(p_S(Z)) = E \times p_S(Z)$$

is irreducible, we see that $Z = E \times p_S(Z)$. Since Z is in the null locus we have

$$0 = \tilde\alpha^2 \cdot Z = \lambda^2 F^2 \cdot Z + 2\lambda F \cdot p_S^* \alpha_S \cdot Z + (p_S^* \alpha_S)^2 \cdot Z = 2\lambda \alpha_S \cdot p_S(Z).$$

Since $\lambda > 0$ we see that $p_S(Z)$ is in the null-locus of α_S, hence it is a (-2)-curve (cf. the proof of [FT17, Thm.1.3]). In conclusion we obtain that

$$\mathrm{Null}(\tilde\alpha) = E \times \mathrm{Null}(\alpha_S)$$

By [FT17, Thm.1.3] there exists a bimeromorphic map $\psi_S : S \to S'$ onto a normal compact surface S' and a Kähler class $\alpha_{S'}$ on S' such that $\alpha_S = \psi_S^* \alpha_{S'}$. We then set

$$\tilde\psi := id_E \times \psi_S : \tilde X = E \times S \to E \times S' =: \tilde X'$$

and $\lambda F + \alpha_{S'}$ is a Kähler class on $\tilde X'$ such that $\tilde\psi^*(\lambda F + \alpha_{S'}) = \tilde\alpha$.

Let us now show that this map descends to X: up to replacing μ by its Galois closure, we have $X = \tilde X/G$ where G is the Galois group of μ. Any automorphism f on $E \times S$ is of the form $f_E \times f_S$ [Bea83a, p.8], thus G acts on the factors E and S. The class $\mu^* \alpha$ is G-invariant, so α_S is invariant under the G-action on S. The Null locus $\mathrm{Null}(\alpha_S)$ being G-invariant, we see that there is an induced G-action on S' that makes ψ_S is G-equivariant. Since $\tilde\psi = id_E \times \psi_S$ there is an induced G-action on $\tilde X'$ that makes $\tilde\psi$ is G-equivariant. We set $Z := \tilde X'/G$ and denote by $\psi : X = X'/G \to Z = \tilde X'/G$ the bimeromorphic morphism induced by $\tilde\psi$. Since ψ is crepant and X has terminal singularities, the space Z has canonical, hence rational singularities.

Denote by $\mu' : \tilde X' \to Z = \tilde X'/G$ the finite cover. Since $\mu^* \alpha$ is G-invariant, the Kähler class $\lambda F + \alpha_{S'}$ is G-invariant and defines a Kähler class α_Z on Z. By construction we have $\alpha = \psi^* \alpha_Z$. $\qquad\square$

Remark. If $\tilde X \simeq E \times S$, then X is in general not a product. Let f_E be a fixpoint free involution (e.g. a translation by 2-torsion point), and f_S an involution with fixed points. Set $X := \tilde X/\langle f_E \times f_S \rangle$ and denote by $p : X \to S/\langle f_S \rangle$ the map induced by the projection p_S. Then p has multiple fibres over the fixed points, so X is not a product.

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Andreas Höring, Université Côte d'Azur, CNRS, LJAD, France

E-mail address: Andreas.Hoering@unice.fr