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► **To cite this version:**

Marc-Antoine Coppo. A note on certain alternating series involving zeta and multiple zeta values. *Journal of Mathematical Analysis and Applications*, 2019, 475, pp.1831-1841. 10.1016/j.jmaa.2019.03.057 . hal-01735381v6

**HAL Id: hal-01735381**

**<https://hal.univ-cotedazur.fr/hal-01735381v6>**

Submitted on 22 Mar 2021 (v6), last revised 6 Oct 2021 (v7)

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# A note on some alternating series involving zeta and multiple zeta values

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*Journal of Mathematical Analysis and Applications*, **475** (2019)

**Abstract** In this article, we study a class of conditionally convergent alternating series including, as a special case, the famous series  $\sum_{n \geq 2} (-1)^n \frac{\zeta(n)}{n}$  which links Euler's constant  $\gamma$  to special values of the Riemann zeta function at positive integers. We give several new relations of the same kind. Among other things, we show the existence of a similar relation for the Apostol-Vu harmonic zeta function which have never been noticed before. We also highlight a deep connection with the Ramanujan summation of certain divergent series which originally motivated this work.

**Keywords** Riemann zeta function; harmonic zeta function; Stirling numbers of the first kind; Stirling numbers of the second kind; Bernoulli numbers; Bernoulli numbers of the second kind; harmonic numbers; Gregory coefficients of higher order; multiple zeta values; Ramanujan summation of divergent series.

**Mathematics Subject Classification (2010)** 11B73; 11B75; 11M06; 11M32; 40G99; 41A58.

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# Introduction

The first part of this article is devoted to the study of the conditionally convergent alternating series  $\nu_k$  defined by

$$\nu_k := \sum_{n=2}^{\infty} (-1)^n \frac{\zeta(n)}{n+k},$$

where  $\zeta(s)$  is the Riemann zeta function and  $k$  denotes an integral parameter. By a classical result (cf. [8, p. 66], [15, p. 62]), it is well known that  $\nu_0$  is Euler's constant

$$\gamma = \lim_{n \rightarrow \infty} \left\{ \sum_{j=1}^n \frac{1}{j} - \ln n \right\} = 0.5772156649 \dots$$

This remarkable connection between  $\gamma$  and the special values at positive integers of the Riemann zeta function goes back to Euler's early works on harmonic series [13]. Less famous but yet fairly well-known (cf. [8, p. 93], [16, Eq. (5.1)], [17, Eq. (1.5)]) is the relation

$$\nu_1 = \frac{\gamma}{2} - \frac{1}{2} \ln 2\pi + 1$$

sometimes called Suryanarayana formula. Recently, Blagouchine [5, p. 413] gave a general expression of these series  $\nu_k$  in the case where  $k$  is a positive integer:

$$\begin{aligned} \nu_k &= \frac{\gamma}{2} - \frac{\ln 2\pi}{k+1} + \frac{1}{k} \\ &+ \sum_{r=1}^{\lfloor \frac{k}{2} \rfloor} (-1)^r \binom{k}{2r-1} \frac{(2r)!}{r(2\pi)^{2r}} \zeta'(2r) + \sum_{r=1}^{\lfloor \frac{k+1}{2} \rfloor - 1} (-1)^r \binom{k}{2r} \frac{(2r)!}{2(2\pi)^{2r}} \zeta(2r+1). \end{aligned} \quad (1)$$

This formula seems quite cumbersome but can be much simplified using the functional equation of  $\zeta$ . After some elementary transformations, we show that equation (1) can be reduced to the following equivalent (but much more pleasant) expression:

$$\nu_k = \frac{\gamma}{k+1} - \frac{1}{2} \ln 2\pi + \sum_{j=1}^{k-1} (-1)^j \binom{k}{j} \zeta'(-j) + C_k, \quad (2)$$

where  $C_k$  is a rational number (see Proposition 1). Moreover, this expression allows to highlight a deep connection between  $\nu_{2k}$  and the sum (in the sense of the Ramanujan summation of divergent series) of the series  $\sum_{n \geq 1} n^{2k} H_n$ , where  $H_n$  is the  $n$ th harmonic number (see Remark 2).

Next, in a second part, we introduce a generalization of these series series  $\nu_k$  replacing the zeta values by certain multiple zeta values. A natural extension may

be defined as follows: for all integers  $k \geq -1$  and  $p \geq 0$ , we consider the class of series  $(\nu_{k,p})$  with

$$\nu_{k,p} := \sum_{n=2}^{\infty} \frac{(-1)^n}{n+k} \zeta(n, \underbrace{1, \dots, 1}_p),$$

where

$$\zeta(s_1, s_2, \dots, s_k) = \sum_{n_1 > n_2 > \dots > n_k \geq 1} \frac{1}{n_1^{s_1} n_2^{s_2} \dots n_k^{s_k}},$$

so that the previous series  $\nu_k$  become  $\nu_{k,0}$ . Then we establish (see Proposition 2) the following identity which is the main result of this work:

$$\nu_{k,p} = \sum_{n=1}^{\infty} \frac{|G_n^{(k+1)}|}{n^{p+1}}, \quad (3)$$

where  $G_n^{(k)}$  denotes the *Gregory coefficients of higher order* recently introduced by Blagouchine [5, 6]. They are defined by

$$G_n^{(k)} := \frac{1}{n!} \sum_{j=1}^n \frac{s(n,j)}{j+k} \quad (k \geq 0, n \geq 1), \quad (4)$$

where  $s(n,j)$  are the Stirling numbers of the first kind. Comprehensive informations on the Stirling numbers of the first and the second kind may be found in [1, 5, 12, 15, 18]. One can prove easily (see Lemma 4) that  $G_n^{(k)} = (-1)^{n+1} |G_n^{(k)}|$ , so that the rational numbers  $G_n^{(k)}$  alternate in sign. As a special case of equation (3), we derive the following result:

$$\nu_{k-1} = \sum_{n=1}^{\infty} \frac{|G_n^{(k)}|}{n} \quad (k \geq 0). \quad (5)$$

In the case  $k = 1$ , we recover the classical Mascheroni's series for  $\gamma$  (cf. [5, p. 406], [15, p. 280]):

$$\gamma = \frac{1}{2} + \frac{1}{24} + \frac{1}{72} + \frac{19}{2880} + \frac{3}{800} + \frac{863}{362880} + \dots$$

Another notable consequence of formula (3) is the deduction of this nice formula (see Example 3):

$$\sum_{n=2}^{\infty} (-1)^n \frac{\zeta_H(n)}{n} = \gamma_1 + \frac{1}{2}\gamma^2 + \frac{\pi^2}{12},$$

where  $\zeta_H(s)$  denotes the Apostol-Vu harmonic zeta function [2, 3, 4] and  $\gamma_1$  is the first Stieltjes constant [7, 8].

Finally, in the last section, we highlight a relation between the series  $\nu_k$ , the Stirling numbers of the second kind  $S(n,k)$ , and the shifted Mascheroni series  $\sigma_r$  whose study was the main subject of [12] (see Proposition 3 and Example 4).

# 1 The case of a positive integer

In this section, we focus on the case of a positive integer  $k$  and give two independent proofs of our formula (2). More precisely, we prove the following proposition:

**Proposition 1.** For any positive integer  $k$ , we have

$$\nu_k = \frac{\gamma}{k+1} - \frac{1}{2} \ln 2\pi + \sum_{j=1}^{k-1} (-1)^j \binom{k}{j} \zeta'(-j) + C_k$$

with

$$C_k = \frac{1}{k} + \sum_{j=1}^{k-1} \binom{k}{j} \frac{B_{j+1} H_j}{j+1}, \quad (6)$$

where  $H_n$  are the harmonic numbers,

$$H_0 = 0, \quad H_n := 1 + \frac{1}{2} + \cdots + \frac{1}{n} \quad (n \geq 1),$$

and  $B_n$  are the Bernoulli numbers defined by means of the exponential generating function

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!} \quad (|x| < 2\pi).$$

In particular,  $B_0 = 1$ ,  $B_1 = -1/2$ ,  $B_2 = 1/6$ ,  $B_{2r+1} = 0$  for  $r \geq 1$ .

*Proof.* We can quite easily derive (2) from (1). Differentiation of the functional equation [3, Eq. (25.4.2)]

$$\zeta(s) = 2(2\pi)^{s-1} \Gamma(1-s) \zeta(1-s) \sin \frac{\pi s}{2}$$

leads to the two relations

$$(-1)^r \frac{(2r)!}{2(2\pi)^{2r}} \zeta(2r+1) = \zeta'(-2r) \quad (r \geq 1),$$

and

$$(-1)^r \frac{(2r)!}{r(2\pi)^{2r}} \zeta'(2r) = -\zeta'(1-2r) + \frac{B_{2r}}{2r} (H_{2r-1} - \gamma - \ln 2\pi) \quad (r \geq 1).$$

Substituting these relations into (1) and grouping together the terms under the two symbols  $\Sigma$ , leads to the expression

$$\nu_k = \frac{\gamma}{k+1} - \frac{1}{2} \ln 2\pi + \sum_{j=1}^{k-1} (-1)^j \binom{k}{j} \zeta'(-j) + C_k,$$

where the rational constant  $C_k$  is given by equation (6).

Another alternative proof of (2), independant from (1), may be deduced from the expansion in powers of  $z$  of the relation [8, p. 93]

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{(-1)^k z^k}{k!} \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{j} \zeta^{\mathcal{R}}(j-k) &= (1-e^z) \sum_{k=0}^{\infty} \frac{(-1)^k z^k}{k!} \zeta'(-k) \\ &+ (1-e^z) \sum_{k=0}^{\infty} \frac{(-1)^k z^k}{k!} \frac{1}{(k+1)^2} \\ &+ \int_0^1 \ln(t+1) e^{-zt} dt \end{aligned}$$

with

$$\zeta^{\mathcal{R}}(j-k) = \begin{cases} \gamma & \text{if } j = k+1 \\ \zeta(j-k) - \frac{1}{j-k-1} & \text{otherwise.} \end{cases}$$

Rewriting the series  $\nu_k$  as

$$\nu_k = \sum_{j=k+2}^{\infty} \frac{(-1)^{j-k}}{j} \zeta(j-k),$$

and using the well-known relations

$$\zeta(0) = -\frac{1}{2}, \quad \zeta(1-2r) = -\frac{B_{2r}}{2r}, \quad \text{and} \quad \zeta'(0) = -\frac{1}{2} \ln 2\pi \quad (\text{cf. [3, p. 605]}),$$

then a careful identification of the terms in  $z^k$  in the previous development leads again, after some simplifications, to formula (2), and provides in addition another equivalent expression for the constant  $C_k$ :

$$C_k = \frac{H_k}{k+1} - \sum_{j=1}^k \frac{B_j}{j(k+1-j)}. \quad (7)$$

An unexpected consequence of this equivalence is the curious identity

$$\frac{H_k}{k+1} - \frac{1}{k} = \sum_{j=0}^{k-1} \frac{B_{j+1}}{j+1} \left\{ \binom{k}{j} H_j + \frac{1}{k-j} \right\} \quad (k \geq 1)$$

whose direct proof does not seem obvious. □

**Example 1.** For the first values of  $k$ , we have the following relations:

$$\begin{aligned}\nu_1 &= \frac{\gamma}{2} - \frac{1}{2} \ln 2\pi + 1, \\ \nu_2 &= \frac{\gamma}{3} - \frac{1}{2} \ln 2\pi - 2\zeta'(-1) + \frac{2}{3}, \\ \nu_3 &= \frac{\gamma}{4} - \frac{1}{2} \ln 2\pi - 3\zeta'(-1) + 3\zeta'(-2) + \frac{7}{12}, \\ \nu_4 &= \frac{\gamma}{5} - \frac{1}{2} \ln 2\pi - 4\zeta'(-1) + 6\zeta'(-2) - 4\zeta'(-3) + \frac{47}{90}, \\ \nu_5 &= \frac{\gamma}{6} - \frac{1}{2} \ln 2\pi - 5\zeta'(-1) + 10\zeta'(-2) - 10\zeta'(-3) + 5\zeta'(-4) + \frac{167}{360}.\end{aligned}$$

**Remark 1.** Starting from the Maclaurin series expansion [3, Eq. (25.8.5)]

$$\psi(x+1) + \gamma = \sum_{n=2}^{\infty} (-1)^n \zeta(n) x^{n-1} \quad (|x| < 1)$$

where  $\psi(x)$  denotes the digamma function (i.e. the logarithmic derivative of the  $\Gamma$ -function), and multiplying each side by  $x^k$  (with  $k \geq 1$ ), then an integration between 0 and 1 gives

$$\nu_k = \frac{\gamma}{k+1} + \int_0^1 x^k \psi(x+1) dx.$$

Thus, it follows from formula (2) that

$$\int_0^1 x^k \psi(x+1) dx = \sum_{j=0}^{k-1} (-1)^j \binom{k}{j} \zeta'(-j) + C_k \quad (k \geq 1).$$

**Remark 2** (Link with the Ramanujan summation: part I). Candelpergher et al. [10, Corollary 1] (see also [8, p. 82]) established that

$$\sum_{n \geq 1}^{\mathcal{R}} H_n = \frac{3}{2} \gamma - \frac{1}{2} \ln 2\pi + \frac{1}{2},$$

and for any positive integer  $p$ ,

$$\sum_{n \geq 1}^{\mathcal{R}} n^p H_n = \left( \frac{1 - B_{p+1}}{p+1} \right) \gamma - \frac{1}{2} \ln 2\pi + \sum_{j=1}^p (-1)^j \binom{p}{j} \zeta'(-j) + R_p \quad \text{with } R_p \in \mathbb{Q},$$

where the symbol  $\sum^{\mathcal{R}}$  denotes the sum of the series in the sense of the Ramanujan summation of divergent series [7, 8, 9, 10]. For  $p = 2k$  (with  $k \geq 1$ ), we have

$B_{p+1} = 0$  and  $R_p = C_p - \frac{B_p}{2p} + \frac{B_p}{2}$ , then, in view of formula (2), these relations may be translated into the following identities:

$$\sum_{n \geq 1}^{\mathcal{R}} H_n = \nu_1 + \gamma - \frac{1}{2},$$

and for  $k \geq 1$ ,

$$\sum_{n \geq 1}^{\mathcal{R}} n^{2k} H_n = \nu_{2k} + \zeta'(-2k) + \frac{1-2k}{2} \zeta(1-2k) = \nu_{2k} + \zeta'(-2k) + (2k-1) \frac{B_{2k}}{4k}. \quad (8)$$

In particular, we have

$$\sum_{n \geq 1}^{\mathcal{R}} n^2 H_n = \nu_2 + \zeta'(-2) + \frac{B_2}{4} = \nu_2 - \frac{\zeta(3)}{4\pi^2} + \frac{1}{24}.$$

## 2 The case $k = -1$

The case  $k = -1$  behaves differently from the previous case and must be studied separately. We recall the identities [11, p. 142]

$$\nu_{-1} = \int_0^1 \frac{\psi(x+1) + \gamma}{x} dx = \sum_{m=1}^{\infty} \frac{\ln(m+1)}{m(m+1)} = - \sum_{n=2}^{\infty} \zeta'(n) = 1.2577468869 \dots$$

Another interesting representation (communicated by I. V. Blagouchine) is

$$\nu_{-1} = \frac{1}{2} \int_{-\infty}^{+\infty} \frac{\zeta(3/2 + ix)}{(1/2 + ix) \cosh(\pi x)} dx.$$

Moreover, we can write yet another relation which will be useful in the next section: let  $\kappa_1$  be the constant

$$\kappa_1 := \sum_{n=1}^{\infty} \frac{|b_n|}{n^2} = 0.5290529699 \dots,$$

where the rational numbers  $b_n$  are the *Bernoulli numbers of the second kind* defined by means of their generating function

$$\frac{x}{\ln(x+1)} = 1 + \sum_{n=1}^{\infty} b_n x^n \quad (|x| < 1).$$

These numbers  $b_n$  were introduced and studied by Jordan [15, p. 265]. Note that several authors quoted here use different notations:  $b_n$  are denoted by  $G_n$  (and



called *Gregory coefficients*) in [5, 6, 7]. The coefficients  $n!b_n$  are sometimes called *Cauchy numbers*. The constants  $\kappa_1$  and  $\nu_{-1}$  are linked by the relation [8, Eq. (3.23) p. 105]

$$\kappa_1 + \frac{1}{2}\zeta(2) = \nu_{-1} + \gamma_1 + \frac{1}{2}\gamma^2 \quad (9)$$

where  $\gamma_1$  denotes the first Stieljes constant [3, 7, 8].

$$\gamma_1 = \lim_{n \rightarrow \infty} \left\{ \sum_{j=1}^n \frac{\ln j}{j} - \frac{1}{2} \ln^2 n \right\} = -0.07281584548 \dots$$

In terms of the Ramanujan summation,  $\gamma_1$  is  $\sum_{n \geq 1}^{\mathcal{R}} \frac{\ln n}{n}$  (cf. [8, p. 67]), whereas  $\kappa_1$  is  $\sum_{n \geq 1}^{\mathcal{R}} \frac{H_n}{n}$  (cf. [8, Eq. (4.29) p. 133]).

### 3 Alternating series involving multiple zeta values

In this section, we consider a more general class of series of the previous type replacing zeta values with certain multiple zeta values. We prove our formula (3) and deduce some interesting consequences.

**Proposition 2.** For all integers  $p \geq 0$  and  $k \geq -1$ , let

$$\nu_{k,p} := \sum_{n=2}^{\infty} \frac{(-1)^n}{n+k} \zeta(n, \underbrace{1, \dots, 1}_p);$$

then

$$\nu_{k,p} = \sum_{n=1}^{\infty} \frac{|G_n^{(k+1)}|}{n^{p+1}},$$

where the rational numbers  $G_n^{(k)}$  are defined by equation (4).

**Corollary 1.** In particular, for  $p = 0$ , we have

$$\nu_{k-1,0} = \nu_{k-1} = \sum_{n=1}^{\infty} \frac{|G_n^{(k)}|}{n} \quad (k \geq 0);$$

and since  $G_n^{(1)} = b_n$ , for  $k = 0$ , we have also

$$\nu_{0,p} = \kappa_p := \sum_{n=1}^{\infty} \frac{|b_n|}{n^{p+1}} \quad (p \geq 0).$$

In order to prove Proposition 2, we begin by stating the following lemmas:

**Lemma 1.** For all integers  $j \geq 1$  and  $p \geq 0$ , we have

$$\int_0^1 \frac{\ln^j(1-x) \ln^p(x)}{x} dx = (-1)^{j+p} j! p! \zeta(j+1, \underbrace{1, \dots, 1}_p). \quad (10)$$

*Proof.* This follows directly from [18, Eqs. (2.27), (2.28)].  $\square$

**Lemma 2.** The Stirling numbers of the first kind  $s(n, j)$  with fixed  $j \geq 1$  admit the (vertical) exponential generating function [1, Eq. (2.8)]

$$\frac{\ln^j(1+x)}{j!} = \sum_{n=j}^{\infty} s(n, j) \frac{x^n}{n!} \quad (|x| < 1). \quad (11)$$

**Lemma 3.** For all integers  $n \geq 1$  and  $p \geq 0$ , we have

$$(-1)^p \int_0^1 x^{n-1} \ln^p(x) dx = \frac{p!}{n^{p+1}} \quad (12)$$

*Proof.* This is nothing else than [7, Eq. (41)] in the case where  $p$  is an integer.  $\square$

**Lemma 4.** For all integers  $n \geq 1$  and  $k \geq 0$ , we have

$$G_n^{(k)} = \frac{(-1)^{n+1}}{n!} \int_0^1 x^k (1-x)_{n-1} dx,$$

where  $(z)_n = z(z+1)(z+2) \cdots (z+n-1)$  is the Pochhammer symbol. In particular, this implies that

$$G_n^{(k)} = (-1)^{n+1} |G_n^{(k)}|. \quad (13)$$

*Proof.* Integration between 0 and 1 of the expansion

$$x^{k-1} x(x-1) \cdots (x-n+1) = \sum_{j=1}^n s(n, j) x^{j+k-1}$$

gives the required result.  $\square$

*Proof of Proposition 2.* Using successively formulas (10)–(13) above, we can write

the following equalities:

$$\begin{aligned}
\nu_{k,p} &= \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j+k+1} \zeta(j+1, \underbrace{1, \dots, 1}_p) \\
&= \frac{(-1)^{p+1}}{p!} \sum_{j=1}^{\infty} \frac{1}{j+k+1} \int_0^1 \frac{\ln^j(1-x) \ln^p(x)}{j! x} dx \\
&= \frac{(-1)^{p+1}}{p!} \sum_{j=1}^{\infty} \frac{1}{j+k+1} \int_0^1 \left( \sum_{n=j}^{\infty} (-1)^n s(n, j) \frac{x^n}{n!} \right) \frac{\ln^p(x)}{x} dx \\
&= \frac{(-1)^{p+1}}{p!} \sum_{j=1}^{\infty} \frac{1}{j+k+1} \sum_{n=j}^{\infty} (-1)^n \frac{s(n, j)}{n!} \int_0^1 x^{n-1} \ln^p(x) dx \\
&= - \sum_{j=1}^{\infty} \frac{1}{j+k+1} \sum_{n=j}^{\infty} (-1)^n \frac{s(n, j)}{n! n^{p+1}} \\
&= - \sum_{n=1}^{\infty} \frac{(-1)^n}{n! n^{p+1}} \sum_{j=1}^n \frac{s(n, j)}{j+k+1} \\
&= \sum_{n=1}^{\infty} (-1)^{n+1} \left( \frac{1}{n!} \sum_{j=1}^n \frac{s(n, j)}{j+k+1} \right) \frac{1}{n^{p+1}} \\
&= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{G_n^{(k+1)}}{n^{p+1}} = \sum_{n=1}^{\infty} \frac{|G_n^{(k+1)}|}{n^{p+1}}.
\end{aligned}$$

This completes the proof.  $\square$

**Example 2.** For the first values of  $k \geq -1$ , we have the following expansions in series containing only positive rational terms:

$$\begin{aligned}
\nu_{-1} &= \sum_{n=1}^{\infty} \frac{|G_n^{(0)}|}{n} = 1 + \frac{1}{8} + \frac{5}{108} + \frac{3}{128} + \frac{251}{18000} + \frac{95}{10368} + \dots, \\
\nu_0 &= \sum_{n=1}^{\infty} \frac{|G_n^{(1)}|}{n} = \frac{1}{2} + \frac{1}{24} + \frac{1}{72} + \frac{19}{2880} + \frac{3}{800} + \frac{863}{362880} + \dots, \\
\nu_1 &= \sum_{n=1}^{\infty} \frac{|G_n^{(2)}|}{n} = \frac{1}{3} + \frac{1}{48} + \frac{7}{1080} + \frac{17}{5760} + \frac{41}{25200} + \frac{731}{725760} + \dots, \\
\nu_2 &= \sum_{n=1}^{\infty} \frac{|G_n^{(3)}|}{n} = \frac{1}{4} + \frac{1}{80} + \frac{1}{270} + \frac{11}{6720} + \frac{89}{100800} + \frac{5849}{10886400} + \dots, \\
\nu_3 &= \sum_{n=1}^{\infty} \frac{|G_n^{(4)}|}{n} = \frac{1}{5} + \frac{1}{120} + \frac{1}{420} + \frac{83}{80640} + \frac{59}{108000} + \frac{397}{1209600} + \dots.
\end{aligned}$$

**Example 3.** Let  $\zeta_H$  be the Apostol-Vu harmonic zeta function [2, 3, 4, 10] defined for  $\text{Re}(s) > 1$  by

$$\zeta_H(s) := \sum_{n=1}^{\infty} \frac{H_n}{n^s}.$$

We recall that  $\zeta_H$  is analytic in the half-plane  $\text{Re}(s) > 1$  and can be extended meromorphically with poles at the integers  $1, 0, -1, -3, -5, \dots$ . The special values at negative even integers are  $\zeta_H(-2k) = B_{2k}/2 - B_{2k}/4k$ . The special values at positive integers are also well-known: the first values are

$$\zeta_H(2) = 2\zeta(3), \quad \zeta_H(3) = \frac{5}{4}\zeta(4),$$

and more generally, they may be computed by means of the following beautiful formula (first discovered by Euler [14] and several times rediscovered afterwards):

$$2\zeta_H(n) = (n+2)\zeta(n+1) - \sum_{r=1}^{n-2} \zeta(r+1)\zeta(n-r) \quad (n \geq 3).$$

Otherwise, by Proposition 2 above, we can write

$$\sum_{n=2}^{\infty} (-1)^n \frac{\zeta_H(n)}{n} = \nu_{0,1} - \nu_{-1} + \zeta(2) = \kappa_1 - \nu_{-1} + \zeta(2),$$

and thus, from equation (9), we derive the following elegant evaluation:

$$\sum_{n=2}^{\infty} (-1)^n \frac{\zeta_H(n)}{n} = \gamma_1 + \frac{1}{2}\gamma^2 + \frac{\pi^2}{12} = 0.916240149\dots \quad (14)$$

Another expression of this constant is  $\zeta''(0) + \frac{1}{2}\ln^2(2\pi) + \frac{\pi^2}{8}$  [3, Eq. (25.6.12)].

**Remark 3** (Link with the Ramanujan summation: part II). For  $s \in \mathbb{C}$ , let  $\zeta_H^{\mathcal{R}}$  be the function  $s \mapsto \sum_{n \geq 1}^{\mathcal{R}} H_n n^{-s}$  where  $\sum^{\mathcal{R}}$  stands for the Ramanujan summation. The function  $\zeta_H^{\mathcal{R}}$  is an entire function linked to the harmonic zeta function  $\zeta_H$  by the relation [10, Eq. (84)]

$$\zeta_H^{\mathcal{R}}(s) = \zeta_H(s) - \int_1^{\infty} x^{-s} (\psi(x+1) + \gamma) dx \quad \text{for } \text{Re}(s) > 1.$$

We have the identities

$$\zeta_H^{\mathcal{R}}(1) = \nu_{0,1} = \kappa_1, \quad \zeta_H^{\mathcal{R}}(0) = \nu_1 + \gamma - \frac{1}{2},$$

and formula (8) may be nicely rewritten

$$\zeta_H^{\mathcal{R}}(-2k) = \zeta_H(-2k) + \zeta'(-2k) + \nu_{2k}.$$

## 4 Link with the shifted Mascheroni series

Let us consider now the forward shifted Mascheroni series which are defined by

$$\sigma_r := \sum_{n=1}^{\infty} \frac{|b_{n+r}|}{n}, \quad \text{for } r = 0, 1, 2, \dots.$$

We have in particular  $\sigma_0 = \nu_0 = \gamma$ . The study of these series  $\sigma_r$  was the main subject of [12]. Among other things, we have established the following decomposition of  $\zeta'(-j)$  on the “basis” of  $\sigma_r$  [12, Proposition 3]:

$$\zeta'(-j) = \sum_{r=2}^{j+1} (-1)^{j-r} (r-1)! S(j, r-1) \sigma_r - \frac{B_{j+1}}{j+1} \gamma - \frac{B_{j+1}}{(j+1)^2}, \quad \text{for } j = 1, 2, 3, \dots,$$

where  $S(j, r)$  are Stirling numbers of the second kind; moreover, for  $j = 0$ , we have also a similar relation:

$$\frac{1}{2} \ln 2\pi = -\zeta'(0) = \sigma_1 + \frac{\gamma}{2} + \frac{1}{2}.$$

Then, substituting these relations into (2) enables us to write each series  $\nu_k$  with  $k \geq 1$  as an integral linear combination of  $\gamma, \sigma_1, \sigma_2, \dots, \sigma_k$  plus a rational number  $D_k$  which is closely linked to  $C_k$ . In this combination, the coefficient of  $\gamma$  is zero since it is equal to  $\frac{1}{k+1} \sum_{j=0}^k \binom{k+1}{j} B_j$  which vanishes by a well-known property of the Bernoulli numbers. Finally, equation (2) may be nicely rewritten in terms of  $\sigma_r$  as follows:

**Proposition 3.** For all integers  $k \geq 1$ , we have the relation

$$\nu_k = D_k - \sigma_1 + \sum_{r=2}^k (-1)^r (r-1)! \left( \sum_{j=r-1}^{k-1} \binom{k}{j} S(j, r-1) \right) \sigma_r \quad (15)$$

with

$$D_k = \frac{1}{k} + \sum_{j=1}^k \binom{k}{j} \frac{B_j H_j}{k+1-j}.$$

**Example 4.** For the first values of  $k$ , we have the following relations:

$$\begin{aligned}\nu_1 &= \frac{1}{2} - \sigma_1, \\ \nu_2 &= \frac{1}{4} - \sigma_1 + 2\sigma_2, \\ \nu_3 &= \frac{5}{24} - \sigma_1 + 6\sigma_2 - 6\sigma_3, \\ \nu_4 &= \frac{13}{72} - \sigma_1 + 14\sigma_2 - 36\sigma_3 + 24\sigma_4, \\ \nu_5 &= \frac{109}{720} - \sigma_1 + 30\sigma_2 - 150\sigma_3 + 240\sigma_4 - 120\sigma_5, \\ \nu_6 &= \frac{23}{180} - \sigma_1 + 62\sigma_2 - 420\sigma_3 + 1560\sigma_4 - 1800\sigma_5 + 720\sigma_6.\end{aligned}$$

## Acknowledgments

The author gratefully acknowledges the referee for his careful reading, constructive comments and helpful suggestions.

## References

- [1] T. Agoh and K. Dilcher, Recurrence relations for Nörlund numbers and Bernoulli numbers of the second kind, *Fibonacci Quarterly* **48** (2010), 4–12.
- [2] E. Alkan, Approximation by special values of harmonic zeta function and log-sine integrals, *Commun. Number Theory Phys.* **7** (2013), 515–550.
- [3] T. M. Apostol, *Zeta and Related Functions*, NIST Handbook of Mathematical Functions, Chapter 25, Cambridge University Press, 2010.
- [4] T. M. Apostol and T. H. Vu, Dirichlet series related to the Riemann zeta function, *J. Number Theory* **19** (1984), 85–102.
- [5] I. V. Blagouchine, Two series expansions for the logarithm of the gamma function involving Stirling numbers and containing only rational coefficients for certain arguments related to  $\pi^{-1}$ , *J. Math. Anal. Appl.* **442** (2016), 404–434.

- [6] I. V. Blagouchine, Three notes on Ser's and Hasse's representations for the zeta-functions, *Integers* **18A** (2018), 1–45.
- [7] I. V. Blagouchine and M-A. Coppo, A note on some constants related to the zeta function, *Ramanujan J.* **47** (2018), 457–473.
- [8] B. Candelpergher, *Ramanujan Summation of Divergent Series*, Lecture Notes in Math. Series, vol. 2185, Springer, 2017.
- [9] B. Candelpergher and M-A. Coppo, A new class of identities involving Cauchy numbers, harmonic numbers and zeta values, *Ramanujan J.* **27** (2012), 305–328.
- [10] B. Candelpergher, H. Gadiyar, and R. Padma, Ramanujan summation and the exponential generating function  $\sum_{k=0}^{\infty} \frac{z^k}{k!} \zeta'(-k)$ , *Ramanujan J.* **21** (2010), 99–122.
- [11] H. Cohen, *Number Theory, Volume II: Analytic and Modern Tools*, Graduate Texts in Math., vol. 240, Springer, 2007.
- [12] M-A. Coppo and P. T. Young, Shifted Mascheroni series and hyperharmonic numbers, *J. Number Theory* **169** (2016), 1–20.
- [13] L. Euler, De progressionibus harmonicis observationes (1734), Eneström-Number E43. *Opera Omnia*, Series 1, vol. 14, pp. 87–100.
- [14] L. Euler, Meditationes circa singulare serierum genus (1775), Eneström-Number E477. *Opera Omnia*, Series 1, vol. 15, pp. 217–267.
- [15] C. Jordan, *Calculus of Finite Differencies*, Chelsea, New York, 1965.
- [16] H. M. Srivastava, Sums of certain series of the Riemann zeta function, *J. Math. Anal. Appl.* **134** (1988), 129–140.
- [17] R. J. Singh and V. P. Verma, Some series involving Riemann zeta function, *Yokohama Math. J.* **31** (1983), 1–4.
- [18] C. Xu, Multiple zeta values and Euler sums, *J. Number Theory* **177** (2017), 443–478.