On certain alternating series involving zeta and multiple zeta values

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Abstract

In this article, we consider various generalizations of Euler's famous relation

$$\gamma = \sum_{n \geq 2} (-1)^n \frac{\zeta(n)}{n}$$

linking Euler's constant γ to special values of the Riemann zeta function at positive integers. Among other things, we highlight the existence of a very similar relation for the Apostol-Vu harmonic zeta function which have never been noticed before.

Keywords: Euler's constant, first Stieltjes constant, Riemann zeta function, harmonic zeta function, Stirling numbers, Bernoulli numbers, Bernoulli numbers of the second kind, harmonic numbers, multiple zeta values.

Introduction

This article is primarily devoted to the study of alternating series ν_k defined by

$$\nu_k := \sum_{n=2}^{\infty} (-1)^n \frac{\zeta(n)}{n+k} \, ,$$

where k denotes an integral parameter. By a classical result (cf. [11] p. 62, [12] p. 532), one knows that ν_0 is Euler's constant

$$\gamma = \lim_{n \to \infty} \left\{ \sum_{j=1}^{n} \frac{1}{j} - \ln n \right\} = 0.5772156649 \ldots$$

This remarkable connection between γ and the special values at positive integers of the Riemann zeta function goes back to Euler's early works on harmonic series[1].

[1] *De progressionibus harmonicis observationes* (1734), Eneström-Number E43.

Less famous but yet fairly well-known (cf. [8] p. 93, [13] Eq. (5.1), [14] Eq. (1.5)) is the following relation:

$$\nu_1 = \frac{\gamma}{2} - \frac{1}{2}\ln 2\pi + 1\,.$$

Recently, Blagouchine ([5]) has given this general expression of the series ν_k in the case where k is a positive integer:

$$\begin{aligned}\nu_k &= \frac{\gamma}{2} - \frac{\ln 2\pi}{k+1} + \frac{1}{k} \\ &+ \sum_{r=1}^{[\frac{k}{2}]}(-1)^r\binom{k}{2r-1}\frac{(2r)!}{r(2\pi)^{2r}}\zeta'(2r) + \sum_{r=1}^{[\frac{k+1}{2}]-1}(-1)^r\binom{k}{2r}\frac{(2r)!}{2(2\pi)^{2r}}\zeta(2r+1)\,. \quad (1)\end{aligned}$$

Though this formula seems quite cumbersome, it may be highly simplified using the functional equation of ζ. After some transformations, we show (see Proposition 1) that formula (1) can be reduced to the following equivalent (but much more pleasant) expression:

$$\nu_k = \frac{\gamma}{k+1} - \frac{1}{2}\ln 2\pi + \sum_{j=1}^{k-1}(-1)^j\binom{k}{j}\zeta'(-j) + C_k\,, \quad (2)$$

where C_k is a rational constant whose explicit expression involves both the Bernoulli numbers and the harmonic numbers (cf. formula (6)).

Next, we introduce some generalizations of these series involving certain multiple zeta values. A natural extension of the series ν_k may be defined as follows: for all integers $k \geq -1$ and $p \geq 0$, we consider the series

$$\nu_{k,p} := \sum_{n=2}^{\infty}\frac{(-1)^n}{n+k}\zeta(n,\underbrace{1,\ldots,1}_{p})\,,$$

where

$$\zeta(s_1,s_2,\cdots,s_k) = \sum_{n_1>n_2>\cdots>n_k\geq 1}\frac{1}{n_1^{s_1}n_2^{s_2}\cdots n_k^{s_k}}\,,$$

in such a way that our series ν_k are nothing else than $\nu_{k,0}$. Then, we establish (see Proposition 2) the following remarkable identity:

$$\nu_{k,p} = \sum_{n=1}^{\infty}\frac{|G_n^{(k+1)}|}{n^{p+1}}\,, \quad (3)$$

where $G_n^{(k)}$ denotes the *Gregory coefficients of higher order* recently introduced in [6]. They are defined by

$$G_n^{(k)} := \frac{1}{n!}\sum_{j=1}^{n}\frac{S_1(n,j)}{j+k} \qquad (k \geq 0, n \geq 1), \quad (4)$$

where $S_1(n,j)$ are the Stirling numbers of the first kind[2]. One can prove easily that $G_n^{(k)} = (-1)^{n+1}|G_n^{(k)}|$, *i.e.* the rationals $G_n^{(k)}$ alternate in sign. As a special case of formula (3) above, we derive the following result:

$$\nu_{k-1} = \sum_{n=1}^{\infty} \frac{|G_n^{(k)}|}{n} \qquad (k \geq 0), \tag{5}$$

which significantly generalizes the Mascheroni series for γ (this classical series (cf. [11] p. 280) is nothing else than formula (5) with $k = 1$).

Another notable consequence of our study is the highlighting of this elegant formula (see Example 3):

$$\sum_{n=2}^{\infty} (-1)^n \frac{\zeta_H(n)}{n} = \gamma_1 + \frac{1}{2}\gamma^2 + \frac{\pi^2}{12},$$

where $\zeta_H(s)$ denotes the Apostol-Vu harmonic zeta function (cf. [2], [3], [4]) and γ_1 is the first Stieltjes constant (cf. [7], [8]).

Finally, in the last section, we highlight the existence of an interesting relation between the series ν_k, the Stirling numbers of the second kind $S_2(n,k)$, and the shifted Mascheroni series σ_r whose study was the main subject of [10] (see Proposition 3 and Example 3).

1 The case of a positive integer

In this section, we focus on the case of a positive integer k and give two independant proofs of our formula (2). More precisely, we prove the following proposition:

Proposition 1. For any positive integer k, then

$$\nu_k = \frac{\gamma}{k+1} - \frac{1}{2}\ln 2\pi + \sum_{j=1}^{k-1} (-1)^j \binom{k}{j} \zeta'(-j) + C_k$$

with

$$C_k = \frac{1}{k} + \sum_{r=1}^{[\frac{k}{2}]} \binom{k}{2r} \frac{B_{2r}\, H_{2r-1}}{k+1-2r} \tag{6}$$

where $H_k = 1 + \frac{1}{2} + \cdots + \frac{1}{k}$ is the kth harmonic number and the B_{2r} are the Bernoulli numbers defined by their exponential generating function

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!} \qquad (|z| < 2\pi).$$

In particular, $B_0 = 1$, $B_1 = -1/2$, $B_{2r+1} = 0$ for $r \geq 1$.

[2]Comprehensive informations on the Stirling numbers of the first and the second kind may be found in [1], [5], [10], [11], [15].

Proof. We can quite easily derive (2) from formula (1). Differentiation of the functional equation

$$\zeta(s) = 2(2\pi)^{s-1}\Gamma(1-s)\zeta(1-s)\sin\frac{\pi s}{2}\,,$$

enables us to write the two relations:

$$(-1)^r\frac{(2r)!}{2(2\pi)^{2r}}\zeta(2r+1) = \zeta'(-2r) \qquad (r \geq 1),$$

and

$$(-1)^r\frac{(2r)!}{r(2\pi)^{2r}}\zeta'(2r) = -\zeta'(1-2r) + \frac{B_{2r}}{2r}\left(H_{2r-1} - \gamma - \ln 2\pi\right) \qquad (r \geq 1)\,.$$

Substituting these relations in (1) and grouping together the terms under the two symbols Σ, leads to the following expression:

$$\nu_k = \frac{\gamma}{k+1} - \frac{1}{2}\ln 2\pi + \sum_{j=1}^{k-1}(-1)^j\binom{k}{j}\zeta'(-j) + C_k$$

with

$$C_k = \frac{1}{k} + \sum_{r=1}^{[\frac{k}{2}]}\binom{k}{2r}\frac{B_{2r}\,H_{2r-1}}{k+1-2r}\,.$$

Another alternative proof of formula (2), independant from (1), may be deduced from the expansion in powers of z of the following relation given in [8] p. 93:

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{(-1)^k z^k}{k!}\sum_{j=1}^{\infty}\frac{(-1)^{j-1}}{j}\zeta^{\mathcal{R}}(j-k) &= (1-e^z)\sum_{k=0}^{\infty}\frac{(-1)^k z^k}{k!}\zeta'(-k)\\ &+ (1-e^z)\sum_{k=0}^{\infty}\frac{(-1)^k z^k}{k!}\frac{1}{(k+1)^2}\\ &+ \int_0^1 \ln(t+1)e^{-zt}\,dt\,,\end{aligned}$$

with

$$\zeta^{\mathcal{R}}(j-k) = \begin{cases}\gamma & \text{if } j = k+1\\ \zeta(j-k) - \frac{1}{j-k-1} & \text{otherwise.}\end{cases}$$

Rewriting the series ν_k under the following form:

$$\nu_k = \sum_{j=k+2}^{\infty}\frac{(-1)^{j-k}}{j}\zeta(j-k)\,,$$

and using the well-known relations (cf. [3]) :

$$\zeta(0) = -\frac{1}{2}\,, \quad \zeta(1-2r) = -\frac{B_{2r}}{2r}\,, \quad \zeta'(0) = -\frac{1}{2}\ln 2\pi\,,$$

a careful identification of the terms in z^k in the previous development leads again to (2) and provides in addition an alternative expression of the constant C_k[3]:

$$C_k = \frac{H_k}{k+1} + \frac{1}{2k} - \sum_{r=1}^{[\frac{k}{2}]} \frac{B_{2r}}{2r(k+1-2r)}\,.$$

□

Example 1. For the first values of k, we have the following relations:

$$\begin{aligned}
\nu_1 &= \frac{\gamma}{2} - \frac{1}{2}\ln 2\pi + 1\,,\\
\nu_2 &= \frac{\gamma}{3} - \frac{1}{2}\ln 2\pi - 2\zeta'(-1) + \frac{2}{3}\,,\\
\nu_3 &= \frac{\gamma}{4} - \frac{1}{2}\ln 2\pi - 3\zeta'(-1) + 3\zeta'(-2) + \frac{7}{12}\,,\\
\nu_4 &= \frac{\gamma}{5} - \frac{1}{2}\ln 2\pi - 4\zeta'(-1) + 6\zeta'(-2) - 4\zeta'(-3) + \frac{47}{90}\,,\\
\nu_5 &= \frac{\gamma}{6} - \frac{1}{2}\ln 2\pi - 5\zeta'(-1) + 10\zeta'(-2) - 10\zeta'(-3) + 5\zeta'(-4) + \frac{167}{360}\,,\\
\nu_6 &= \frac{\gamma}{7} - \frac{1}{2}\ln 2\pi - 6\zeta'(-1) + 15\zeta'(-2) - 20\zeta'(-3) + 15\zeta'(-4) - 6\zeta'(-5) + \frac{349}{840}\,.
\end{aligned}$$

2 The case $k = -1$

The case $k = -1$ behaves differently from the previous case and must be studied by other tools. We recall the identities (cf. [8] p. 105, [9] p. 142):

$$\nu_{-1} = \int_0^1 \frac{\psi(x+1)+\gamma}{x}\,dx = \sum_{m=1}^{\infty} \frac{\ln(m+1)}{m(m+1)} = -\sum_{n=2}^{\infty} \zeta'(n) = 1,2577468869\ldots\,,$$

[3] An unexpected consequence is the equation

$$\sum_{r=1}^{[\frac{k}{2}]} \binom{k}{2r} \frac{B_{2r}\,H_{2r-1}}{k+1-2r} = \frac{H_k}{k+1} - \frac{1}{2k} - \sum_{r=1}^{[\frac{k}{2}]} \frac{B_{2r}}{2r(k+1-2r)}$$

whose direct proof does not seem obvious.

where ψ is the digamma function[4]. Moreover, we can also write another interesting relation which will be useful in the next section: let κ_1 be the constant

$$\kappa_1 := \sum_{n=1}^{\infty} \frac{|b_n|}{n^2} = 0,5290529699\ldots,$$

where the rational numbers b_n are the *Bernoulli numbers of the second kind*[5] defined by their generating function

$$\frac{x}{\ln(x+1)} = 1 + \sum_{n=1}^{\infty} b_n\, x^n \qquad (|x| < 1);$$

we recall the relation (cf. [7] Eq. (37), [8] Eq. (3.23) p. 105):

$$\kappa_1 + \frac{1}{2}\zeta(2) = \gamma_1 + \frac{1}{2}\gamma^2 + \nu_{-1}\,, \tag{7}$$

where γ_1 denotes the first Stieljes constant (cf. [7], [8]):

$$\gamma_1 = \lim_{n\to\infty} \left\{ \sum_{j=1}^{n} \frac{\ln j}{j} - \frac{1}{2}\ln^2 n \right\} = -0.07281584548\ldots$$

3 Alternating series involving multiple zeta values

In this section, we consider more general series of the same type involving certain multiple zeta values, we prove our formula (3) and draw some interesting consequences.

Proposition 2. For all integers $p \geq 0$ and $k \geq -1$, let

$$\nu_{k,p} := \sum_{n=2}^{\infty} \frac{(-1)^n}{n+k}\zeta(n, \underbrace{1, \ldots, 1}_{p})\,;$$

[4] Another interesting identity (communicated by I. V. Blagouchine) is

$$\nu_{-1} = \frac{1}{2}\int_{-\infty}^{+\infty} \frac{\zeta(3/2 + ix)}{(1/2 + ix)\cosh(\pi x)}\, dx.$$

[5] These numbers were introduced and studied by Jordan ([11] p. 265 ff.); see also [1], [10], [16]. Note that several authors quoted in reference use different notations: numbers b_n are noted G_n (and called *Gregory coefficients*) in [5], [6], [7], they are noted $\beta_n/n!$ in [8], and $b_n/n!$ in [9].

then

$$\nu_{k,p} = \sum_{n=1}^{\infty} \frac{|G_n^{(k+1)}|}{n^{p+1}}\,,$$

where $G_n^{(k)}$ are the Gregory coefficients of higher order defined by Eq. (4). In particular, we have

$$\nu_{k-1,0} = \nu_{k-1} = \sum_{n=1}^{\infty} \frac{|G_n^{(k)}|}{n} \qquad (k \geq 0);$$

since $G_n^{(1)} = b_n$, we have also

$$\nu_{0,p} = \kappa_p := \sum_{n=1}^{\infty} \frac{|b_n|}{n^{p+1}} \qquad (p \geq 0)\,.$$

Proof. In order to prove our Proposition 2, we use the following lemmas:

Lemma 1. For all integers $n \geq 1$ and $k \geq 0$, then

$$G_n^{(k)} = \frac{(-1)^{n+1}}{n!} \int_0^1 x^k (1-x)_{n-1}\, dx\,, \tag{8}$$

where $(z)_n = z(z+1)(z+2)\cdots(z+n-1)$ is the Pochhammer symbol. In particular, this implies that

$$G_n^{(k)} = (-1)^{n+1} |G_n^{(k)}|\,.$$

Proof. Integration between 0 and 1 of the development

$$x^{k-1} x(x-1)\cdots(x-n+1) = \sum_{j=1}^{n} S_1(n,j) x^{j+k-1}$$

gives the required result. □

Lemma 2 ([7] Eq. (41))**.** For all integers $n \geq 1$ and $p \geq 0$, then

$$(-1)^p \int_0^1 x^{n-1} \ln^p(x)\, dx = \frac{p!}{n^{p+1}} \tag{9}$$

Lemma 3 ([15] Eq. (2.27) and (2.28))**.** For all integers $j \geq 1$ and $p \geq 0$, then

$$\int_0^1 \frac{\ln^j(1-x)\, \ln^p(x)}{x}\, dx = (-1)^{j+p}\, j!\, p!\, \zeta(j+1, \underbrace{1,\ldots,1}_{p})\,. \tag{10}$$

This enable us to write the following equalities:

$$
\begin{aligned}
\sum_{n=2}^{\infty} \frac{(-1)^n}{n+k} \zeta(n, \underbrace{1, \ldots, 1}_{p}) &= \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j+k+1} \zeta(j+1, \underbrace{1, \ldots, 1}_{p}) \\
&= \frac{(-1)^{p+1}}{p!} \sum_{j=1}^{\infty} \frac{1}{j+k+1} \int_0^1 \frac{\ln^j(1-x)}{j!} \frac{\ln^p(x)}{x} \, dx \\
&= \frac{(-1)^{p+1}}{p!} \sum_{j=1}^{\infty} \frac{1}{j+k+1} \int_0^1 \left(\sum_{n=j}^{\infty} (-1)^n S_1(n,j) \frac{x^n}{n!} \right) \frac{\ln^p(x)}{x} \, dx \\
&= -\sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \sum_{j=1}^{n} \frac{S_1(n,j)}{j+k+1} \frac{(-1)^p}{p!} \int_0^1 x^{n-1} \ln^p(x) \, dx \\
&= \sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{1}{n!} \sum_{j=1}^{n} \frac{S_1(n,j)}{j+k+1} \right) \frac{1}{n^{p+1}} \\
&= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{G_n^{(k+1)}}{n^{p+1}} = \sum_{n=1}^{\infty} \frac{|G_n^{(k+1)}|}{n^{p+1}} \,.
\end{aligned}
$$

□

Example 2. For $-1 \leq k \leq 6$, we have the following expansions in series containing only positive rational terms:

$$
\begin{aligned}
\nu_{-1} &= 1 + \frac{1}{8} + \frac{5}{108} + \frac{3}{128} + \frac{251}{18000} + \frac{95}{10368} + \cdots \,, \\
\nu_0 &= \frac{1}{2} + \frac{1}{24} + \frac{1}{72} + \frac{19}{2880} + \frac{3}{800} + \frac{863}{362\,880} + \cdots \,, \\
\nu_1 &= \frac{1}{3} + \frac{1}{48} + \frac{7}{1080} + \frac{17}{5760} + \frac{41}{25\,200} + \frac{731}{725\,760} + \cdots \,, \\
\nu_2 &= \frac{1}{4} + \frac{1}{80} + \frac{1}{270} + \frac{11}{6720} + \frac{89}{100\,800} + \frac{5849}{10\,886\,400} + \cdots \,, \\
\nu_3 &= \frac{1}{5} + \frac{1}{120} + \frac{1}{420} + \frac{83}{80\,640} + \frac{59}{108\,000} + \frac{397}{1\,209\,600} + \cdots \,, \\
\nu_4 &= \frac{1}{6} + \frac{1}{168} + \frac{5}{3024} + \frac{17}{24\,192} + \frac{557}{1\,512\,000} + \frac{5249}{23\,950\,080} + \cdots \,, \\
\nu_5 &= \frac{1}{7} + \frac{1}{224} + \frac{11}{9072} + \frac{41}{80\,640} + \frac{439}{1\,663\,200} + \frac{311}{1\,995\,840} + \cdots \,, \\
\nu_6 &= \frac{1}{8} + \frac{1}{288} + \frac{1}{1080} + \frac{73}{190\,080} + \frac{47}{237\,600} + \frac{2581}{22\,239\,360} + \cdots \,,
\end{aligned}
$$

Example 3. Let ζ_H be the Apostol-Vu harmonic zeta function ([2], [3], [4], [8] p. 72) defined for $\mathrm{Re}(s) > 1$ by

$$\zeta_H(s) = \sum_{n\geq 1} \frac{H_n}{n^s}\,.$$

The special values at positive integers of this harmonic zeta function are well-known: the first values are

$$\zeta_H(2) = 2\zeta(3),\ \zeta_H(3) = \frac{5}{4}\zeta(4);$$

more generally, they may be computed by means of the beautiful formula (first discovered by Euler[6] and several times rediscovered afterwards):

$$2\zeta_H(n) = (n+2)\zeta(n+1) - \sum_{r=1}^{n-2} \zeta(r+1)\zeta(n-r) \qquad (n \geq 3).$$

Otherwise, by Proposition 2 above, we can write

$$\sum_{n=2}^{\infty}(-1)^n \frac{\zeta_H(n)}{n} = \nu_{0,1} - \nu_{-1} + \zeta(2) = \kappa_1 - \nu_{-1} + \zeta(2)\,,$$

and thus, using formula (7), we derive the following elegant evaluation:

$$\sum_{n=2}^{\infty}(-1)^n \frac{\zeta_H(n)}{n} = \gamma_1 + \frac{1}{2}\gamma^2 + \frac{\pi^2}{12} = 0,916240149\ldots^{7} \qquad (11)$$

4 Link with shifted Mascheroni series

Let us consider by now the forward shifted Mascheroni series which are defined by

$$\sigma_r := \sum_{n=1}^{\infty} \frac{|b_{n+r}|}{n} \qquad (r = 0, 1, 2, \cdots)\,.$$

The study of these series was the main subject of [10]. Among other things, we have established the following decomposition of $\zeta'(-j)$ on the "basis" of σ_r (cf. [10], Proposition 3):

$$\zeta'(-j) = \sum_{r=2}^{j+1}(-1)^{j-r}(r-1)!S_2(j, r-1)\sigma_r - \frac{B_{j+1}}{j+1}\gamma - \frac{B_{j+1}}{(j+1)^2} \qquad (j = 1, 2, \cdots),$$

[6] *Meditationes circa singulare serierum genus* (1775), Eneström-Number E477.

[7] Another alternative expression of this constant is $\zeta''(0)+\frac{1}{2}\ln^2(2\pi)+\frac{\pi^2}{8}$ (cf. [3] Eq. (25.6.12)).

where $S_2(j,r)$ are Stirling numbers of the second kind; moreover we have the relation ([16], Corollary 9):

$$\frac{1}{2}\ln(2\pi) = \sigma_1 + \frac{\gamma}{2} + \frac{1}{2}\,.$$

Then, substituting these relations in (2) enables us to write each series ν_k as an integral linear combination of $\sigma_0 = \gamma$, $\sigma_1, \sigma_2, \cdots, \sigma_k$ plus a rational constant D_k which is closely linked to C_k. In this combination, the coefficient of γ is zero since it is equal to $\frac{1}{k+1}\sum_{j=0}^{k}\binom{k+1}{j}B_j$ which vanishes by a well-known property of the Bernoulli numbers. Finally, formula (2) may be nicely rewritten in terms of σ_r as follows:

Proposition 3. For each integer $k \geq 1$,

$$\nu_k = D_k + \sum_{r=1}^{k}(-1)^r(r-1)!\left(\sum_{j=r-1}^{k-1}\binom{k}{j}S_2(j,r-1)\right)\sigma_r \tag{12}$$

with

$$D_k = C_k - \frac{1}{2} + \sum_{r=1}^{[\frac{k}{2}]}\binom{k}{2r}\frac{B_{2r}}{2r(k+1-2r)} = \frac{1}{k} - \frac{1}{2} + \sum_{r=1}^{[\frac{k}{2}]}\binom{k}{2r}\frac{B_{2r}H_{2r}}{k+1-2r}\,.$$

Example 4. For the first values of k, we have the following relations:

$$\begin{aligned}
\nu_1 &= \frac{1}{2} - \sigma_1\,,\\
\nu_2 &= \frac{1}{4} - \sigma_1 + 2\sigma_2\,,\\
\nu_3 &= \frac{5}{24} - \sigma_1 + 6\sigma_2 - 6\sigma_3\,,\\
\nu_4 &= \frac{13}{72} - \sigma_1 + 14\sigma_2 - 36\sigma_3 + 24\sigma_4\,,\\
\nu_5 &= \frac{109}{720} - \sigma_1 + 30\sigma_2 - 150\sigma_3 + 240\sigma_4 - 120\sigma_5\,,\\
\nu_6 &= \frac{23}{180} - \sigma_1 + 62\sigma_2 - 420\sigma_3 + 1560\sigma_4 - 1800\sigma_5 + 720\sigma_6\,.
\end{aligned}$$

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