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Abstract In this study, we apply the Ramanujan summation method to a certain class of Euler sums and provide new reflection formulas that extend the well-known relations of symmetry between reciprocal linear Euler sums.

Keywords Euler sums, analytic continuation, Ramanujan summation of series, harmonic numbers, series identities with special numbers.

Mathematics Subject Classification (2020) 11B75, 11M06, 11M32, 40G99.

Introduction

The study of Euler sums has a fairly long history dating back to the middle of the 18th century. In response to a letter from Goldbach dated from december 1742, Euler considered infinite sums of the form

$$\mathcal{S}_{p,q} = \sum_{n=1}^{\infty} \frac{H_n^{(p)}}{n^q} ,$$

where p and q are positive integers, and $H_n^{(p)} = \sum_{k=1}^{n} \frac{1}{k^p}$ are generalized harmonic numbers. For $p = 1$, the generalized harmonic numbers reduce to classical harmonic numbers $H_n = H_n^{(1)}$. The importance of harmonic numbers comes from the fact that they appear (sometimes quite unexpectedly) in different branches of number theory and combinatorics. In our times, the sums $\mathcal{S}_{p,q}$ are called the *linear Euler sums*. Euler discovered that for all pairs (p, q) with $p = 1$, or $p = q$, or $p + q$

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odd, these sums have expressions in terms of zeta values (i.e. the values of the Riemann zeta function $\zeta(s) = \sum_{n\geq 1} n^{-s}$ at positive integers), a remarkable result that will be also found and completed later by Nielsen [12]. Among the beautiful formulas due to Euler[1], the following two are particularly noteworthy:

– The reflection formula [10, 12]:

$$\mathcal{S}_{p,q} + \mathcal{S}_{q,p} = \zeta(p)\zeta(q) + \zeta(p+q) \quad \text{for } p \geq 2 \text{ and } q \geq 2$$

that allows to express $\mathcal{S}_{q,p}$ as a function of $\mathcal{S}_{p,q}$ and vice versa. In particular, for $p = q$, it results that

$$2\mathcal{S}_{p,p} = (\zeta(p))^2 + \zeta(2p)\,;$$

– Euler's formula [10, 12]:

$$\mathcal{S}_{1,2} = 2\zeta(3) \quad \text{and} \quad 2\mathcal{S}_{1,p} = (p+2)\zeta(p+1) - \sum_{j=1}^{p-2} \zeta(p-j)\zeta(j+1) \quad \text{for } p > 2$$

that will be several times rediscovered throughout the 20th century (see [1, p. 259] and [10, Remark 3.1] for historical details).

Ramanujan's method of summation of series appears in Chapter VI of the second *Notebook* [1, Chapter 6]. Because of the ambiguities (observed by Hardy[2]) contained in the definition of the "constant of a series" that made its use very tricky, Ramanujan's method, based on the Euler-MacLaurin summation formula, had fallen into neglect. The method has known a revival of interest at the end of the 20th century when a clear and rigorous definition of the sum of a series in the sense of Ramanujan summation was given at the same time as the link with usual summation was completely clarified [6]. The reader will find in the recent monograph [3] a masterful synthesis of main definitions, fundamental properties, and scope of application of the Ramanujan summation.

Ramanujan's method is particulary well appropriate to Euler sums and allows to treat at once the convergence case and the divergence case. In the remainder of this article, we give a complete evaluation of the sums in the sense of Ramanujan summation (noted $\zeta^{\mathcal{R}}(1,p)$, $\zeta^{\mathcal{R}}(p,1)$ and $\zeta^{\mathcal{R}}(p,p)$) corresponding respectively to the Euler sums $\mathcal{S}_{1,p}$, $\mathcal{S}_{p,1}$ and $\mathcal{S}_{p,p}$ which enables us to extend the classical formulas mentioned above.

1. These formulas appear without proof in Euler's masterwork dated 1776 *Meditationes circa singulare serierum genus.*

2. In Chapter XIII of Hardy's classical treatise on divergent series.

1 Ramanujan summation of Euler sums

Let us recall that the generalized harmonic numbers $H_n^{(p)}$ are defined by

$$H_n^{(p)} = \sum_{k=1}^{n} \frac{1}{k^p} \quad \text{for } n \geq 1 \text{ and } p \geq 1\,.$$

When $p = 1$, they reduce to classical harmonic numbers denoted $H_n = H_n^{(1)}$. It is convenient to express them in the form

$$H_n = \psi(n+1) + \gamma\,,$$

where $\psi(z) = \partial \ln(\Gamma(z))$ is the digamma function and $\gamma = -\psi(1)$ is the Euler constant; in the same way, we have for $p \geq 2$ the following expression [5, 7]:

$$H_n^{(p)} = \frac{(-1)^{p-1}}{(p-1)!}\partial^{p-1}\psi(n+1) + \zeta(p)\,.$$

Definition 1. For any positive integer p, the function $s \mapsto \zeta^{\mathcal{R}}(p,s)$ is defined as the analytic continuation of the function defined in the half-plane $\mathrm{Re}(s) > 1$ by

$$\sum_{n=1}^{+\infty} H_n^{(p)}\, n^{-s} - \int_1^{+\infty} \psi_p(x)\, x^{-s}\, dx\,,$$

where $\psi_1(x) = \psi(x+1) + \gamma$, and

$$\psi_p(x) = \frac{(-1)^{p-1}}{(p-1)!}\partial^{p-1}\psi(x+1) + \zeta(p) \quad \text{for } p \geq 2\,.$$

It follows from [3, Theorem 9] that this function can be analytically continued as an entire function in the whole $\mathbb{C}$. Thus, for each integer $q \in \mathbb{Z}$, the value $\zeta^{\mathcal{R}}(p,q)$ is well-defined and may be interpreted as the $\mathcal{R}$-sum (i.e. the sum in the sense of Ramanujan summation) of the (possibly divergent) series $\sum_{n\geq 1} H_n^{(p)}\, n^{-q}$.

For $p = 1$, the function $s \mapsto \zeta^{\mathcal{R}}(1,s)$ is closely linked to the harmonic zeta function ζ_H [5] which is defined for $\mathrm{Re}(s) > 1$ by

$$\zeta_H(s) = \sum_{n=1}^{\infty} H_n\, n^{-s}\,,$$

through the relation

$$\zeta^{\mathcal{R}}(1,s) = \zeta_H(s) - \int_1^{\infty} x^{-s}\left(\psi(x+1) + \gamma\right)\, dx \qquad \text{for } \mathrm{Re}(s) > 1\,.$$

In particular, since

$$\zeta_H(p) = \mathcal{S}_{1,p} \quad \text{for } p \geq 2\,,$$

it follows that

$$\zeta^{\mathcal{R}}(1,p) = \mathcal{S}_{1,p} - \int_1^{\infty} \frac{\psi(x+1)+\gamma}{x^p}\, dx \qquad (p \geq 2)\,. \tag{1}$$

2 Evaluation of $\zeta^{\mathcal{R}}(1,p)$

Definition 2. For any positive integer p, let τ_p be the real constant defined by the series representation

$$\tau_p = \sum_{k=1}^{\infty} (-1)^{k+p} \frac{\zeta(k+p)}{k} \,. \tag{2}$$

Remark 1. The sequence $\{\tau_p\}_p$ first appeared in [5, 9]. The constant $\tau_1 = 1.25774688\ldots$ has been thoroughly studied by Boyadzhiev [2] (see also [7, Ex. 92 (b) p. 142]), and the constants τ_p for $p = 2, 3, \ldots$ have been extensively evaluated in [9, Section 4].

To give an evaluation of the $\mathcal{R}$-sum $\zeta^{\mathcal{R}}(1,p)$, we first prove the following lemma:

Lemma 1. For $p > 2$, we have the relation

$$\int_1^{\infty} \frac{\psi(x+1)+\gamma}{x^p}\, dx = \sum_{j=1}^{p-2} \frac{(-1)^{p-j}}{j} \zeta(p-j) - (-1)^p \zeta'(p) - \tau_p \,. \tag{3}$$

For $p = 2$, this relation reduces to

$$\int_1^{\infty} \frac{\psi(x+1)+\gamma}{x^2}\, dx = -\zeta'(2) - \tau_2 \,.$$

Proof. For $p \geq 2$, the convergent series $\sum_{n\geq 1} \frac{\ln(n+1)}{n^p}$ may be splitted into the two series

$$\sum_{n=1}^{\infty} \frac{\ln(n+1)}{n^p} = \sum_{n=1}^{\infty} \frac{\ln(n)}{n^p} + \sum_{n=1}^{\infty} \frac{1}{n^p} \ln\left(1+\frac{1}{n}\right) .$$

The well-known expansion of $\ln(1+1/n)$ in power series leads to the identity

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \ln\left(1+\frac{1}{n}\right) = \sum_{n=1}^{\infty} \frac{1}{n^p} \left[\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \left(\frac{1}{n}\right)^k\right] = (-1)^{p-1}\, \tau_p \,,$$

then it follows that

$$\sum_{n=1}^{\infty} \frac{\ln(n+1)}{n^p} = -\zeta'(p) - (-1)^p\, \tau_p \,. \tag{4}$$

On the other side, for $p > 2$, the finite Taylor expansion of the logarithm allows us to write

$$\ln(x+1) = \sum_{j=1}^{p-2} \frac{(-1)^{j-1}}{j} x^j + (-1)^p x^{p-1} \int_1^{\infty} \frac{1}{t^{p-1}(t+x)}\, dt \,,$$

and thus, for any positive integer n, we have

$$\frac{\ln(n+1)}{n^p} = \sum_{j=1}^{p-2} \frac{(-1)^{j-1}}{j} \frac{1}{n^{p-j}} + (-1)^p \int_1^\infty \frac{1}{t^{p-1} n(t+n)}\, dt\,.$$

By summing this identity, we get

$$\sum_{n=1}^{\infty} \frac{\ln(n+1)}{n^p} = \sum_{j=1}^{p-2} \frac{(-1)^{j-1}}{j} \zeta(p-j) + (-1)^p \int_1^\infty \frac{\psi(x+1)+\gamma}{x^p}\, dx\,.$$

Hence, formula (3) follows from (4) by substitution. For $p=2$, it reduces to

$$\int_1^\infty \frac{\psi(x+1)+\gamma}{x^2}\, dx = \sum_{n=1}^{\infty} \frac{\ln(n+1)}{n^2} = -\zeta'(2) - \tau_2\,. \tag{5}$$

□

Proposition 1. For any positive integer $p > 2$, we have

$$\zeta^{\mathcal{R}}(1,p) = \mathcal{S}_{1,p} - \sum_{j=1}^{p-2} \frac{(-1)^{p-j}}{j} \zeta(p-j) + (-1)^p \zeta'(p) + \tau_p\,, \tag{6}$$

where τ_p is defined by formula (2). For $p = 2$, it reduces to

$$\zeta^{\mathcal{R}}(1,2) = \mathcal{S}_{1,2} + \zeta'(2) + \tau_2 = 2\zeta(3) + \zeta'(2) + \tau_2\,.$$

Corollary 1. For $p \geq 2$, we have the formula

$$\zeta^{\mathcal{R}}(1,2p) = (p+1)\zeta(2p+1) - \sum_{j=1}^{p-1} \zeta(2p-j)\zeta(j+1) - \sum_{j=1}^{2p-2} \frac{(-1)^j}{j} \zeta(2p-j) + \zeta'(2p) + \tau_{2p}\,. \tag{7}$$

Proof. By formula (1), we have the relation

$$\zeta^{\mathcal{R}}(1,p) = \mathcal{S}_{1,p} - \int_1^\infty \frac{\psi(x+1)+\gamma}{x^p}\, dx\,,$$

and, by (3), we have

$$(-1)^p \int_1^\infty \frac{\psi(x+1)+\gamma}{x^p}\, dx = -\zeta'(p) - (-1)^p \tau_p + \sum_{j=1}^{p-2} \frac{(-1)^j}{j} \zeta(p-j)\,.$$

Then formula (6) follows immediately by substitution, and (7) results from the expression of $S_{1,2p}$ given by Euler's formula:

$$\mathcal{S}_{1,2p} = (p+1)\zeta(2p+1) - \sum_{j=1}^{p-1} \zeta(2p-j)\zeta(j+1) \quad (p>1)\,. \tag{8}$$

□

Example 1.

$$\begin{aligned}
\zeta^{\mathcal{R}}(1,2) &= 2\zeta(3) + \zeta'(2) + \tau_2\,,\\
\zeta^{\mathcal{R}}(1,4) &= 3\zeta(5) - \zeta(3)\zeta(2) + \zeta(3) - \frac{1}{2}\zeta(2) + \zeta'(4) + \tau_4\,,\\
\zeta^{\mathcal{R}}(1,6) &= 4\zeta(7) - \zeta(3)\zeta(4) - \zeta(2)\zeta(5) + \zeta(5) - \frac{1}{2}\zeta(4) + \frac{1}{3}\zeta(3) - \frac{1}{4}\zeta(2)\\
&\quad + \zeta'(6) + \tau_6\,.
\end{aligned}$$

Remark 2. We point out here the analogy between our formula (7) and the "dual" formula

$$\zeta^{\mathcal{R}}(1,-2p) = \frac{1-2p}{2}\zeta(1-2p) + \zeta'(-2p) + \nu_{2p}$$

given in [8, Eq. (8)], where ν_p is defined for $p \geq -1$ by the series representation

$$\nu_p = \sum_{k=1}^{\infty}(-1)^{k+1}\,\frac{\zeta(k+1)}{k+p+1} \quad (\text{note that } \nu_{-1} \text{ coincides with } \tau_1)\,.$$

3 Evaluation of $\zeta^{\mathcal{R}}(p,1)$

We now give an evaluation of the "reciprocal" sum $\zeta^{\mathcal{R}}(p,1)$ (note that, as a divergent series, $S_{p,1}$ is not defined).

Definition 3. Let σ_p defined by $\sigma_2 = 1$ and, for $p > 2$,

$$\sigma_p = \frac{1+(-1)^p}{p} + \sum_{j=1}^{p-2}(-1)^j\zeta(p-j)\left[\frac{(j-1)!(p-1-j)!}{(p-1)!} - \frac{1}{j}\right]\,. \tag{9}$$

Example 2. The first values of σ_p are

$$\begin{aligned}
\sigma_2 &= 1\,,\\
\sigma_3 &= \frac{1}{2}\zeta(2)\,,\\
\sigma_4 &= \frac{2}{3}\zeta(3) - \frac{1}{3}\zeta(2) + \frac{1}{2}\,,\\
\sigma_5 &= \frac{3}{4}\zeta(4) - \frac{5}{12}\zeta(3) + \frac{1}{4}\zeta(2)\,,\\
\sigma_6 &= \frac{4}{5}\zeta(5) - \frac{9}{20}\zeta(4) + \frac{9}{10}\zeta(3) - \frac{1}{5}\zeta(2) + \frac{1}{3}\,.
\end{aligned}$$

Remark 3. We can deduce from [4, Eq. (27)] another interesting expression of σ_p. Let us define the infinite sum $Z(i,j)$ by

$$Z(i,j) = \sum_{n=1}^{\infty}\frac{1}{n^i\,(n+1)^j} \quad \text{for } i,j \geq 1\,.$$

Partial fraction decomposition of $\frac{1}{n^i\,(n+1)^j}$ shows that $Z(i,j)$ have an expression as $\mathbb{Z}$-linear combinations of zeta values and integers. Explicitly,

$$\begin{aligned}
Z(1,j) &= j - \sum_{r=0}^{j-2} \zeta(j-r) \qquad (j \geq 2)\,, \\
Z(i,1) &= (-1)^{i-1} + \sum_{r=0}^{i-2} (-1)^r \zeta(i-r) \qquad (i \geq 2)\,, \\
Z(i,j) &= (-1)^i \sum_{r=0}^{j-2} \binom{i+r-1}{i-1} \zeta(j-r) + \sum_{r=0}^{i-2} (-1)^r \binom{j+r-1}{j-1} \zeta(i-r) \\
&\quad + (-1)^{i-1} \binom{i+j-1}{j-1} \qquad (i,j \geq 2)\,.
\end{aligned}$$

Then, formula [4, Eq. (27)] may be translated into the identity

$$\sigma_p = \sum_{i+j=p} \frac{1}{j}\, Z(i,j)\,.$$

Proposition 2. For any positive integer $p \geq 2$, we have

$$\zeta^{\mathcal{R}}(p,1) = \gamma\zeta(p) + \zeta(p+1) - \mathcal{S}_{1,p} - \sigma_p - \zeta'(p) - (-1)^p \tau_p\,, \tag{10}$$

where τ_p and σ_p are respectively defined by formulas (2) and (9).

Corollary 2. For $p \geq 2$, we have the formula

$$\zeta^{\mathcal{R}}(2p,1) = \gamma\zeta(2p) - p\,\zeta(2p+1) + \sum_{j=1}^{p-1} \zeta(2p-j)\zeta(j+1) - \sigma_{2p} - \zeta'(2p) - \tau_{2p}\,. \tag{11}$$

Proof. By summing (in the sense of Ramanujan summation) the following equations :

$$\frac{H_n^{(p)}}{n} - \frac{1}{n}\zeta(p) = -\frac{1}{n} \sum_{m=n+1}^{+\infty} \frac{1}{m^p} = \frac{1}{n} \frac{(-1)^{p-1}}{(p-1)!} \partial^{p-1}\psi(n+1)\,,$$

we get

$$\begin{aligned}
\sum_{n\geq 1}^{\mathcal{R}} \left(\frac{H_n^{(p)}}{n} - \frac{\zeta(p)}{n} \right) &= -\sum_{n\geq 1}^{\mathcal{R}} \frac{1}{n} \sum_{m=n+1}^{+\infty} \frac{1}{m^p} \\
&= \sum_{n\geq 1}^{\mathcal{R}} \frac{1}{n} \frac{(-1)^{p-1}}{(p-1)!} \partial^{p-1}\psi(n+1) \\
&= -\sum_{n=1}^{+\infty} \frac{1}{n} \sum_{m=n+1}^{+\infty} \frac{1}{m^p} + \frac{(-1)^p}{(p-1)!} \int_1^{+\infty} \partial^{p-1}\psi(x+1) \frac{1}{x}\, dx\,,
\end{aligned}$$

where the symbol $\sum_{n\geq 1}^{\mathcal{R}}$ denotes the $\mathcal{R}$-sum of the series (see [3] for a precise definition). Since

$$\sum_{n\geq 1}^{+\infty}\frac{1}{n}\sum_{m=n+1}^{+\infty}\frac{1}{m^p}=\sum_{n=1}^{+\infty}\frac{H_n}{n^p}-\zeta(p+1)\,,$$

this can be rewritten

$$\sum_{n\geq 1}^{\mathcal{R}}\left(\frac{H_n^{(p)}}{n}-\frac{\zeta(p)}{n}\right)=\zeta(p+1)-\sum_{n=1}^{+\infty}\frac{H_n}{n^p}+\frac{(-1)^p}{(p-1)!}\int_1^{+\infty}\partial^{p-1}\psi(x+1)\frac{1}{x}\,dx\,.$$

Thus we have

$$\sum_{n\geq 1}^{\mathcal{R}}\frac{H_n^{(p)}}{n}=\gamma\zeta(p)+\zeta(p+1)-\sum_{n=1}^{+\infty}\frac{H_n}{n^p}+\frac{(-1)^p}{(p-1)!}\int_1^{+\infty}\partial^{p-1}\psi(x+1)\frac{1}{x}\,dx\,,$$

i.e.

$$\zeta^{\mathcal{R}}(p,1)=\gamma\zeta(p)+\zeta(p+1)-S_{1,p}+\frac{(-1)^p}{(p-1)!}\int_1^{+\infty}\frac{\partial^{p-1}\psi(x+1)}{x}\,dx\,. \tag{12}$$

We evaluate the integral in the right member of (12) by integrating $p-1$ times by parts. When $p=2$, this is just

$$\int_1^{+\infty}\frac{\partial\psi(x+1)}{x}\,dx=\int_1^{\infty}\frac{\psi(x+1)+\gamma}{x^2}\,dx\,-1\,,$$

which, by (5), is $-\zeta'(2)-\tau_2-1$. Hence, by (12), we have

$$\zeta^{\mathcal{R}}(2,1)=\gamma\zeta(2)+\zeta(3)-\mathcal{S}_{1,2}-\sigma_2-\zeta'(2)-\tau_2\,.$$

We now assume that $p>2$. We have the identity

$$\partial^{p-k}\psi(2)=(-1)^{p-k}(p-k)!+(-1)^{p-k+1}(p-k)!\zeta(p-k+1)\qquad(p-k\geq 1)$$

[7, Proposition 9.6.41] from which results the relation

$$\frac{(-1)^p}{(p-1)!}\int_1^{+\infty}\frac{\partial^{p-1}\psi(x+1)}{x}\,dx=(-1)^p\int_1^{\infty}\frac{\psi(x+1)+\gamma}{x^p}\,dx$$
$$+\frac{1}{(p-1)!}\sum_{k=0}^{p-3}(-1)^k k!(p-k-2)!\,\zeta(p-k-1)-\frac{1}{(p-1)!}\sum_{k=0}^{p-2}(-1)^k k!(p-k-2)!\,.$$

In this expression, the last term can be simplified by means of the formula

$$\frac{1}{(p-1)!}\sum_{k=0}^{p-2}(-1)^k k!(p-k-2)!=\frac{1}{p-1}\sum_{k=0}^{p-2}\frac{(-1)^k}{\binom{p-2}{k}}=\frac{1+(-1)^p}{p}$$

[11, Eq. (14)]. After reindexation, we can also write

$$\frac{1}{(p-1)!}\sum_{k=0}^{p-3}(-1)^k k!(p-k-2)!\,\zeta(p-k-1) = -\sum_{j=1}^{p-2}(-1)^j\frac{(j-1)!(p-j-1)!}{(p-1)!}\,\zeta(p-j)\,.$$

Moreover, by (3), we have

$$(-1)^p\int_1^\infty \frac{\psi(x+1)+\gamma}{x^p}\,dx = \sum_{j=1}^{p-2}\frac{(-1)^j}{j}\,\zeta(p-j) - \zeta'(p) - (-1)^p\tau_p\,.$$

Thanks to these simplifications, formula (12) can then be rewritten

$$\zeta^{\mathcal{R}}(p,1) = \gamma\zeta(p) + \zeta(p+1) - \mathcal{S}_{1,p} - \zeta'(p) - (-1)^p\tau_p - \sigma_p\,,$$

with

$$\sigma_p = \frac{1+(-1)^p}{p} + \sum_{j=1}^{p-2}(-1)^j\frac{(j-1)!(p-j-1)!}{(p-1)!}\,\zeta(p-j) - \sum_{j=1}^{p-2}\frac{(-1)^j}{j}\,\zeta(p-j)\,.$$

This completes the demonstration of the expected formula (10). Formula (11) is immediately deduced from Euler's formula (8). □

Example 3.

$$\begin{aligned}
\zeta^{\mathcal{R}}(2,1) &= \gamma\zeta(2) - \zeta(3) - 1 - \zeta'(2) - \tau_2\\
\zeta^{\mathcal{R}}(3,1) &= \gamma\zeta(3) - \frac{1}{4}\zeta(4) - \frac{1}{2}\zeta(2) - \zeta'(3) + \tau_3\,,\\
\zeta^{\mathcal{R}}(4,1) &= \gamma\zeta(4) - 2\zeta(5) + \zeta(3)\zeta(2) - \frac{2}{3}\zeta(3) + \frac{1}{3}\zeta(2) - \frac{1}{2} - \zeta'(4) - \tau_4\,,\\
\zeta^{\mathcal{R}}(5,1) &= \gamma\zeta(5) - \frac{3}{4}\zeta(6) - \frac{3}{4}\zeta(4) + \frac{1}{2}(\zeta(3))^2 + \frac{5}{12}\zeta(3) - \frac{1}{4}\zeta(2) - \zeta'(5) + \tau_5\,.
\end{aligned}$$

4 Values of $\zeta^{\mathcal{R}}(p,p)$

The following formula [3, Eq. (3.23)] allows to extend formula (10) to the case $p=1$. We have

$$\zeta^{\mathcal{R}}(1,1) = \frac{1}{2}\gamma^2 - \frac{1}{2}\zeta(2) + \gamma_1 + \tau_1\,, \tag{13}$$

where γ_1 is the first Stieltjes constant and τ_1 is the constant defined by (2).

A new proof of this formula is given below.

Proof. The relation

$$\zeta^{\mathcal{R}}(1,1) = \frac{1}{2}\gamma^2 + \frac{1}{2}\zeta(2) - \frac{1}{2} + \frac{1}{2}\int_0^1 \psi^2(x+1)\,dx$$

[3, Eq. (2.6)] is a direct consequence of [3, Theorem 3]. Since $\psi(x+1) = \psi(x)+1/x$, this relation can be rewritten

$$\int_0^1 \left(\psi^2(x) + 2\frac{\psi(x)}{x} + \frac{1}{x^2}\right) dx = 2\zeta^{\mathcal{R}}(1,1) - \gamma^2 - \zeta(2) + 1\,.$$

Moreover, from [7, p. 145], we have

$$\int_0^1 \left(\psi^2(x) - \frac{2\gamma}{x} - \frac{1}{x^2}\right) dx = 2\gamma_1 - 2\zeta(2) + 1\,.$$

Subtracting these two expressions, we obtain the following

$$2\int_0^1 \left((\psi(x)+\gamma)\frac{1}{x} + \frac{1}{x^2}\right) dx = \zeta^{\mathcal{R}}(1,1) - \gamma^2 - \zeta(2) + 1 - (2\gamma_1 - 2\zeta(2) + 1)\,.$$

Since

$$(\psi(x)+\gamma)\frac{1}{x} + \frac{1}{x^2} = \frac{\psi(x+1)+\gamma}{x}\,,$$

we deduce the relation

$$2\int_0^1 \frac{\psi(x+1)+\gamma}{x}\,dx = 2\zeta^{\mathcal{R}}(1,1) + \zeta(2) - \gamma^2 - 2\gamma_1\,.$$

An integration of the expansion of ψ in power series

$$\psi(x+1) + \gamma = \sum_{n=2}^{\infty} (-1)^n \zeta(n) x^{n-1} \qquad (|x| < 1)$$

shows that the integral in the left member of the previous relation is nothing else than the series τ_1, i.e.

$$\int_0^1 \frac{\psi(x+1)+\gamma}{x}\,dx = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\zeta(n+1)}{n} = \tau_1.$$

Hence we obtain formula (13) after division by 2. □

Remark 4. For $p \geq 2$, the $\mathcal{R}$-sums $\zeta^{\mathcal{R}}(p,p)$ may be easily evaluated by means of the relation

$$\zeta^{\mathcal{R}}(p,p) = \mathcal{S}_{p,p} - \int_1^{\infty} \frac{\psi_p(x)}{x^p}\,dx\,,$$

with

$$\psi_p(x) = \frac{(-1)^{p-1}}{(p-1)!}\partial^{p-1}\psi(x+1) + \zeta(p)\,,$$

and the expression

$$\mathcal{S}_{p,p} = \frac{1}{2}\zeta(p)^2 + \frac{1}{2}\zeta(2p)$$

which results directly from the reflection formula. By performing $p-1$ successive integrations by parts, we deduce an expression of $\zeta^{\mathcal{R}}(p,p)$ in terms of zeta values $\zeta(2p), \zeta(2p-2), \cdots, \zeta(2)$, as well as $\zeta'(2p-1)$, τ_{2p-1} and a rational constant. For the first values, we then obtain

$$\zeta^{\mathcal{R}}(2,2) = \frac{7}{4}\zeta(4) + \zeta(2) + 2\zeta'(3) - 2\tau_3 - 1\,, \tag{14}$$

$$\zeta^{\mathcal{R}}(3,3) = \frac{1}{2}\zeta(3)^2 + \frac{1}{2}\zeta(6) + \frac{5}{2}\zeta(3) - 6\zeta(4) - \frac{3}{2}\zeta(2) - 6\zeta'(5) + 6\tau_5 + 1\,. \tag{15}$$

The general formula is given by

$$\begin{aligned}\zeta^{\mathcal{R}}(p,p) = {} & \frac{1}{2}\zeta(p)^2 + \frac{1}{2}\zeta(2p) - \frac{\zeta(p)}{p-1} \\ & + (-1)^p\binom{2p-2}{p-1}\left[\sum_{j=1}^{2p-3}\frac{(-1)^{j+1}}{j}\zeta(2p-1-j) + \zeta'(2p-1) - \tau_{2p-1}\right] \\ & + \frac{1}{((p-1)!)^2}\sum_{k=2}^{p-1}(-1)^k(p-k)!(p+k-3)!\,\zeta(p+1-k) \\ & - \frac{1}{((p-1)!)^2}\sum_{k=2}^{p}(-1)^k(p-k)!(p+k-3)! \qquad (p \geq 3)\,. \end{aligned} \tag{16}$$

Remark 5. Using [3, Theorem 18], one can show that

$$\zeta^{\mathcal{R}}(1,1) = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}\,c_n}{n!\,n^2} \quad \text{and} \quad \zeta^{\mathcal{R}}(2,2) = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}\,c_n}{n!\,n}\sum_{j=1}^{n}\frac{H_j}{j^2}\,,$$

where

$$c_n = \int_0^1 x(x-1)\cdots(x-n+1)\,dx \qquad (n \geq 1)$$

are the Cauchy numbers.

5 Reflection formulas

5.1 The even case

Theorem (first part). *For any integer $p \geq 1$, we have*

$$\zeta^{\mathcal{R}}(1,2p) + \zeta^{\mathcal{R}}(2p,1) = \gamma\zeta(2p) + \zeta(2p+1) - \sum_{j=0}^{2p-2} (-1)^j A_j\, \zeta(2p-j) - \frac{1}{p} \tag{17}$$

with

$$A_0 = 0\,, \quad \textit{and} \quad A_j = \frac{(j-1)!(2p-1-j)!}{(2p-1)!} \quad \textit{for } j \geq 1\,.$$

Proof. By adding identities (7) and (11), we get for $p \geq 2$,

$$\zeta^{\mathcal{R}}(2p,1) + \zeta^{\mathcal{R}}(1,2p) = \gamma\zeta(2p) + \zeta(2p+1) - \sum_{j=1}^{2p-2} \frac{(-1)^j}{j}\, \zeta(2p-j) - \sigma_{2p}\,.$$

Thus, for $p > 1$, formula (17) follows immediately by replacing σ_{2p} by its expression given by (9) and is extendable to the case $p = 1$ by setting $A_0 = 0$. $\square$

Example 4. We have the following relations:

$$\zeta^{\mathcal{R}}(1,2) + \zeta^{\mathcal{R}}(2,1) = \gamma\zeta(2) + \zeta(3) - 1\,,$$
$$\zeta^{\mathcal{R}}(1,4) + \zeta^{\mathcal{R}}(4,1) = \gamma\zeta(4) + \zeta(5) + \frac{1}{3}\zeta(3) - \frac{1}{6}\zeta(2) - \frac{1}{2}\,,$$
$$\zeta^{\mathcal{R}}(1,6) + \zeta^{\mathcal{R}}(6,1) = \gamma\zeta(6) + \zeta(7) + \frac{1}{5}\zeta(5) - \frac{1}{20}\zeta(4) + \frac{1}{30}\zeta(3) - \frac{1}{20}\zeta(2) - \frac{1}{3}\,.$$

5.2 The odd case

Theorem (second part). *For any integer $p \geq 2$, we have*

$$\begin{aligned}\zeta^{\mathcal{R}}(1,2p-1) &+ \zeta^{\mathcal{R}}(2p-1,1) \\ &= \gamma\zeta(2p-1) + \zeta(2p) - \sum_{j=1}^{2p-3} (-1)^j C_j\, \zeta(2p-1-j) \\ &\qquad - 2\zeta'(2p-1) + 2\tau_{2p-1}\end{aligned} \tag{18}$$

with

$$C_j = \frac{(j-1)!(2p-2-j)!}{(2p-2)!} - \frac{2}{j} \quad \textit{for } j \geq 1\,.$$

Proof. By adding identities (6) and (10), we get

$$\zeta^{\mathcal{R}}(p,1)+\zeta^{\mathcal{R}}(1,p)=\gamma\zeta(p)+\zeta(p+1)-\sigma_p-(-1)^p\sum_{j=1}^{p-2}\frac{(-1)^j}{j}\,\zeta(p-j)$$
$$+(1-(-1)^p)\tau_p+((-1)^p-1)\zeta'(p)\,.$$

Hence we have the following relation

$$\zeta^{\mathcal{R}}(2p-1,1)+\zeta^{\mathcal{R}}(1,2p-1)=\zeta(2p)+\gamma\zeta(2p-1)-\sigma_{2p-1}+\sum_{j=1}^{2p-3}\frac{(-1)^j}{j}\zeta(2p-1-j)$$
$$-2\zeta'(2p-1)+2\tau_{2p-1}\,,$$

from which formula (18) is derived by replacing σ_{2p-1} by its expression given by (9). Note that, in the odd case, the constant term of σ_{2p-1} is null. $\square$

Example 5. We have the following relations:

$$\zeta^{\mathcal{R}}(1,3)+\zeta^{\mathcal{R}}(3,1)=\gamma\zeta(3)+\zeta(4)-\frac{3}{2}\zeta(2)-2\zeta'(3)+2\tau_3\,,$$
$$\zeta^{\mathcal{R}}(1,5)+\zeta^{\mathcal{R}}(5,1)=\gamma\zeta(5)+\zeta(6)-\frac{7}{4}\zeta(4)+\frac{11}{12}\zeta(3)-\frac{7}{12}\zeta(2)-2\zeta'(5)+2\tau_5\,,$$
$$\zeta^{\mathcal{R}}(1,7)+\zeta^{\mathcal{R}}(7,1)=\gamma\zeta(7)+\zeta(8)-\frac{11}{6}\zeta(6)+\frac{29}{30}\zeta(5)-\frac{17}{30}\zeta(4)+\frac{2}{5}\zeta(3)-\frac{11}{30}\zeta(2)$$
$$-2\zeta'(7)+2\tau_7\,.$$

Remark 6. The reader will note that the reflection formula linking $\zeta^{\mathcal{R}}(1,p)$ and $\zeta^{\mathcal{R}}(p,1)$ involves the constants $\zeta'(p)$ and τ_p only in the odd case.

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