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**Abstract.** In this study, we apply the Ramanujan summation method to a certain class of Euler sums and provide new reflection formulas that extend the well-known relation of symmetry between reciprocal linear Euler sums.

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## Introduction

The study of Euler sums has a fairly long history dating back to the middle of the 18th century. In response to a letter from Goldbach dated from december 1742, Euler considered infinite sums of the form

$$\mathcal{S}_{p,q} = \sum_{n=1}^{\infty} \frac{H_n^{(p)}}{n^q} \,,$$

where $p$ and $q$ are positive integers, and $H_n^{(p)} = \sum_{k=1}^{n} \frac{1}{k^p}$ are generalized harmonic numbers. For $p = 1$, the generalized harmonic numbers reduce to classical harmonic numbers $H_n = H_n^{(1)}$. The importance of harmonic numbers comes from the fact that they appear (sometimes quite unexpectedly) in different branches of number theory and combinatorics. In our times, the sums $\mathcal{S}_{p,q}$ are called the *linear Euler sums* (cf. [12]). Euler discovered that for all pairs $(p, q)$ with $p = 1$, or $p = q$,

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or $p+q$ odd, these sums have expressions in terms of zeta values (i.e. the values of Riemann's zeta function $\zeta(s) = \sum_{n\geq 1} n^{-s}$ at positive integers), a remarkable result that will be also found and completed later by Nielsen[1].

Among the elegant formulas already known to Euler[2], the following two are particularly noteworthy:

– The reflection formula (cf. [10, 12])

$$\mathcal{S}_{p,q} + \mathcal{S}_{q,p} = \zeta(p)\zeta(q) + \zeta(p+q) \qquad (p, q \geq 2)\,,$$

which allows to express $\mathcal{S}_{q,p}$ as a function of $\mathcal{S}_{p,q}$ and vice versa.

– Euler's formula (cf. [10, 12])

$$\mathcal{S}_{1,2} = 2\zeta(3) \quad \text{and} \quad 2\mathcal{S}_{1,p} = (p+2)\zeta(p+1) - \sum_{j=1}^{p-2} \zeta(p-j)\zeta(j+1) \qquad (p > 2)\,,$$

that will be several times rediscovered throughout the 20th century (see [10, Remark 3.1] for historical details).

Ramanujan's method of summation of series appears in chapter VI of the second *Notebook*. Because of the ambiguities (observed by Hardy[3]) contained in the definition of the "constant of a series" that made its use very tricky, Ramanujan's method, based on the Euler-MacLaurin summation formula, had fallen into neglect. The method has known a revival of interest at the end of the 20th century when a clear and rigorous definition of the sum of a series (in the sense of Ramanujan summation) was given at the same time as the link with usual summation was completely clarified. The reader will find in [4] a masterful synthesis of main definitions, fundamental properties, and scope of application of Ramanujan summation.

Ramanujan's method is particulary appropriate to Euler sums and allows to treat both the convergence case and the divergence case. In the remainder of this article, we give a complete evaluation of the sums (in the sense of Ramanujan summation) corresponding respectively to $\mathcal{S}_{1,p}$ and $\mathcal{S}_{p,1}$ which allows to extend the classical reflection formula mentioned above (see Propositions 1 et 2 and Theorem 1).

1. N. Nielsen, Handbuch der Theorie der Gammafunction, Teubner, Leipzig, 1906.
2. These formulas are given without proof in Euler's masterwork dated 1776 *Meditationes circa singulare serierum genus.*
3. G. H. Hardy, Divergent Series, Clarendon press, Oxford, 1949, Chapter XIII.

# 1 Ramanujan summation of Euler sums

Let us recall that the generalized harmonic numbers $H_n^{(p)}$ are defined by

$$H_n^{(p)} = \sum_{k=1}^{n} \frac{1}{k^p} \quad \text{for } n, p = 1, 2, 3, \ldots .$$

When $p = 1$, they reduce to classical harmonic numbers denoted $H_n = H_n^{(1)}$ . It is convenient to express them in the form

$$H_n = \psi(n+1) + \gamma \,,$$

where $\psi(z) = \partial \ln(\Gamma(z))$ is the digamma function and $\gamma = -\psi(1)$ is Euler's constant; in the same way (cf. [3, 7]), we have

$$H_n^{(p)} = \frac{(-1)^{p-1}}{(p-1)!} \partial^{p-1} \psi(n+1) + \zeta(p) \quad (p \geq 2) \,.$$

**Definition 1.** For any positive integer $p$, the function $s \mapsto \zeta^{\mathcal{R}}(p, s)$ is defined as the analytic continuation of the function defined in the half-plane $\mathrm{Re}(s) > 1$ by

$$\sum_{n=1}^{+\infty} H_n^{(p)} \, n^{-s} - \int_1^{+\infty} \psi_p(x) \, x^{-s} \, dx \,,$$

where $\psi_1(x) = \psi(x+1) + \gamma$, and

$$\psi_p(x) = \frac{(-1)^{p-1}}{(p-1)!} \partial^{p-1} \psi(x+1) + \zeta(p) \quad (p \geq 2) \,.$$

It follows from [4, Theorem 9] that this function can be analytically continued as an entire function in the whole $\mathbb{C}$.

Thus, the values $\zeta^{\mathcal{R}}(p, q)$ at integers $q \in \mathbb{Z}$ are well-defined and can be interpreted as the $\mathcal{R}$-sum (i.e. the sum in the sense of Ramanujan summation) of the (possibly divergent) series $\sum_{n \geq 1} H_n^{(p)} n^{-q}$.

The function $s \mapsto \zeta^{\mathcal{R}}(1, s)$ is closely linked to the harmonic zeta function $\zeta_H$ (cf. [6]) defined for $\mathrm{Re}(s) > 1$ by

$$\zeta_H(s) = \sum_{n=1}^{\infty} H_n \, n^{-s} \,,$$

through the relation

$$\zeta^{\mathcal{R}}(1, s) = \zeta_H(s) - \int_1^{\infty} x^{-s} \left(\psi(x+1) + \gamma\right) \, dx \qquad \text{for } \mathrm{Re}(s) > 1 \,.$$

Since

$$\zeta_H(p) = \mathcal{S}_{1,p} \quad \text{for } p = 2, 3, \ldots,$$

it follows that

$$\zeta^{\mathcal{R}}(1,p) = \mathcal{S}_{1,p} - \int_1^{\infty} \frac{\psi(x+1)+\gamma}{x^p}\, dx \qquad (p \geq 2)\,. \tag{1}$$

# 2 Evaluation of $\zeta^{\mathcal{R}}(1,p)$

**Definition 2.** For any positive integer $p$, let $\tau_p$ defined by the series representation

$$\tau_p = \sum_{k=1}^{\infty} (-1)^{k+p} \frac{\zeta(k+p)}{k}\,. \tag{2}$$

**Remark 1.** Series representations $\tau_p$ $(p = 1, 2, \ldots)$ have been extensively studied in [9] and also appear in [2] and [6].

To give an evaluation of the $\mathcal{R}$-sum $\zeta^{\mathcal{R}}(1,p)$, we first prove the following lemma:

**Lemma 1.** For $p > 2$, we have the relation

$$\int_1^{\infty} \frac{\psi(x+1)+\gamma}{x^p}\, dx = \sum_{j=1}^{p-2} \frac{(-1)^{p-j}}{j} \zeta(p-j) - (-1)^p \zeta'(p) - \tau_p \tag{3}$$

which, for $p = 2$, reduces to

$$\int_1^{\infty} \frac{\psi(x+1)+\gamma}{x^2}\, dx = -\zeta'(2) - \tau_2\,.$$

*Proof.* For $p \geq 2$, the convergent series $\sum_{n\geq 1} \frac{\ln(n+1)}{n^p}$ may be splitted into the two series

$$\sum_{n=1}^{\infty} \frac{\ln(n+1)}{n^p} = \sum_{n=1}^{\infty} \frac{\ln(n)}{n^p} + \sum_{n=1}^{\infty} \frac{1}{n^p} \ln\left(1 + \frac{1}{n}\right)\,.$$

The well-known expansion of $\ln(1 + 1/n)$ in power series leads to the identity

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \ln\left(1+\frac{1}{n}\right) = \sum_{n=1}^{\infty} \frac{1}{n^p} \left[\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \left(\frac{1}{n}\right)^k\right] = (-1)^{p-1}\, \tau_p\,,$$

then it follows that

$$\sum_{n=1}^{\infty} \frac{\ln(n+1)}{n^p} = -\zeta'(p) - (-1)^p\, \tau_p\,. \tag{4}$$

On the other side, for $p > 2$, the finite Taylor expansion of the logarithm allows to write

$$\ln(x+1) = \sum_{j=1}^{p-2} \frac{(-1)^{j-1}}{j} x^j + (-1)^p x^{p-1} \int_1^\infty \frac{1}{t^{p-1}(t+x)}\, dt\,,$$

and thus, for any positive integer $n$, we have

$$\frac{\ln(n+1)}{n^p} = \sum_{j=1}^{p-2} \frac{(-1)^{j-1}}{j} \frac{1}{n^{p-j}} + (-1)^p \int_1^\infty \frac{1}{t^{p-1} n(t+n)}\, dt\,.$$

By summing this identity, we get

$$\sum_{n=1}^{\infty} \frac{\ln(n+1)}{n^p} = \sum_{j=1}^{p-2} \frac{(-1)^{j-1}}{j} \zeta(p-j) + (-1)^p \int_1^\infty \frac{\psi(x+1)+\gamma}{x^p}\, dx\,.$$

Hence, formula (3) follows from (4) by substitution. For $p = 2$, it reduces to

$$\int_1^\infty \frac{\psi(x+1)+\gamma}{x^2}\, dx = \sum_{n=1}^{\infty} \frac{\ln(n+1)}{n^2} = -\zeta'(2) - \tau_2\,. \tag{5}$$

□

**Proposition 1.** For any positive integer $p > 2$, we have

$$\zeta^{\mathcal{R}}(1,p) = \mathcal{S}_{1,p} - \sum_{j=1}^{p-2} \frac{(-1)^{p-j}}{j}\, \zeta(p-j) + (-1)^p \zeta'(p) + \tau_p\,, \tag{6}$$

where $\tau_p$ is defined by formula (2), and for $p = 2$,

$$\zeta^{\mathcal{R}}(1,2) = 2\zeta(3) + \zeta'(2) + \tau_2\,.$$

**Corollary 1.** For $p > 1$, we have the formula

$$\zeta^{\mathcal{R}}(1,2p) = (p+1)\zeta(2p+1) - \sum_{j=1}^{p-1} \zeta(2p-j)\zeta(j+1) - \sum_{j=1}^{2p-2} \frac{(-1)^j}{j} \zeta(2p-j) + \zeta'(2p) + \tau_{2p}\,. \tag{7}$$

*Proof.* By formula (1), we have the relation

$$\zeta^{\mathcal{R}}(1,p) = \mathcal{S}_{1,p} - \int_1^\infty \frac{\psi(x+1)+\gamma}{x^p}\, dx\,,$$

and, by (3), we have

$$(-1)^p \int_1^\infty \frac{\psi(x+1)+\gamma}{x^p}\, dx = -\zeta'(p) - (-1)^p \tau_p + \sum_{j=1}^{p-2} \frac{(-1)^j}{j}\, \zeta(p-j)\,.$$

Then formula (6) follows immediately by substitution, and (7) results from the expression of $S_{1,2p}$ given by Euler's formula:

$$\mathcal{S}_{1,2p} = (p+1)\zeta(2p+1) - \sum_{j=1}^{p-1} \zeta(2p-j)\zeta(j+1) \quad (p>1)\,. \tag{8}$$

□

**Example 1.**

$$\begin{aligned}
\zeta^{\mathcal{R}}(1,2) &= 2\zeta(3) + \zeta'(2) + \tau_2\,,\\
\zeta^{\mathcal{R}}(1,4) &= 3\zeta(5) - \zeta(3)\zeta(2) + \zeta(3) - \frac{1}{2}\zeta(2) + \zeta'(4) + \tau_4\,,\\
\zeta^{\mathcal{R}}(1,6) &= 4\zeta(7) - \zeta(3)\zeta(4) - \zeta(2)\zeta(5) + \zeta(5) - \frac{1}{2}\zeta(4) + \frac{1}{3}\zeta(3) - \frac{1}{4}\zeta(2)\\
&\quad + \zeta'(6) + \tau_6\,.
\end{aligned}$$

**Remark 2.** We point out an analogy between our formula (7) and the "dual" formula given in [8, Eq. (8)]

$$\zeta^{\mathcal{R}}(1,-2p) = \frac{1-2p}{2}\zeta(1-2p) + \zeta'(-2p) + \nu_{2p}$$

where $\nu_p$ is defined for $p \geq -1$ by the series representation

$$\nu_p = \sum_{n=2}^{\infty} (-1)^n \frac{\zeta(n)}{n+p} \quad (\text{note that } \nu_{-1} = \tau_1)\,.$$

# 3 Evaluation of $\zeta^{\mathcal{R}}(p,1)$

We now give an evaluation of the "reciprocal" sum $\zeta^{\mathcal{R}}(p,1)$ (note that, as a divergent series, $S_{p,1}$ is not defined).

**Definition 3.** Let $\sigma_p$ defined by $\sigma_2 = 1$ and, for $p > 2$,

$$\sigma_p = \frac{1+(-1)^p}{p} + \sum_{j=1}^{p-2} (-1)^j \zeta(p-j) \left[ \frac{(j-1)!(p-1-j)!}{(p-1)!} - \frac{1}{j} \right]\,. \tag{9}$$

**Example 2.** The first values of $\sigma_p$ are

$$\begin{aligned}
\sigma_2 &= 1\,,\\
\sigma_3 &= \frac{1}{2}\zeta(2)\,,\\
\sigma_4 &= \frac{2}{3}\zeta(3) - \frac{1}{3}\zeta(2) + \frac{1}{2}\,,\\
\sigma_5 &= \frac{3}{4}\zeta(4) - \frac{5}{12}\zeta(3) + \frac{1}{4}\zeta(2)\,,\\
\sigma_6 &= \frac{4}{5}\zeta(5) - \frac{9}{20}\zeta(4) + \frac{9}{10}\zeta(3) - \frac{1}{5}\zeta(2) + \frac{1}{3}\,.
\end{aligned}$$

**Remark 3.** We can deduce from [5, Eq. (27)] another interesting expression of $\sigma_p$. Let us define the infinite sum $Z(i,j)$ by

$$Z(i,j) = \sum_{n=1}^{\infty} \frac{1}{n^i\,(n+1)^j} \qquad \text{for } i,j \geq 1\,.$$

Partial fraction decomposition of $\frac{1}{n^i\,(n+1)^j}$ shows that $Z(i,j)$ have an expression as $\mathbb{Z}$-linear combinations of zeta values and integers. Explicitly,

$$\begin{aligned}
Z(1,j) &= j - \sum_{r=0}^{j-2} \zeta(j-r) \qquad (j \geq 2)\,,\\
Z(i,1) &= (-1)^{i-1} + \sum_{r=0}^{i-2} (-1)^r \zeta(i-r) \qquad (i \geq 2)\,,\\
Z(i,j) &= (-1)^i \sum_{r=0}^{j-2} \binom{i+r-1}{i-1} \zeta(j-r) + \sum_{r=0}^{i-2} (-1)^r \binom{j+r-1}{j-1} \zeta(i-r)\\
&\quad + (-1)^{i-1} \binom{i+j-1}{j-1} \qquad (i,j \geq 2)\,.
\end{aligned}$$

Then, formula [5, Eq. (27)] may be translated into the identity

$$\sigma_p = \sum_{i+j=p} \frac{1}{j}\, Z(i,j)\,.$$

**Proposition 2.** For any positive integer $p \geq 2$, we have

$$\zeta^{\mathcal{R}}(p,1) = \gamma\zeta(p) + \zeta(p+1) - \mathcal{S}_{1,p} - \sigma_p - \zeta'(p) - (-1)^p \tau_p\,, \tag{10}$$

where $\tau_p$ and $\sigma_p$ are respectively defined by formulas (2) and (9).

**Corollary 2.** For $p > 1$, we have the formula

$$\zeta^{\mathcal{R}}(2p,1) = \gamma\zeta(2p) - p\,\zeta(2p+1) + \sum_{j=1}^{p-1} \zeta(2p-j)\zeta(j+1) - \sigma_{2p} - \zeta'(2p) - \tau_{2p}\,. \tag{11}$$

*Proof.* By summing (in the sense of Ramanujan summation) the following equations :

$$\frac{H_n^{(p)}}{n} - \frac{1}{n}\zeta(p) = -\frac{1}{n}\sum_{m=n+1}^{+\infty}\frac{1}{m^p} = \frac{1}{n}\frac{(-1)^{p-1}}{(p-1)!}\partial^{p-1}\psi(n+1)\,,$$

we get

$$\begin{aligned}\sum_{n\geq 1}^{\mathcal{R}}\left(\frac{H_n^{(p)}}{n} - \frac{\zeta(p)}{n}\right) &= -\sum_{n\geq 1}^{\mathcal{R}}\frac{1}{n}\sum_{m=n+1}^{+\infty}\frac{1}{m^p}\\ &= \sum_{n\geq 1}^{\mathcal{R}}\frac{1}{n}\frac{(-1)^{p-1}}{(p-1)!}\partial^{p-1}\psi(n+1)\\ &= -\sum_{n=1}^{+\infty}\frac{1}{n}\sum_{m=n+1}^{+\infty}\frac{1}{m^p} + \frac{(-1)^p}{(p-1)!}\int_1^{+\infty}\partial^{p-1}\psi(x+1)\frac{1}{x}\,dx\,,\end{aligned}$$

where the symbol $\sum_{n\geq 1}^{\mathcal{R}}$ denotes the $\mathcal{R}$-sum of the series (see [4] for a precise definition). Since

$$\sum_{n\geq 1}^{+\infty}\frac{1}{n}\sum_{m=n+1}^{+\infty}\frac{1}{m^p} = \sum_{n=1}^{+\infty}\frac{H_n}{n^p} - \zeta(p+1)\,,$$

this can be rewritten

$$\sum_{n\geq 1}^{\mathcal{R}}\left(\frac{H_n^{(p)}}{n} - \frac{\zeta(p)}{n}\right) = \zeta(p+1) - \sum_{n=1}^{+\infty}\frac{H_n}{n^p} + \frac{(-1)^p}{(p-1)!}\int_1^{+\infty}\partial^{p-1}\psi(x+1)\frac{1}{x}\,dx\,.$$

Thus we have

$$\sum_{n\geq 1}^{\mathcal{R}}\frac{H_n^{(p)}}{n} = \gamma\zeta(p) + \zeta(p+1) - \sum_{n=1}^{+\infty}\frac{H_n}{n^p} + \frac{(-1)^p}{(p-1)!}\int_1^{+\infty}\partial^{p-1}\psi(x+1)\frac{1}{x}\,dx\,,$$

i.e.

$$\zeta^{\mathcal{R}}(p,1) = \gamma\zeta(p) + \zeta(p+1) - S_{1,p} + \frac{(-1)^p}{(p-1)!}\int_1^{+\infty}\frac{\partial^{p-1}\psi(x+1)}{x}\,dx\,. \tag{12}$$

We evaluate the integral in the right member of (12) by integrating $p-1$ times by parts. When $p=2$, this is just

$$\int_1^{+\infty}\frac{\partial\psi(x+1)}{x}\,dx = \int_1^{\infty}\frac{\psi(x+1)+\gamma}{x^2}\,dx\; - 1\,,$$

which, by (5), is $-\zeta'(2) - \tau_2 - 1$. Hence, by (12), we have

$$\zeta^{\mathcal{R}}(2,1) = \gamma\zeta(2) + \zeta(3) - \mathcal{S}_{1,2} - \sigma_2 - \zeta'(2) - \tau_2\,.$$

We now assume that $p>2$. From

$$\partial^{p-k}\psi(2)=(-1)^{p-k}(p-k)!+(-1)^{p-k+1}(p-k)!\zeta(p-k+1)\qquad(p-k\geq 1)$$

(cf. [7, Proposition 9.6.41]) results the relation

$$\frac{(-1)^p}{(p-1)!}\int_1^{+\infty}\frac{\partial^{p-1}\psi(x+1)}{x}\,dx=(-1)^p\int_1^{\infty}\frac{\psi(x+1)+\gamma}{x^p}\,dx$$
$$+\frac{1}{(p-1)!}\sum_{k=0}^{p-3}(-1)^k k!(p-k-2)!\,\zeta(p-k-1)-\frac{1}{(p-1)!}\sum_{k=0}^{p-2}(-1)^k k!(p-k-2)!\,.$$

In this expression, the last term is much more simplified by means of the formula

$$\frac{1}{(p-1)!}\sum_{k=0}^{p-2}(-1)^k k!(p-k-2)!=\frac{1}{p-1}\sum_{k=0}^{p-2}\frac{(-1)^k}{\binom{p-2}{k}}=\frac{1+(-1)^p}{p}$$

(cf. [11, Eq. (14)]). After reindexation, we have

$$\frac{1}{(p-1)!}\sum_{k=0}^{p-3}(-1)^k k!(p-k-2)!\,\zeta(p-k-1)=-\sum_{j=1}^{p-2}(-1)^j\frac{(j-1)!(p-j-1)!}{(p-1)!}\,\zeta(p-j)\,.$$

Moreover, by (3), we have

$$(-1)^p\int_1^{\infty}\frac{\psi(x+1)+\gamma}{x^p}\,dx=\sum_{j=1}^{p-2}\frac{(-1)^j}{j}\,\zeta(p-j)-\zeta'(p)-(-1)^p\tau_p\,.$$

Thanks to these simplifications, formula (12) can now be rewritten

$$\zeta^{\mathcal{R}}(p,1)=\gamma\zeta(p)+\zeta(p+1)-\mathcal{S}_{1,p}-\zeta'(p)-(-1)^p\tau_p-\sigma_p\,,$$

with

$$\sigma_p=\frac{1+(-1)^p}{p}+\sum_{j=1}^{p-2}(-1)^j\frac{(j-1)!(p-j-1)!}{(p-1)!}\,\zeta(p-j)-\sum_{j=1}^{p-2}\frac{(-1)^j}{j}\,\zeta(p-j)\,.$$

This completes the demonstration of the expected formula (10). Formula (11) is immediately deduced from Euler's formula (8). □

**Example 3.**

$$\zeta^{\mathcal{R}}(2,1)=\gamma\zeta(2)-\zeta(3)-1-\zeta'(2)-\tau_2$$
$$\zeta^{\mathcal{R}}(3,1)=\gamma\zeta(3)-\frac{1}{4}\zeta(4)-\frac{1}{2}\zeta(2)-\zeta'(3)+\tau_3\,,$$
$$\zeta^{\mathcal{R}}(4,1)=\gamma\zeta(4)-2\zeta(5)+\zeta(3)\zeta(2)-\frac{2}{3}\zeta(3)+\frac{1}{3}\zeta(2)-\frac{1}{2}-\zeta'(4)-\tau_4\,,$$
$$\zeta^{\mathcal{R}}(5,1)=\gamma\zeta(5)-\frac{3}{4}\zeta(6)-\frac{3}{4}\zeta(4)+\frac{1}{2}(\zeta(3))^2+\frac{5}{12}\zeta(3)-\frac{1}{4}\zeta(2)-\zeta'(5)+\tau_5\,.$$

## 4 A formula for $\zeta^{\mathcal{R}}(1,1)$

The following formula (cf. [4, Eq. (3.23)]) allows to extend formula (10) to the case $p = 1$. We have

$$\zeta^{\mathcal{R}}(1,1) = \frac{1}{2}\gamma^2 - \frac{1}{2}\zeta(2) + \gamma_1 + \tau_1\,, \tag{13}$$

where $\gamma_1$ is the first Stieltjes constant and $\tau_1 = 1.25774688\ldots$ is defined by (2).

A new proof of this formula is given below.

*Proof.* The relation

$$\zeta^{\mathcal{R}}(1,1) = \frac{1}{2}\gamma^2 + \frac{1}{2}\zeta(2) - \frac{1}{2} + \frac{1}{2}\int_0^1 \psi^2(x+1)\,dx$$

(cf. [4, Eq. (2.6) p. 40]) is a direct consequence of [4, Theorem 3]. Since $\psi(x+1) = \psi(x) + 1/x$, this relation can be rewritten

$$\int_0^1 \left(\psi^2(x) + 2\frac{\psi(x)}{x} + \frac{1}{x^2}\right) dx = 2\zeta^{\mathcal{R}}(1,1) - \gamma^2 - \zeta(2) + 1\,.$$

Moreover, from[7, p. 145], we have

$$\int_0^1 \left(\psi^2(x) - \frac{2\gamma}{x} - \frac{1}{x^2}\right) dx = 2\gamma_1 - 2\zeta(2) + 1\,.$$

Subtracting these two expressions, we obtain the following

$$2\int_0^1 \left((\psi(x)+\gamma)\frac{1}{x} + \frac{1}{x^2}\right) dx = \zeta^{\mathcal{R}}(1,1) - \gamma^2 - \zeta(2) + 1 - (2\gamma_1 - 2\zeta(2) + 1)\,.$$

Since

$$(\psi(x)+\gamma)\frac{1}{x} + \frac{1}{x^2} = \frac{\psi(x+1)+\gamma}{x}\,,$$

we deduce the relation

$$2\int_0^1 \frac{\psi(x+1)+\gamma}{x}\,dx = 2\zeta^{\mathcal{R}}(1,1) + \zeta(2) - \gamma^2 - 2\gamma_1\,.$$

An integration of the expansion of $\psi$ in power series

$$\psi(x+1) + \gamma = \sum_{n=2}^{\infty} (-1)^n \zeta(n) x^{n-1} \qquad (|x| < 1)$$

shows that the integral in the left member of the previous relation is nothing else than the series $\tau_1$, i.e.

$$\int_0^1 \frac{\psi(x+1)+\gamma}{x}\,dx = \sum_{n=1}^{\infty} (-1)^{n+1}\frac{\zeta(n+1)}{n} = \tau_1.$$

Hence we obtain formula (13) after division by 2. $\square$

**Remark 4.** The $\mathcal{R}$-sum $\zeta^{\mathcal{R}}(1,1)$ coincide with the constant $\kappa_1 = 0.5290529\dots$ studied in detail in [1]. In particular, we have the following integral representation

$$\zeta^{\mathcal{R}}(1,1) = \int_0^1 \frac{\ln x - \mathrm{li}(1-x) + \gamma}{x}\, dx\,,$$

where $\mathrm{li}(x)$ denotes the logarithmic integral (cf. [1, Eq. (33)–(37)]).

# 5 Reflection formulas

## 5.1 The even case

**Theorem 1** (first part). *For any integer $p \geq 2$, we have*

$$\zeta^{\mathcal{R}}(1,2p) + \zeta^{\mathcal{R}}(2p,1) = \gamma\zeta(2p) + \zeta(2p+1) - \sum_{j=1}^{2p-2} (-1)^j A_j\, \zeta(2p-j) - \frac{1}{p} \tag{14}$$

*with*

$$A_j = \frac{(j-1)!(2p-1-j)!}{(2p-1)!}\,.$$

*Proof.* By adding identities (7) and (11), we get

$$\zeta^{\mathcal{R}}(2p,1) + \zeta^{\mathcal{R}}(1,2p) = \gamma\zeta(2p) + \zeta(2p+1) - \sum_{j=1}^{2p-2} \frac{(-1)^j}{j}\, \zeta(2p-j) - \sigma_{2p}\,.$$

Formula (14) follows immediately by replacing $\sigma_{2p}$ by its expression given by (9). □

**Example 4.** We have the following relations:

$$\begin{aligned}
&\zeta^{\mathcal{R}}(1,2) + \zeta^{\mathcal{R}}(2,1) = \gamma\zeta(2) + \zeta(3) - 1\,,\\
&\zeta^{\mathcal{R}}(1,4) + \zeta^{\mathcal{R}}(4,1) = \gamma\zeta(4) + \zeta(5) + \frac{1}{3}\zeta(3) - \frac{1}{6}\zeta(2) - \frac{1}{2}\,,\\
&\zeta^{\mathcal{R}}(1,6) + \zeta^{\mathcal{R}}(6,1) = \gamma\zeta(6) + \zeta(7) + \frac{1}{5}\zeta(5) - \frac{1}{20}\zeta(4) + \frac{1}{30}\zeta(3) - \frac{1}{20}\zeta(2) - \frac{1}{3}\,.
\end{aligned}$$

## 5.2 The odd case

**Theorem 1** (second part). *For any integer $p \geq 2$, we have*

$$\begin{aligned}
&\zeta^{\mathcal{R}}(1,2p-1) + \zeta^{\mathcal{R}}(2p-1,1)\\
&\qquad = \gamma\zeta(2p-1) + \zeta(2p) - \sum_{j=1}^{2p-3} (-1)^j C_j\, \zeta(2p-1-j)\\
&\qquad\qquad - 2\zeta'(2p-1) + 2\tau_{2p-1}
\end{aligned} \tag{15}$$

*with*

$$C_j = \frac{(j-1)!(2p-2-j)!}{(2p-2)!} - \frac{2}{j}\,.$$

*Proof.* By adding identities (6) and (10), we get

$$\zeta^{\mathcal{R}}(p,1) + \zeta^{\mathcal{R}}(1,p) = \gamma\zeta(p) + \zeta(p+1) - \sigma_p - (-1)^p \sum_{j=1}^{p-2} \frac{(-1)^j}{j}\,\zeta(p-j) + (1-(-1)^p)\tau_p + ((-1)^p - 1)\zeta'(p)\,.$$

Hence we have the following relation

$$\zeta^{\mathcal{R}}(2p-1,1) + \zeta^{\mathcal{R}}(1,2p-1) = \zeta(2p) + \gamma\zeta(2p-1) - \sigma_{2p-1} + \sum_{j=1}^{2p-3} \frac{(-1)^j}{j}\zeta(2p-1-j) - 2\zeta'(2p-1) + 2\tau_{2p-1}$$

Formula (15) follows by replacing $\sigma_{2p-1}$ by its expression given by (9). Note that, in the odd case, the constant term of $\sigma_{2p-1}$ is null. □

**Example 5.** We have the following relations:

$$\zeta^{\mathcal{R}}(1,3) + \zeta^{\mathcal{R}}(3,1) = \gamma\zeta(3) + \zeta(4) - \frac{3}{2}\zeta(2) - 2\zeta'(3) + 2\tau_3\,,$$
$$\zeta^{\mathcal{R}}(1,5) + \zeta^{\mathcal{R}}(5,1) = \gamma\zeta(5) + \zeta(6) - \frac{7}{4}\zeta(4) + \frac{11}{12}\zeta(3) - \frac{7}{12}\zeta(2) - 2\zeta'(5) + 2\tau_5\,,$$
$$\zeta^{\mathcal{R}}(1,7) + \zeta^{\mathcal{R}}(7,1) = \gamma\zeta(7) + \zeta(8) - \frac{11}{6}\zeta(6) + \frac{29}{30}\zeta(5) - \frac{17}{30}\zeta(4) + \frac{2}{5}\zeta(3) - \frac{11}{30}\zeta(2) - 2\zeta'(7) + 2\tau_7\,.$$

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