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Laurent expansion of harmonic zeta functions

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Abstract. In this study, we determine certain constants which naturally occur in the Laurent expansion of harmonic zeta functions.

Keywords. Zeta values, Harmonic numbers, Harmonic zeta functions, Laurent expansion, Ramanujan summation of divergent series.

Mathematics Subject Classification (2020): 11M06; 11M41; 30B40; 30B50; 40G99.

Introduction

Let us consider an analytic function f defined in the half-plane $\mathrm{Re}(z) > 0$ and the function ζ_f defined in a half-plane $\mathrm{Re}(s) > \sigma$ by the Dirichlet series $\sum_{n\geq 1} \frac{f(n)}{n^s}$ whose meromorphic continuation is supposed to have a pole of order m at $s = a$. The purpose of this study is to examine how the constant C_a involved in the Laurent expansion

$$\zeta_f(s) = \sum_{n=1}^{m} \frac{b_n}{(s-a)^n} + C_a + O(s-a)$$

in a neighborhood of $s = a$ is linked to the sum of the series $\sum_{n\geq 1} \frac{f(n)}{n^a}$ in the sense of Ramanujan's summation. To what extent does the knowledge of one enable to determine the other?

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In the case where $f(n) = 1$ for all n, ζ_f is Riemann ζ function. It is well-known that this famous function can be continued as an analytic function in $\mathbb{C}\backslash\{1\}$ and, in a neighborhood of its pole $s = 1$, may be written

$$\zeta(s) = \frac{1}{s-1} + \gamma + O(s-1)\,,$$

where γ denotes Euler's constant

$$\gamma = \lim_{N\to\infty}\left\{\sum_{n=1}^{N}\frac{1}{n} - \ln N\right\} = 0.5772156649\ldots$$

Furthermore, the Ramanujan summation method (cf. [6]) enable to sum the series $\sum_{n\geq 1}\frac{1}{n^s}$ for all complex values of s, and the sum of the series at $s = 1$ is nothing else than Euler's constant γ (cf. [6, Eq. (1.24)]). Thus, in this particular simple case, the constant C_1 at $s = 1$ and the sum in the sense of Ramanujan of the series at this point are the same.

In the first part of this study, we examine the more difficult case of the analytic function ζ_H defined for $\mathrm{Re}(s) > 1$ by

$$\zeta_H(s) = \sum_{n=1}^{+\infty}\frac{H_n}{n^s}\,,$$

where $H = (H_n)$ is the sequence of harmonic numbers. Apostol-Vu [2] and Matsuoka [12] have shown that this function, called the *harmonic zeta function*, can be continued as a meromorphic function with a double pole at $s = 1$, and an infinite number of simple poles at $s = 0$ and $s = 1 - 2k$ for each integer $k \geq 1$. The Ramanujan summation method allows to sum the series $\sum_{n\geqslant 1}\frac{H_n}{n^s}$ for all values of s, which make possible, for each pole $s = a$, to give an expression of the constant C_a in terms of the sum $\sum_{n\geqslant 1}^{\mathcal{R}}\frac{H_n}{n^a}$ of the series in the sense of Ramanujan's summation method. Regarding the poles at negative integers, we also find the results previously obtained by Boyadzhiev et al. [5] using a different method. Nevertheless, our method has the advantage of reformulating these results in a more pleasant way, while computing at the same time the value of the constant at $s = 1$ (see formula (6)) that had not been done in [5] nor, as far as we know, in any other article.

In the second part of this paper, we extend our study to a more general class of harmonic zeta functions denoted ζ_{H^p} for each integer $p \geq 2$, which are defined by the sequence of generalized harmonic numbers $H^p = (H_n^{(p)})$. This meromorphic function ζ_{H^p} has simple poles at $s = 1$, and $s = m - p$ for $m = 2, 1, 0, -2, -4, -6$, etc. In the simplest case where $p = 2$, we obtain a complete determination of the constant C_a at the corresponding pole $s = a$ for all values of a (Propositions 3 and

4) as well as a determination of the special values of the function at negative odd integers (Proposition 5). In the general case, we give an expression of the constant C_1 at $s = 1$ for all integers $p \geq 2$ (Proposition 6). Finally, we indicate a method to evaluate the constants at the poles $s = m - p$ of ζ_{H^p} based on the one already used in the case $p = 2$.

1 The harmonic zeta function ζ_H

If $H = (H_n)$ is the sequence of harmonic numbers

$$H_n := \sum_{j=1}^{n} \frac{1}{j}\,,$$

then, for each integer $n \geq 1$, we recall that

$$H_n = \psi(n+1) + \gamma\,,$$

where $\psi = \Gamma'/\Gamma$ denotes the digamma function (cf. [8, p. 95]).

Definition 1. We call *harmonic zeta function* and note ζ_H[1] the analytic function defined in the half-plane $\mathrm{Re}(s) > 1$ by

$$\zeta_H(s) = \sum_{n=1}^{+\infty} \frac{H_n}{n^s}\,.$$

Let us remind that the special values of ζ_H at positive integers are given by Euler's formula (cf. [2, 13]):

$$2\zeta_H(p) = (p+2)\zeta(p+1) - \sum_{k=1}^{p-2} \zeta(k+1)\zeta(p-k) \qquad (p \geq 2).$$

Apostol-Vu [2] and Matsuoka [12] provided the meromorphic continuation of this function in $\mathbb{C}$ and investigated its values and poles at the negative integers. In particular, the special values at negative even integers are given by Matsuoka's formula[2]

$$\zeta_H(-2k) = -\frac{B_{2k}}{4k} + \frac{B_{2k}}{2} \qquad (k \geq 1)\,,$$

where the B_k are Bernoulli numbers.

[1] The function ζ_H is noted H in [2] and h in [5].

[2] Boyadzhiev et al. point out that the value $\zeta_H(-2k) = -\frac{B_{2k}}{4k}$ given by Apostol-Vu (cf. [2, Eq. (7)]) is incomplete; they write the correct formula (cf. [5, Eq. (16)]) which matches with that of Matsuoka.

Definition 2. Let the symbol $\sum_{n\geqslant 1}^{\mathcal{R}}$ denotes the $\mathcal{R}$-sum of the series (i.e. the sum in the sense of Ramanujan summation method), then the function

$$s \mapsto \sum_{n\geqslant 1}^{\mathcal{R}} \frac{H_n}{n^s}$$

is defined by analytic continuation of the function defined in the half-plane $\mathrm{Re}(s) > 1$ by

$$\zeta_H(s) - \int_1^{+\infty} \frac{\psi(x+1)+\gamma}{x^s}\, dx\,.$$

It results from [6, Theorem 9] that this function is analytic in the whole $\mathbb{C}$.

Theorem 1. *If $1 < \mathrm{Re}(s) < 2$, then the function ζ_H can be decomposed as*

$$\zeta_H(s) = \sum_{n\geqslant 1}^{\mathcal{R}} \frac{H_n}{n^s} - \int_0^1 \frac{\psi(x+1)+\gamma}{x^s}\, dx - \frac{\pi}{\sin(\pi s)}\,\zeta(s)\,. \tag{1}$$

Proof. For $\mathrm{Re}(s) > 1$, we have (by Definition 2 above) the relation

$$\zeta_H(s) = \sum_{n\geqslant 1}^{\mathcal{R}} \frac{H_n}{n^s} + \int_1^{+\infty} \frac{\psi(x+1)+\gamma}{x^s}\, dx\,.$$

Moreover, since

$$\psi(x+1)+\gamma = O(x) \text{ at } 0\,,$$

then, for $1 < \mathrm{Re}(s) < 2$, we can split up the last integral into the difference

$$\int_0^{+\infty} \frac{\psi(x+1)+\gamma}{x^s}\, dx - \int_0^1 \frac{\psi(x+1)+\gamma}{x^s}\, dx\,.$$

For $x \in]-1,1[$, the expansion in power series

$$\psi(x+1)+\gamma = \sum_{n\geq 1} (-1)^{n+1}\zeta(n+1)x^n\,,$$

gives for $x \in]0,1[$ the expansion

$$\frac{\psi(x+1)+\gamma}{x} = \sum_{n\geq 0} (-1)^n \zeta(n+2)x^n\,.$$

This enable to evaluate the first integral by means of the Ramanujan master's theorem (cf. [1, Theorem 3.2]). We apply this theorem to the function φ defined by

$\varphi(s) = \zeta(s+2)$ (this function verifies the hypotheses of the theorem for $\mathrm{Re}(s) \geq -\delta$ with $\delta = 1-\varepsilon$, for any ε with $0 < \varepsilon < 1$). For $0 < \mathrm{Re}(s) < 1-\varepsilon$, this gives

$$\int_0^{+\infty} x^{s-1}\left(\frac{\psi(x+1)+\gamma}{x}\right) dx = \frac{\pi}{\sin(\pi s)}\zeta(2-s)\,,$$

or equivalently, for $1+\varepsilon < \mathrm{Re}(s) < 2$,

$$\int_0^{+\infty} \frac{\psi(x+1)+\gamma}{x^s}\, dx = \frac{-\pi}{\sin(\pi s)}\zeta(s)\,.$$

This completes the proof of Theorem 1. □

Remark 1. The decomposition of ζ_H given by formula (1) immediately provides the meromorphic continuation of the harmonic zeta function in the half-plane $\mathrm{Re}(s) < 2$. Indeed, since the function

$$s \mapsto \sum_{n\geqslant 1}^{\mathcal{R}} \frac{H_n}{n^s}$$

is analytic in the entire complex plane (cf. [6, Theorem 9]), the values of this function at points $s = -k$ ($k = -1, 0, 1, 2, 3, \ldots$) are nothing else than the sums, in the sense of Ramanujan summation, of the divergent series $\sum_{n\geqslant 1} n^k H^n$. Furthermore, since

$$\psi(x+1)+\gamma = O(x) \text{ at } 0\,,$$

the function

$$s \mapsto \int_0^1 \frac{\psi(x+1)+\gamma}{x^s}\, dx$$

is analytic for $\mathrm{Re}(s) < 2$. In formula (1), the function

$$s \mapsto \frac{-\pi}{\sin(\pi s)}\,\zeta(s)$$

is the only one to have singularities in this half-plane. It admits a double pole at $s = 1$ and simple poles at $s = 0, -1, -3, -5, \ldots$

1.1 The constant at $s = 1$

We first write the expansion at $s = 1$ of the function $s \mapsto \dfrac{-\pi}{\sin(\pi s)}\,\zeta(s)$.

Proposition 1. In a neighborhood of $s = 1$, we have the representation

$$\frac{-\pi}{\sin(\pi s)}\,\zeta(s) = \frac{1}{(s-1)^2} + \frac{\gamma}{(s-1)} + \zeta(2) - \gamma_1 + O(s-1) \tag{2}$$

where γ_1 is the first Stieltjes constant:

$$\gamma_1 = \lim_{N\to\infty}\left\{\sum_{n=1}^{N}\frac{\ln n}{n} - \frac{1}{2}\ln^2(N)\right\}.$$

Proof. We have

$$\begin{aligned}
\frac{-\pi}{\sin(\pi s)} = \frac{\pi}{\sin(\pi(s-1))} &= \frac{\exp(i\pi(s-1))}{s-1}\frac{2i\pi(s-1)}{\exp(2i\pi(s-1))-1} \\
&= \frac{1}{s-1}(\sum_{k\geq 0} i^k\pi^k\frac{1}{k!}(s-1)^k)(\sum_{k\geq 0} i^k(2\pi)^k\frac{B_k}{k!}(s-1)^k) \\
&= \frac{1}{s-1}\sum_{k\geq 0}(\sum_{i+j=2k}\frac{2^jB_j}{i!j!})(-1)^k\pi^{2k}(s-1)^{2k} \\
&= \frac{1}{s-1} + \sum_{k\geq 1}(-1)^k\left(\sum_{j=0}^{2k}\frac{2^jB_j}{(2k-j)!j!}\right)\pi^{2k}(s-1)^{2k-1},
\end{aligned}$$

where the B_k are Bernoulli numbers. It follows that

$$\frac{-\pi}{\sin(\pi s)} = \frac{1}{s-1} + \frac{\pi^2}{6}(s-1) + \cdots$$

On the other hand, we have the expansion

$$\zeta(s) = \frac{1}{s-1} + \gamma - \gamma_1(s-1) + \cdots$$

Making the product of these expansions leads immediately to formula (2). □

Lemma 1. For each integer $p \geq 1$, let

$$\tau_p := \sum_{n=1}^{\infty}(-1)^{n+p}\,\frac{\zeta(n+p)}{n}, \tag{3}$$

then

$$\tau_1 = \int_0^1 \frac{\psi(x+1)+\gamma}{x}\,dx,$$

and for $p \geq 2$,

$$\tau_p = \int_0^1 \frac{\psi(x+1)+\gamma-\sum_{j=1}^{p-1}(-1)^{j-1}\zeta(j+1)x^j}{x^p}\,dx.$$

Proof. For $x \in]0\,,1[$, it follows from the expansion in power series

$$\psi(x+1)+\gamma = \sum_{n=1}^{\infty}(-1)^{n-1}\zeta(n+1)x^n$$

that

$$\frac{\psi(x+1)+\gamma}{x} = \sum_{n=1}^{\infty}(-1)^{n-1}\zeta(n+1)x^{n-1}\,,$$

and

$$x^{-p}(\psi(x+1)+\gamma-\sum_{j=1}^{p-1}(-1)^{j-1}\zeta(j+1)x^j) = \sum_{n=p}^{\infty}(-1)^{n-1}\zeta(n+1)x^{n-p} \qquad (p\geq 2)\,.$$

For $0<a<1$, we have

$$\sum_{n=p}^{\infty}(-1)^{n+1}\frac{\zeta(n+1)}{n-p+1}a^{n-p+1} = \int_0^a \sum_{n=p}^{\infty}(-1)^{n-1}\zeta(n+1)x^{n-p}\,dx$$

(the permutation of $\int$ et Σ is justified by the fact that $\displaystyle\sum_{n=p}^{\infty}\frac{\zeta(n+1)}{n-p+1}\,a^{n-p+1} < +\infty$).
By Leibniz criterion, the alternating series $\sum_{n\geq p}(-1)^{n+1}\frac{\zeta(n+1)}{n-p+1}$ is convergent, hence by Abel's lemma for power series, we have

$$\lim_{a\to 1}\sum_{n=p}^{\infty}(-1)^{n+1}\frac{\zeta(n+1)}{n-p+1}\,a^{n-p+1} = \sum_{n=p}^{\infty}(-1)^{n+1}\frac{\zeta(n+1)}{n-p+1} = \tau_p\,,$$

this leads to the desired result. □

Remark 2. The constant

$$\tau_1 = \int_0^1 \frac{\psi(x+1)+\gamma}{x}\,dx = 1.2577468869\ldots$$

has been thoroughly studied by Boyadzhiev [4] who provided several remarkable formulas (see also [8, p. 142] and [9, p. 1836]). The series τ_p for $p=2,3,4,\ldots$ have been extensively studied in [11], they also appear in [4] and [10].

We can deduce from formulae (1) and (2) the following corollary:

Corollary 1. In a neighborhood of $s=1$, the function ζ_H is represented as

$$\zeta_H(s) = \frac{1}{(s-1)^2}+\frac{\gamma}{(s-1)}+C_1+O(s-1)$$

with

$$C_1 = \sum_{n\geqslant 1}^{\mathcal{R}}\frac{H_n}{n}-\tau_1+\zeta(2)-\gamma_1\,. \tag{4}$$

This expression of C_1 may be highly simplified by means of the following lemma:

Lemma 2. We have the relation

$$\sum_{n\geqslant 1}^{\mathcal{R}} \frac{H_n}{n} = \tau_1 + \gamma_1 + \frac{\gamma^2}{2} - \frac{\zeta(2)}{2}\,. \tag{5}$$

It follows from (5) that the value of the constant C_1 is

$$C_1 = \frac{1}{2}\gamma^2 + \frac{1}{2}\zeta(2)\,. \tag{6}$$

Proof of Lemma 2. We start from the relation

$$\sum_{n\geqslant 1}^{\mathcal{R}} \frac{H_n}{n} = \frac{1}{2}\gamma^2 + \frac{1}{2}\zeta(2) - \frac{1}{2} + \frac{1}{2}\int_0^1 \psi^2(x+1)\,dx$$

(cf. [6, Eq. (2.6)]) which is a direct consequence of [6, Theorem 3]. Since $\psi(x+1) = \psi(x) + 1/x$, this relation can be rewritten

$$2\sum_{n\geq 1}^{\mathcal{R}} \frac{H_n}{n} - \gamma^2 - \zeta(2) + 1 = \int_0^1 \left(\psi^2(x) + 2\frac{\psi(x)}{x} + \frac{1}{x^2}\right) dx\,.$$

Moreover, from [8, p. 145], we have

$$\int_0^1 \left(\psi^2(x) - \frac{2\gamma}{x} - \frac{1}{x^2}\right) dx = 2\gamma_1 - 2\zeta(2) + 1\,.$$

Subtracting these two expressions, we obtain the following

$$2\sum_{n\geqslant 1}^{\mathcal{R}} \frac{H_n}{n} - \gamma^2 - \zeta(2) + 1 - (2\gamma_1 - 2\zeta(2) + 1) = 2\int_0^1 \left((\psi(x)+\gamma)\frac{1}{x} + \frac{1}{x^2}\right) dx\,.$$

Since

$$(\psi(x)+\gamma)\frac{1}{x} + \frac{1}{x^2} = \frac{\psi(x+1)+\gamma}{x}\,,$$

we deduce the relation

$$2\sum_{n\geqslant 1}^{\mathcal{R}} \frac{H_n}{n} + \zeta(2) - \gamma^2 - 2\gamma_1 = 2\int_0^1 \frac{\psi(x+1)+\gamma}{x}\,dx = 2\tau_1$$

which is nothing else than formula (5) after division by 2. □

Remark 3. a) In particular, formulas (5) and (6) above show that the value of the constant C_1 differs from that of $\sum_{n\geqslant 1}^{\mathcal{R}} \frac{H_n}{n}$. By numerical calculation we find

$$\sum_{n\geqslant 1}^{\mathcal{R}} \frac{H_n}{n} = 0.5290529699\ldots,$$

whereas

$$C_1 = 0.989055995\ldots$$

b) From the knowledge of the asymptotic expansion of $\sum_{n=1}^{N} \frac{H_n}{n}$ at infinity:

$$\sum_{n=1}^{N} \frac{H_n}{n} = \frac{1}{2}\ln^2(N) + \gamma\ln(N) + \frac{1}{2}\gamma^2 + \frac{1}{2}\zeta(2) + o(1),$$

we can easily deduce an asymptotic representation of C_1:

$$C_1 = \lim_{N\to\infty}\left\{\sum_{n=1}^{N} \frac{H_n}{n} - \frac{1}{2}\ln^2(N) - \gamma\ln(N)\right\}.$$

c) More generally, if the numbers $\tilde{\gamma}_k$ (that we propose to call *harmonic Stieltjes constants*) are defined by the Laurent expansion

$$\zeta_H(s) = \frac{1}{(s-1)^2} + \frac{\gamma}{(s-1)} + \sum_{k=0}^{\infty} \frac{(-1)^k}{k!}\tilde{\gamma}_k(s-1)^k \qquad (0 < |s-1| < 1),$$

then one can prove (via an adaptation of [3]) that

$$\tilde{\gamma}_k = \lim_{N\to\infty}\left\{\sum_{n=1}^{N} \frac{H_n \ln^k(n)}{n} - \frac{1}{k+2}\ln^{k+2}(N) - \frac{\gamma}{k+1}\ln^{k+1}(N)\right\}.$$

1.2 The constants at zero and negative integers

The constants at zero and at the negative poles of ζ_H have been evaluated by Boyadzhiev et al. (cf. [5, Corollaries 2 and 3]). We now find again these results by another method. Indeed, formula (1) gives in a neighborhood of $s = -k$ (with $k = 0, 1, 2, \ldots$) the representation

$$\zeta_H(s) = (-1)^{k-1}\frac{\zeta(-k)}{s+k} + \sum_{n\geqslant 1}^{\mathcal{R}} n^k H_n - \nu_k + (-1)^{k-1}\zeta'(-k) + O(s+k), \qquad (7)$$

where ν_k is defined (cf. [9]) by

$$\nu_k := \sum_{n=2}^{\infty}(-1)^n \frac{\zeta(n)}{n+k} = \int_0^1 x^k\left(\psi(x+1)+\gamma\right)dx \qquad (k \geq 0).$$

Let us recall (cf. [8, p. 76]) that $\zeta(-k) = -\frac{B_{k+1}}{k+1}$ $(k \geq 1)$ and $\zeta(0) = -\frac{1}{2}$. We can deduce from formula (7) the following corollary:

Corollary 2. a) In a neighborhood of $s = 0$, we have the representation

$$\zeta_H(s) = -\frac{\zeta(0)}{s} + C_0 + O(s)$$

with

$$C_0 = \sum_{n\geqslant 1}^{\mathcal{R}} H_n - \nu_0 - \zeta'(0)\,.$$

Since we know that $\nu_0 = \gamma$, $\zeta'(0) = -\frac{1}{2}\log(2\pi)$, and

$$\sum_{n\geqslant 1}^{\mathcal{R}} H_n = \frac{3}{2}\gamma + \frac{1}{2} - \frac{1}{2}\log(2\pi)$$

(cf. [6, p. 44]), it follows that a simpler expression of C_0 is

$$C_0 = \frac{1}{2}\gamma + \frac{1}{2}\,. \tag{8}$$

b) In a neighborhood of $s = -1$, we have the representation

$$\zeta_H(s) = \frac{\zeta(-1)}{s+1} + C_{-1} + O(s+1)\,,$$

with

$$C_{-1} = \sum_{n\geqslant 1}^{\mathcal{R}} nH_n - \nu_1 + \zeta'(-1)\,.$$

Since

$$\sum_{n\geqslant 1}^{\mathcal{R}} nH_n = \frac{5}{12}\gamma - \frac{1}{2}\log(2\pi) - \zeta'(-1) + \frac{7}{8}\,,$$

and

$$\nu_1 = \frac{1}{2}\gamma - \frac{1}{2}\log(2\pi) + 1$$

(cf. [6, p. 93–99], [9]), it follows that a simpler expression of C_{-1} is

$$C_{-1} = -\frac{1}{12}\gamma - \frac{1}{8}\,. \tag{9}$$

Formula (9) can be generalized as follows:

Proposition 2. In a neighborhood of $s = 1 - 2k$ with $k \geq 1$, the function ζ_H is represented as

$$\zeta_H(s) = \frac{\zeta(1-2k)}{s+2k-1} + C_{1-2k} + O(s+2k-1)$$

with

$$C_{1-2k} = -\frac{B_{2k}}{2k}\gamma + D_{2k-1}\,,$$

where D_{2k-1} are the rationals:

$$D_1 = -\frac{1}{8}\,,$$

and

$$D_{2k-1} = \frac{1}{2k}\left[H_{2k-1}B_{2k} + \sum_{j=0}^{2k-2}\binom{2k}{j}\frac{B_jB_{2k-j}}{2k-j}\right] \qquad (k \geq 2)\,. \tag{10}$$

Proof. From formula (7), we have

$$C_{1-2k} = \sum_{n\geqslant 1}^{\mathcal{R}} n^{2k-1}H_n - \nu_{2k-1} + \zeta'(1-2k)\,.$$

This may be rewritten

$$C_{1-2k} = D_{2k-1} - \gamma\frac{B_{2k}}{2k}$$

with

$$D_{2k-1} = \sum_{n\geqslant 1}^{\mathcal{R}} n^{2k-1}H_n - \nu_{2k-1} + \zeta'(1-2k) + \gamma\frac{B_{2k}}{2k}\,.$$

From Corollary 2, b), we have $D_1 = -\frac{1}{8}$, and for $k \geq 2$, we deduce from [5, Eq. (22)] the relation

$$D_{2k-1} = H_{2k-1}\frac{B_{2k}}{2k} + \sum_{j=0}^{2k-2}\binom{2k-1}{j}\frac{B_jB_{2k-j}}{(2k-j)^2}$$

which leads to (10). □

Example 1. In particular, we calculate $D_3 = \frac{1}{288}$. Hence the value of the constant at pole $s = -3$ is

$$C_{-3} = \frac{1}{120}\gamma + \frac{1}{288}\,.$$

Remark 4. We can define numbers D_k for all integers $k \geq 0$ by

$$D_k := \sum_{n\geqslant 1}^{\mathcal{R}} n^kH_n - \nu_k + (-1)^{k-1}\zeta'(-k) + \gamma\frac{B_{k+1}}{k+1}\,.$$

By Corollary 2, a), we have $D_0 = \frac{1}{2}$, and formula (7) shows that, for $k \geq 1$, the $\mathcal{R}$-sum $\sum_{n\geqslant 1}^{\mathcal{R}} n^{2k}H_n$ is linked to the expansion of ζ_H at $s = -2k$ by

$$\zeta_H(s) = \sum_{n\geqslant 1}^{\mathcal{R}} n^{2k}H_n - \nu_{2k} - \zeta'(-2k) + O(s+2k)\,.$$

It follows immediately (since $B_{2k+1} = 0$) that

$$D_{2k} = \zeta_H(-2k)\,.$$

From Matsuoka's formula, we deduce the following identity:

$$D_{2k} = -\frac{B_{2k}}{4k} + \frac{B_{2k}}{2} = (2k-1)\frac{B_{2k}}{4k}\,. \tag{11}$$

Hence formulas (10) and (11) show that D_k are rational numbers for all k. In the case where k is positive and even, D_k is the value of ζ_H at $s = -k$, whereas in the odd case, D_k is the rational part of C_{-k} at pole $s = -k$.

2 The generalized harmonic zeta function ζ_{H^p}

For each integer $p > 1$, we now consider the sequence $H^p = (H_n^{(p)})$ of harmonic generalized numbers

$$H_n^{(p)} := \sum_{j=1}^{n} \frac{1}{j^p}\,.$$

Nota bene. In the remainder of the text, we adopt the lighter notation H_n^p in place of $H_n^{(p)}$. To avoid confusions, we will take care to write $(H_n)^p$ the p-th power of H_n.

For each integer $n \geq 1$, we recall (cf. [8, p. 95]) that

$$H_n^p = \psi_p(n)$$

where ψ_p is the analytic function defined in the half-plane $\mathrm{Re}(x) > 1$ by

$$\psi_p(x) = \zeta(p) + \frac{(-1)^{p-1}}{(p-1)!}\partial^{p-1}\psi(x+1)\,.$$

Definition 3. Let $p \geq 2$ be an integer. We call *harmonic zeta function of order p* and note ζ_{H^p} the analytic function defined in the half-plane $\mathrm{Re}(s) > 1$ by

$$\zeta_{H^p}(s) = \sum_{n=1}^{+\infty} \frac{H_n^p}{n^s}\,.$$

Definition 4. The function

$$s \mapsto \sum_{n\geqslant 1}^{\mathcal{R}} \frac{H_n^p}{n^s}$$

is defined by analytic continuation of the function defined for $\mathrm{Re}(s) > 1$ by

$$\zeta_{H^p}(s) - \int_1^{+\infty} \frac{\psi_p(x)}{x^s}\,dx\,.$$

It results from [6, Theorem 9] that this function is analytic in the whole $\mathbb{C}$.

We can establish for the function ζ_{H^p} a result very similar to Theorem 1.

Theorem 2. *If* $1 < \mathrm{Re}(s) < 2$*, then the function* ζ_{H^p} *can be decomposed as*

$$\zeta_{H^p}(s) = \sum_{n\geqslant 1}^{\mathcal{R}} \frac{H_n^p}{n^s} - \int_0^1 \frac{\psi_p(x)}{x^s}\,dx - \frac{\pi}{\sin(\pi s)}\frac{\Gamma(s+p-1)}{\Gamma(s)\Gamma(p)}\zeta(s+p-1)\,. \tag{12}$$

Proof. As in Theorem 1, for $1 < \mathrm{Re}(s) < 2$, we have the relation

$$\sum_{n=1}^{+\infty} \frac{H_n^p}{n^s} = \sum_{n\geqslant 1}^{\mathcal{R}} \frac{H_n^p}{n^s} + \int_0^{+\infty} \frac{\psi_p(x)}{x^s}\,dx - \int_0^1 \frac{\psi_p(x)}{x^s}\,dx\,.$$

By deriving $p-1$ times the expansion of $x \mapsto \psi(x+1)$, we obtain for $x \in]0,1[$, the expansion in power series

$$\frac{\psi_p(x)}{x} = \sum_{n\geq 0}(-1)^n \frac{(n+2)\cdots(n+p)}{(p-1)!}\zeta(n+p+1)x^n\,.$$

Then an application of the Ramanujan master's theorem gives for $1 < \mathrm{Re}(s) < 2$,

$$\begin{aligned}\int_0^{+\infty} \frac{\psi_p(x)}{x^s}\,dx &= \frac{-\pi}{\sin(\pi s)}\frac{s(s+1)\cdots(s+p-2)}{(p-1)!}\zeta(s+p-1)\\ &= \frac{-\pi}{\sin(\pi s)}\frac{\Gamma(s+p-1)}{\Gamma(s)\Gamma(p)}\zeta(s+p-1).\end{aligned}$$

□

Moreover, formula (12) provides the meromorphic continuation of ζ_{H^p} in the half-plane $\mathrm{Re}(s) < 2$. This function has only simple poles at $s = 1$ and $s = m - p$ with

$$m = 2,\ 1,\ 0,\ -2,\ -2-2k \quad \text{for each } k \geq 1\,,$$

which are the poles of the function

$$s \mapsto \frac{-\pi}{\sin(\pi s)}\frac{\Gamma(s+p-1)}{\Gamma(s)\Gamma(p)}\zeta(s+p-1)\,.$$

2.1 The case $p = 2$

We deduce immediately from Theorem 2 the decomposition

$$\zeta_{H^2}(s) = \sum_{n\geqslant 1}^{\mathcal{R}} \frac{H_n^2}{n^s} - \int_0^1 \frac{\zeta(2) - \psi'(x+1)}{x^s}\,dx - \frac{\pi}{\sin(\pi s)}s\,\zeta(s+1) \qquad (1 < \mathrm{Re}(s) < 2)\,. \tag{13}$$

The poles of ζ_{H^2} are all simple poles located at points

$$s = 1, 0, -1, -2, -4, -6, \dots$$

which are the poles of the function

$$s \mapsto \frac{-\pi}{\sin(\pi s)} s\, \zeta(s+1)\,.$$

We now fully determine the constants at the poles of ζ_{H^2}.

Proposition 3. a) In a neighborhood of $s = 1$, ζ_{H^2} is represented as

$$\zeta_{H^2}(s) = \frac{\zeta(2)}{s-1} + C_1^{(2)} + O(s-1)$$

with

$$C_1^{(2)} = \gamma\zeta(2) - \zeta(3)\,. \tag{14}$$

b) In a neighborhood of $s = 0$, ζ_{H^2} is represented as

$$\zeta_{H^2}(s) = -\frac{1}{s} + C_0^{(2)} + O(s)$$

with

$$C_0^{(2)} = \frac{1}{2}\zeta(2) - \gamma - 1\,. \tag{15}$$

c) In a neighborhood of $s = -1$, ζ_{H^2} is represented as

$$\zeta_{H^2}(s) = \frac{1}{2(s+1)} + C_{-1}^{(2)} + O(s+1)$$

with

$$C_{-1}^{(2)} = -\frac{1}{12}\zeta(2) + \frac{1}{2}\gamma + \frac{1}{4}\,. \tag{16}$$

Proof. a) We deduce from (13) that an expression of the constant at $s = 1$ is

$$C_1^{(2)} = \sum_{n\geqslant 1}^{\mathcal{R}} \frac{H_n^2}{n} + \zeta(2) + \zeta'(2) + \int_0^1 \frac{\psi'(x+1) - \zeta(2)}{x}\, dx\,.$$

We have calculated the value of $\sum_{n\geqslant 1}^{\mathcal{R}} \frac{H_n^2}{n}$ (cf. [10, Eq. (3)]). We have

$$\sum_{n\geqslant 1}^{\mathcal{R}} \frac{H_n^2}{n} = \gamma\zeta(2) - \zeta(3) - 1 - \zeta'(2) - \tau_2$$

with

$$\tau_2 = \sum_{n=1}^{\infty} (-1)^n \frac{\zeta(n+2)}{n}\,.$$

We evaluate the integral $\displaystyle\int_0^1 \frac{\psi'(x+1)-\zeta(2)}{x}\,dx$ by an integration by parts, that gives

$$\int_0^1 \frac{\psi'(x+1)-\zeta(2)}{x}\,dx = 1-\zeta(2)+\int_0^1 \frac{\psi(x+1)+\gamma-\zeta(2)x}{x^2}\,dx\,.$$

By Lemma 1, we have seen that

$$\int_0^1 \frac{\psi(x+1)+\gamma-\zeta(2)x}{x^2}\,dx = \tau_2\,.$$

Thus, after cancellation of the term τ_2, we obtain the simpler expression

$$C_1^{(2)} = \gamma\zeta(2)-\zeta(3)\,.$$

b) We deduce from (13) that an expression of the constant at $s=0$ is

$$C_0^{(2)} = \sum_{n\geqslant 1}^{\mathcal{R}} H_n^2 - \gamma - \zeta(2) + 1\,.$$

We have calculated the value of $\sum_{n\geqslant 1}^{\mathcal{R}} H_n^2$ (cf. [6, p. 44]). We have

$$\sum_{n\geqslant 1}^{\mathcal{R}} H_n^2 = \frac{3}{2}\zeta(2) - 2\,.$$

Therefore

$$C_0^{(2)} = \frac{1}{2}\zeta(2) - \gamma - 1\,.$$

c) We deduce from (13) that an expression of the constant at $s=-1$ is

$$C_{-1}^{(2)} = \sum_{n\geqslant 1}^{\mathcal{R}} nH_n^2 + \frac{1}{2}\log(2\pi) + \frac{1}{2} - \gamma - \frac{1}{2}\zeta(2)\,.$$

We have calculated the value of the $\mathcal{R}$-sum $\sum_{n\geqslant 1}^{\mathcal{R}} nH_n^2$ (cf. [6, p. 82]). We have

$$\sum_{n\geqslant 1}^{\mathcal{R}} nH_n^2 = \frac{5}{12}\zeta(2) - \frac{1}{2}\log(2\pi) + \frac{3}{2}\gamma - \frac{1}{4}\,.$$

It follows that

$$C_{-1}^{(2)} = -\frac{1}{12}\zeta(2) + \frac{1}{2}\gamma + \frac{1}{4}\,.$$

□

Regarding the determination of the constants at poles $s = -2, -4, -6, \ldots$ of ζ_{H^2}, we prove the following result:

Proposition 4. In a neighborhood of $s = -2k$, ζ_{H^2} is represented as

$$\zeta_{H^2}(s) = -\frac{B_{2k}}{s+2k} + C^{(2)}_{-2k} + O(s+2k)$$

with

$$C^{(2)}_{-2k} = -B_{2k}\gamma + D^{(2)}_{2k}\,, \tag{17}$$

where

$$D^{(2)}_2 = -\frac{5}{36}\,,$$

and

$$D^{(2)}_{2k} = B_{2k}H_{2k-1} + \sum_{j=0}^{2k-2}\binom{2k}{j}\frac{B_jB_{2k-j}}{2k-j} - \sum_{j=0}^{2k}(-1)^j\binom{2k}{j}\frac{B_jB_{2k-j}}{j+1} \qquad (k \geq 2)\,. \tag{18}$$

The proof of Proposition 4 is based on the following lemma:

Lemma 3. For all integers $k \geq 1$, we have the relation

$$\sum_{n\geqslant 1}^{\mathcal{R}} n^k H_n^2 = k\sum_{n\geqslant 1}^{\mathcal{R}} n^{k-1}H_n + \frac{1-B_{k+1}}{k+1}\zeta(2) - b_k \tag{19}$$

with

$$b_k = 1 + \sum_{j=0}^{k}(-1)^j\binom{k}{j}\frac{B_jB_{k-j}}{j+1}\,.$$

Proof. We can write the following identities

$$\begin{aligned}\sum_{n\geqslant 1}^{\mathcal{R}} H_n e^{-nz} &= \sum_{n=1}^{\infty} H_n e^{-nz} - \int_1^{\infty}(\psi(x+1)+\gamma)e^{-xz}\,dx \\ &= -\frac{\log(1-e^{-z})}{1-e^{-z}} - \frac{e^{-z}}{z}\gamma - \int_1^{\infty}\psi(x+1)e^{-xz}\,dx\end{aligned}$$

and

$$\begin{aligned}\sum_{n\geqslant 1}^{\mathcal{R}} H_n^2 e^{-nz} &= \sum_{n=1}^{\infty} H_n^2 e^{-nz} + \int_1^{\infty}(\psi'(x+1)-\zeta(2))e^{-xz}\,dx \\ &= \frac{1}{1-e^{-z}}\mathrm{Li}_2(e^{-z}) - \frac{e^{-z}}{z}\zeta(2) + \int_1^{\infty}\psi'(x+1)e^{-xz}\,dx\,.\end{aligned}$$

By integration by parts of the last integral, this may be rewritten

$$\sum_{n\geqslant 1}^{\mathcal{R}} H_n^2 e^{-nz} = \frac{1}{1-e^{-z}}\mathrm{Li}_2(e^{-z}) - \frac{e^{-z}}{z}\zeta(2) - e^{-z}(1-\gamma) + z\int_1^\infty \psi(x+1)e^{-xz}\,dx\,.$$

Thus, after cancellation of the integral term in both formulas, we obtain the relation

$$\sum_{n\geqslant 1}^{\mathcal{R}} H_n^2 e^{-nz} = -z\sum_{n\geqslant 1}^{\mathcal{R}} H_n e^{-nz} + \frac{1}{1-e^{-z}}\mathrm{Li}_2(e^{-z}) - z\frac{\log(1-e^{-z})}{1-e^{-z}} - \frac{e^{-z}}{z}\zeta(2) - e^{-z}\,. \tag{20}$$

Then, using the following expansions:

$$\begin{aligned} -z\frac{\log(1-e^{-z})}{1-e^{-z}} &= \frac{-z}{1-e^{-z}}\log\frac{1-e^{-z}}{z} + \frac{-z}{1-e^{-z}}\log z \\ &= \frac{-z}{1-e^{-z}}\sum_{k=1}^{\infty} B_k\frac{1}{k}\frac{z^k}{k!} + \frac{-z}{1-e^{-z}}\log z \end{aligned}$$

and

$$\frac{1}{1-e^{-z}}\mathrm{Li}_2(e^{-z}) = \frac{1}{1-e^{-z}}\left(\zeta(2) + z\log z - z + z\sum_{k=1}^{\infty} B_k\frac{1}{k(k+1)}\frac{z^k}{k!}\right)$$

(cf. [7, Eq. (133)]), we can simplify this relation as follows:

$$\begin{aligned} \sum_{n\geqslant 1}^{\mathcal{R}} H_n^2 e^{-nz} = -z\sum_{n\geqslant 1}^{\mathcal{R}} H_n e^{-nz} + (\frac{1}{1-e^{-z}} - \frac{e^{-z}}{z})\zeta(2) + \frac{-z}{1-e^{-z}} \\ + \frac{-z}{1-e^{-z}}\sum_{k=1}^{\infty} B_k\frac{1}{k+1}\frac{z^k}{k!} - e^{-z}\,. \end{aligned}$$

Expanding each term in powers of z, this translates into the relation (valid for $k \geq 1$):

$$\sum_{n\geq 1}^{\mathcal{R}} n^k H_n^2 = k\sum_{n\geqslant 1}^{\mathcal{R}} n^{k-1}H_n + (\frac{1-B_{k+1}}{k+1})\zeta(2) - 1 - B_k - \sum_{j=1}^{k}\binom{k}{j}\frac{(-1)^j}{j+1}B_jB_{k-j}\,,$$

which is nothing else than (19). □

We are now in a position to prove Proposition 4.

Proof of Proposition 4. We deduce from (13) that an expression of the constant at $s = -2k$ is

$$C_{-2k}^{(2)} = \sum_{n\geqslant 1}^{\mathcal{R}} n^{2k}H_n^2 + \frac{B_{2k}}{2k} + 2k\zeta'(1-2k) - \frac{1}{2k+1}\zeta(2) + \int_0^1 x^{2k}\psi'(x+1)\,dx\,.$$

We now evaluate this last integral. An integration by parts gives

$$\int_0^1 x^{2k}\psi'(x+1)\,dx = 1-\gamma-2k\int_0^1 x^{2k-1}\psi(x+1)\,dx\,.$$

We can write

$$\int_0^1 x^{2k-1}\psi(x+1)\,dx = \nu_{2k-1}-\frac{1}{2k}\,\gamma\,,$$

with

$$\nu_{2k-1} = \sum_{n\geqslant 1}^{\mathcal{R}} n^{2k-1}H_n + \zeta'(1-2k) + \frac{B_{2k}}{2k}\,\gamma - D_{2k-1}\,,$$

where $D_1 = -\frac{1}{8}$, and D_{2k-1} is given for $k \geq 2$ by formula (10) . It follows that

$$\int_0^1 x^{2k}\psi'(x+1)\,dx = 1-2k\zeta'(1-2k)-B_{2k}\gamma-2k\sum_{n\geqslant 1}^{\mathcal{R}} n^{2k-1}H_n+2kD_{2k-1}\,.$$

We can now write a simpler expression of $C^{(2)}_{-2k}$ using Lemma 3. Indeed, we deduce from (19) the following relation between the $\mathcal{R}$-sums:

$$\sum_{n\geqslant 1}^{\mathcal{R}} n^{2k}H_n^2 = 2k\sum_{n\geqslant 1}^{\mathcal{R}} n^{2k-1}H_n + \frac{1}{2k+1}\zeta(2) - b_{2k}$$

with

$$b_{2k} = 1+\sum_{j=0}^{2k}(-1)^j\binom{2k}{j}\frac{B_jB_{2k-j}}{j+1}\,.$$

It follows that

$$C^{(2)}_{-2k} = -B_{2k}\gamma + D^{(2)}_{2k}$$

with

$$D^{(2)}_{2k} := \frac{B_{2k}}{2k} + 2kD_{2k-1} - b_{2k} + 1\,.$$

This last expression gives

$$D^{(2)}_2 = -\frac{5}{36}\,,$$

and translates into formula (18) for $k \geq 2$. □

Example 2. The value of the constant at $s = -4$ is

$$C^{(2)}_{-4} = \frac{1}{30}\gamma - \frac{1}{600}\,.$$

In addition, formula (13) allows us to evaluate the special values of ζ_{H^2} at odd negative integers $s = -3, -5, -7, \ldots$. More precisely, we show the following result:

Proposition 5. For each integer $k \geq 2$, we have

$$\zeta_{H^2}(1-2k) = -\frac{B_{2k}}{2k}\,\zeta(2) + \frac{k(2k-3)}{4(k-1)}\,B_{2k-2}\,. \tag{21}$$

Proof. From (13) we deduce that

$$\zeta_{H^2}(1-2k) = (1-2k)\zeta'(2-2k) - \frac{1}{2k}\zeta(2) + \int_0^1 x^{2k-1}\psi'(x+1)\,dx + \sum_{n\geqslant 1}^{\mathcal{R}} n^{2k-1}H_n^2\,.$$

Proceeding in the same manner as in the proof of Proposition 4, we obtain

$$\zeta_{H^2}(1-2k) = \sum_{n\geqslant 1}^{\mathcal{R}} n^{2k-1}H_n^2 - (2k-1)\sum_{n\geqslant 1}^{\mathcal{R}} n^{2k-2}H_n - \frac{1}{2k}\zeta(2) + 1 + (2k-1)D_{2k-2}\,,$$

where D_{2k} is given by formula (11). Using the relation (19) linking the $\mathcal{R}$-sums $\sum_{n\geqslant 1}^{\mathcal{R}} n^{2k-1}H_n^2$ and $\sum_{n\geqslant 1}^{\mathcal{R}} n^{2k-2}H_n$, we deduce the following formula:

$$\zeta_{H^2}(1-2k) = -\frac{B_{2k}}{2k}\zeta(2) + \frac{(2k-1)(2k-3)B_{2k-2}}{4(k-1)} - \sum_{j=1}^{2k-2}(-1)^j\binom{2k-1}{j}\frac{B_jB_{2k-1-j}}{j+1}$$

which reduces to formula (21) since almost all terms of the sum Σ are null. $\square$

Example 3. The value of ζ_{H^2} at $s=-3$ is

$$\zeta_{H^2}(-3) = \frac{1}{120}\zeta(2) + \frac{1}{12}\,.$$

2.2 The constant at $s=1$ in the general case

The following proposition extends our formula (14) established in the case $p=2$.

Proposition 6. In a neighborhood of $s=1$, for all integers $p \geq 2$, the function ζ_{H^p} is represented as

$$\zeta_{H^p}(s) = \frac{\zeta(p)}{s-1} + C_1^{(p)} + O(s-1)\,, \tag{22}$$

with

$$C_1^{(p)} = \gamma\zeta(p) + \zeta(p+1) - \zeta_H(p)\,. \tag{23}$$

Proof. For $p=2$, formula (23) is nothing else than (14) because $\zeta_H(2) = 2\zeta(3)$. We suppose now that $p \geq 3$. We deduce from (12) that, in a neighborhhood of $s=1$, we have the representation

$$\zeta_{H^p}(s) = \frac{\zeta(p)}{s-1} + C_1^{(p)} + O(s-1)\,,$$

where

$$C_1^{(p)} = H_{p-1}\,\zeta(p) + \zeta'(p) + (-1)^p I_p + \sum_{n\geqslant 1}^{\mathcal{R}} \frac{H_n^p}{n}\,, \tag{24}$$

with

$$I_p := \frac{1}{(p-1)!}\int_0^1 \frac{\partial^{p-1}\psi(x+1) - (-1)^p (p-1)!\,\zeta(p)}{x}\,dx\,.$$

We evaluate I_p by integrating $p-1$ times by parts, this leads to the relation

$$\begin{aligned}(-1)^p I_p = (-1)^p \tau_p - H_{p-1}\zeta(p)\\ - \frac{1}{(p-1)!}\sum_{j=1}^{p-2}(-1)^j \zeta(p-j)(p-j-1)! \sum_{k=1}^{p-j-1}\frac{(k+j-1)!}{k!}\\ + \frac{1}{(p-1)!}\sum_{k=0}^{p-2}(-1)^k k!(p-k-2)!\,,\end{aligned}$$

with (by Lemma 1)

$$(-1)^p \tau_p = \sum_{n=1}^{\infty}(-1)^n \frac{\zeta(n+p)}{n} = (-1)^p \int_0^1 \frac{\psi(x+1) + \gamma - \sum_{j=1}^{p-1}(-1)^{j-1}\zeta(j+1)x^j}{x^p}\,dx\,.$$

We obtain a much simpler expression of I_p by means of the following identities (cf. [14]):

$$\frac{1}{(p-1)!}\sum_{k=0}^{p-2}(-1)^k k!(p-k-2)! = \frac{1}{p-1}\sum_{k=0}^{p-2}\frac{(-1)^k}{\binom{p-2}{k}} = \frac{1+(-1)^p}{p}\,,$$

and

$$\sum_{k=1}^{p-j-1}\frac{(k+j-1)!}{k!} = (j-1)!\sum_{k=1}^{p-j-1}\binom{k+j-1}{j-1} = (j-1)!\left(\binom{p-1}{j} - 1\right)\,.$$

This allows us to write

$$(-1)^p I_p = (-1)^p \tau_p - H_{p-1}\zeta(p) + \sigma_p \tag{25}$$

with

$$\sigma_p = \frac{1+(-1)^p}{p} + \sum_{j=1}^{p-2}(-1)^j \zeta(p-j)\left[\frac{(p-j-1)!(j-1)!}{(p-1)!} - \frac{1}{j}\right]\,.$$

Moreover, we have shown that

$$\sum_{n\geqslant 1}^{\mathcal{R}}\frac{H_n^p}{n} = \gamma\zeta(p) + \zeta(p+1) - \zeta_H(p) - \sigma_p - \zeta'(p) - (-1)^p\tau_p \tag{26}$$

(cf. [10, Eq. (9)]). By substituting (25) and (26) into (24), we then obtain (23). □

Example 4. The values of the constants of functions ζ_{H^3} and ζ_{H^4} at pole $s=1$ are

$$C_1^{(3)} = \gamma\zeta(3) - \frac{1}{4}\zeta(4)$$

and

$$C_1^{(4)} = \gamma\zeta(4) - 2\zeta(5) + \zeta(3)\zeta(2)\,.$$

Remark 5. Formula (23) extends to the case $p=1$ provided that $\zeta(1)$ must be replaced by γ (the constant of ζ at $s=1$) and $\zeta_H(1)$ by C_1 (the constant of ζ_H at $s=1$). This leads to the equation

$$C_1 = \gamma^2 + \zeta(2) - C_1$$

which admits formula (6) as solution.

2.3 A method to evaluate the constants at poles $s = m - p$

In the general case, we have seen that all poles of ζ_{H^p} (apart from 1) are located at points $s = m - p$ with

$$m = 2,\ 1,\ 0,\ -2,\ -4,\ -6, \ldots$$

In a neighborhood of $s = m - p$, we have the representation

$$\zeta_{H^p}(s) = \frac{A_m}{s+p-m} + B_m - \int_0^1 \psi_p(x) x^{p-m}\,dx + \sum_{n\geqslant 1}^{\mathcal{R}} n^{p-m} H_n^p + O(s+p-m)\,,$$

where A_m et B_m are the constants defined by the expansion

$$\frac{-\pi}{\sin(\pi s)} \frac{\Gamma(s+p-1)}{\Gamma(s)\Gamma(p)} \zeta(s+p-1) = \frac{A_m}{s+p-m} + B_m + O(s+p-m)\,.$$

To evaluate the $\mathcal{R}$-sum $\sum_{n\geqslant 1}^{\mathcal{R}} n^{p-m} H_n^p$, we can use the same method as in the proof of Lemma 3. We have the following extension of (20):

$$\begin{aligned}
\sum_{n\geqslant 1}^{\mathcal{R}} H_n^p e^{-nz} &= (-1)^{p-1} \frac{1}{(p-1)!} z^{p-1} \sum_{n\geqslant 1}^{\mathcal{R}} H_n e^{-nz} + \frac{1}{1-e^{-z}} \mathrm{Li}_p(e^{-z}) \\
&\quad + (-1)^{p-1} \frac{1}{(p-1)!} z^{p-1} \Big(\frac{\log(1-e^{-z})}{1-e^{-z}}\Big) \\
&\quad + (-1)^{p-1} \frac{1}{(p-1)!} e^{-z} \Big(z^{p-2} + \sum_{k=2}^{p-1} z^{k-2} (-1)^{p-k+1} (p-k)! \zeta(p-k+1)\Big) \\
&\quad - \frac{e^{-z}}{z} \zeta(p) + (-1)^{p-1} \frac{1}{(p-1)!} \sum_{k=2}^{p-1} z^{k-2} (-1)^{p-k} (p-k)!\,.
\end{aligned}$$

For example, for $p = 3$, we get

$$\sum_{n\geqslant 1}^{\mathcal{R}} H_n^3 e^{-nz} = \frac{1}{2}z^2 \sum_{n\geqslant 1}^{\mathcal{R}} H_n e^{-nz} + \frac{1}{1-e^{-z}}\mathrm{Li}_3(e^{-z}) + \frac{1}{2}z^2\left(\frac{\log(1-e^{-z})}{1-e^{-z}}\right)$$
$$+ \frac{1}{2}e^{-z}(z+\zeta(2)) - \frac{e^{-z}}{z}\zeta(3) - \frac{1}{2}\,.$$

This enables to express the $\mathcal{R}$-sum $\sum_{n\geqslant 1}^{\mathcal{R}} n^k H_n^p$ in terms of $\sum_{n\geqslant 1}^{\mathcal{R}} n^k H_n$. All constants C_{m-p} can then be determined by the same method used in section 2.1.

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