On certain alternating series involving zeta and multiple zeta values

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Introduction

This article is primarily devoted to the alternating series ν_k defined by

$$\nu_k := \sum_{j=2}^{\infty}(-1)^j \frac{\zeta(j)}{j+k}\,,$$

where k denotes a complex parameter. By a classical result (cf. e.g. [Er], p. 45, Eq. (3) or [Jo], p. 62) which goes back to Euler's early works on zeta, one knows that $\nu_0 = \gamma$, where γ denotes the Euler-Mascheroni constant. It is less famous but yet fairly well-known (cf. [Sr], p. 135, Eq. (5.1), [SV], Eq. (1.5), [Ca], p. 93) that

$$\nu_1 = \frac{\gamma}{2} - \frac{1}{2}\ln 2\pi + 1\,.$$

In a recent paper, Blagouchine ([B1]) has obtained the following general expression for ν_k in the case where k is a positive integer:

$$\begin{aligned}\nu_k = {} & \frac{\gamma}{2} - \frac{\ln 2\pi}{k+1} + \frac{1}{k} \\ & + \sum_{r=1}^{[\frac{k}{2}]}(-1)^r \binom{k}{2r-1}\frac{(2r)!}{r(2\pi)^{2r}}\zeta'(2r) + \sum_{r=1}^{[\frac{k+1}{2}]-1}(-1)^r \binom{k}{2r}\frac{(2r)!}{2(2\pi)^{2r}}\zeta(2r+1)\,. \qquad (1)\end{aligned}$$

However, this formula is quite cumbersome. Using the functional equation of zeta, we give an equivalent but simpler expression for this series in this case. We

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also examine the series ν_k for complex values of the parameter k, a study which, as far as we know, have never been made before. Furthermore, we study certain natural generalizations of these series involving multiple zeta values.

In the case where k is a positive integer, we show that the series ν_k admits the following explicit evaluation:

$$\nu_k = \frac{\gamma}{k+1} - \frac{1}{2}\ln 2\pi + \sum_{j=1}^{k-1}(-1)^j\binom{k}{j}\zeta'(-j) + C_k\,, \tag{2}$$

where C_k is a rational constant for which we give two different, even though equivalent, expressions (cf. Proposition 1).

A further interesting generalization of the series ν_k may be defined as follows: for any natural number p, we consider the series

$$\nu_{k,p} := \sum_{j=2}^{\infty}\frac{(-1)^j}{j+k}\zeta(j,\underbrace{1,\ldots,1}_{p})\,,$$

where

$$\zeta(s_1,s_2,\cdots,s_k) = \sum_{n_1>n_2>\cdots>n_k\geq 1}\frac{1}{n_1^{s_1}n_2^{s_2}\cdots n_k^{s_k}}\,,$$

in such a way that $\nu_k = \nu_{k,0}$. Then, for any integer $k \geq -1$, we show (cf. Proposition 2) the following identity:

$$\nu_{k,p} = \sum_{n=1}^{\infty}\frac{|G_n^{(k+1)}|}{n^{p+1}}\,. \tag{3}$$

Here, the numbers $G_n^{(k)}$ are the *Gregory coefficients of higher order* introduced in [B2]. They may be either defined by

$$G_n^{(k)} := \frac{1}{n!}\sum_{m=1}^{n}\frac{S_1(n,m)}{m+k}\,, \quad \text{for } k = 0,1,2,\cdots \tag{4}$$

where $S_1(n,m)$ are the Stirling numbers of the first kind, or equivalently by the integral formula

$$G_n^{(k)} = \frac{(-1)^n}{n!}\int_0^1 x^{k-1}(-x)_n\,dx\,, \tag{5}$$

where $(z)_n = z(z+1)(z+2)\cdots(z+n-1)$ is the Pochhammer symbol. It follows from this last expression that $G_n^{(k)} = (-1)^{n+1}|G_n^{(k)}|$. Furthermore, $G_n^{(1)} = G_n$, where G_n are the *Gregory coefficients*, also known as the *Bernoulli numbers of the second kind* [1].

[1]Several authors quoted in reference use different notations for these numbers, they are noted b_n in [Jo], [CY], and $\beta_n/n!$ in [Ca].

As a special case of the identity above, we deduce that

$$\nu_{k-1} = \sum_{n=1}^{\infty} \frac{|G_n^{(k)}|}{n}, \quad \text{for } k = 0, 1, 2, \cdots \tag{6}$$

and we note that this formula nicely generalizes the famous Mascheroni series for γ (which is nothing else than the case $k = 1$).

Finally, the extension to the complex case is studied in the last section. We provide two beautiful integral formulas for the same series:

$$\nu_k = \frac{1}{2}\int_{-\infty}^{+\infty} \frac{\zeta\left(3/2 \pm ix\right)}{\left(3/2 \pm ix + k\right)\cosh \pi x}\, dx\,, \tag{7}$$

and

$$\nu_k = \frac{\gamma}{k+1} - \frac{1}{(k+1)^2} - \frac{1}{2}\int_{-\infty}^{+\infty} \frac{\zeta\left(1/2 \pm ix\right)}{\left(1/2 \pm ix + k\right)\cosh \pi x}\, dx\,, \tag{8}$$

which are valid for $\mathrm{Re}(k) > -\frac{3}{2}$ and $\mathrm{Re}(k) > -\frac{1}{2}$ respectively. The second representation seems especially interesting because the integral runs over the whole critical line.

In appendix, we return to the case of a positive integer k and highlight the existence of an interesting relation between the series ν_k, the Stirling numbers of the second kind $S_2(n, k)$ and the shifted Mascheroni series σ_r which have been recently studied in [CY].

1 The case of a positive integer

In this section, we study the case where the parameter k is a positve integer and give two independant proofs of our formula (2). More precisely, we prove the following proposition:

Proposition 1. For any positive integer k, then

$$\nu_k = \frac{\gamma}{k+1} - \frac{1}{2}\ln 2\pi + \sum_{j=1}^{k-1} (-1)^j \binom{k}{j} \zeta'(-j) + C_k$$

with

$$\begin{aligned} C_k &= \frac{1}{k} + \sum_{r=1}^{[\frac{k}{2}]} \binom{k}{2r} \frac{B_{2r}\, H_{2r-1}}{k+1-2r} \\ &= \frac{H_k}{k+1} + \frac{1}{2k} - \sum_{r=1}^{[\frac{k}{2}]} \frac{B_{2r}}{2r(k+1-2r)} \end{aligned} \tag{9}$$

where H_n are the harmonic numbers defined by

$$H_n = \sum_{m=1}^{n} \frac{1}{m}\,, \quad \text{for } n = 1, 2, 3, \cdots$$

and B_n are the Bernoulli numbers defined by their exponential generating series

$$\sum_{n=0}^{\infty} B_n \frac{z^n}{n!} = \frac{z}{e^z - 1}\,, \quad \text{for } |z| < 2\pi\,.$$

In particular, $B_0 = 1$, $B_1 = -1/2$, $B_{2r+1} = 0$ for $r \geq 1$.

Proof. We can quite easily deduce (2) from formula (1). A differentiation of the functional equation

$$\zeta(s) = 2(2\pi)^{s-1}\Gamma(1-s)\zeta(1-s)\sin\frac{\pi s}{2}\,,$$

leads to the relations

$$(-1)^r \frac{(2r)!}{2(2\pi)^{2r}}\zeta(2r+1) = \zeta'(-2r)\,, \quad \text{for } r = 1, 2, 3, \cdots$$

and

$$(-1)^r \frac{(2r)!}{r(2\pi)^{2r}}\zeta'(2r) = -\zeta'(1-2r) + \frac{B_{2r}}{2r}\left(H_{2r-1} - \gamma - \ln 2\pi\right)\,, \quad \text{for } r = 1, 2, 3, \cdots\,.$$

Substituting these relations in (1) and aggregating the terms under the two Σ then gives

$$\nu_k = \frac{\gamma}{k+1} - \frac{1}{2}\ln 2\pi + \sum_{j=1}^{k-1}(-1)^j \binom{k}{j}\zeta'(-j) + C_k$$

with

$$C_k = \frac{1}{k} + \sum_{r=1}^{[\frac{k}{2}]} \binom{k}{2r} \frac{B_{2r}\, H_{2r-1}}{k+1-2r}\,.$$

Another proof of formula (2), independant from (1), may also be deduced from the following development given in [Ca] p. 93:

$$\begin{aligned}\sum_{k=0}^{\infty} \frac{(-1)^k z^k}{k!} \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{j}\zeta^{\mathcal{R}}(j-k) = (1-e^z) &\sum_{k=0}^{\infty} \frac{(-1)^k z^k}{k!}\zeta'(-k) \\ &+ (1-e^z)\sum_{k=0}^{\infty} \frac{(-1)^k z^k}{k!}\frac{1}{(k+1)^2} \\ &+ \int_0^1 \ln(t+1)e^{-zt}\,dt\,,\end{aligned}$$

with

$$\zeta^{\mathcal{R}}(j-k) = \begin{cases} \gamma & \text{if } j = k+1 \\ \zeta(j-k) - \frac{1}{j-k-1} & \text{otherwise.} \end{cases}$$

Rewriting the series ν_k under the following form:

$$\nu_k = \sum_{j=k+2}^{\infty} \frac{(-1)^{j-k}}{j} \zeta(j-k)\,,$$

and using the well-known relations:

$$\zeta(0) = -\frac{1}{2}\,, \quad \zeta'(0) = -\frac{1}{2}\ln 2\pi\,, \quad \zeta(1-2r) = -\frac{B_{2r}}{2r}\,,$$

as well as the combinatorial identity

$$\sum_{j=0}^{k} (-1)^j \binom{k}{j} \frac{1}{(j+1)^2} = \frac{H_{k+1}}{k+1}\,,$$

then, a careful identification of the terms in $\frac{z^k}{k!}$ in the previous development leads to the same formula (2) with a simpler expression of the constant C_k:

$$C_k = \frac{H_k}{k+1} + \frac{1}{2k} - \sum_{r=1}^{[\frac{k}{2}]} \frac{B_{2r}}{2r(k+1-2r)}\,.$$

□

Example 1. For the first six values of k, we obtain the following relations:

$$\begin{aligned}
\nu_1 &= \frac{\gamma}{2} - \frac{1}{2}\ln 2\pi + 1\,, \\
\nu_2 &= \frac{\gamma}{3} - \frac{1}{2}\ln 2\pi - 2\zeta'(-1) + \frac{2}{3}\,, \\
\nu_3 &= \frac{\gamma}{4} - \frac{1}{2}\ln 2\pi - 3\zeta'(-1) + 3\zeta'(-2) + \frac{7}{12}\,, \\
\nu_4 &= \frac{\gamma}{5} - \frac{1}{2}\ln 2\pi - 4\zeta'(-1) + 6\zeta'(-2) - 4\zeta'(-3) + \frac{47}{90}\,, \\
\nu_5 &= \frac{\gamma}{6} - \frac{1}{2}\ln 2\pi - 5\zeta'(-1) + 10\zeta'(-2) - 10\zeta'(-3) + 5\zeta'(-4) + \frac{167}{360}\,, \\
\nu_6 &= \frac{\gamma}{7} - \frac{1}{2}\ln 2\pi - 6\zeta'(-1) + 15\zeta'(-2) - 20\zeta'(-3) + 15\zeta'(-4) - 6\zeta'(-5) + \frac{349}{840}\,.
\end{aligned}$$

2 Alternating series involving multiple zeta values

In this section, we generalize the series ν_k to certain multiple zeta values and prove our formulae (3) and (6).

Proposition 2. For all natural numbers $p \geq 0$ and integers $k \geq -1$, then

$$\nu_{k,p} := \sum_{j=2}^{\infty} \frac{(-1)^j}{j+k} \zeta(j, \underbrace{1, \ldots, 1}_{p}) = \sum_{n=1}^{\infty} \frac{|G_n^{(k+1)}|}{n^{p+1}},$$

where $G_n^{(k)}$ are the Gregory coefficients of higher order defined by Eq. (5) or (6). In particular, for $k \geq 0$,

$$\nu_{k-1} = \sum_{n=1}^{\infty} \frac{|G_n^{(k)}|}{n}.$$

Example 2. For the first values of k, we obtain the following developments in series containing only positive rational terms:

$$\nu_{-1} = 1 + \frac{1}{8} + \frac{5}{108} + \frac{3}{128} + \frac{251}{18000} + \frac{95}{10368} + \ldots,$$
$$\nu_0 = \frac{1}{2} + \frac{1}{24} + \frac{1}{72} + \frac{19}{2880} + \frac{3}{800} + \frac{863}{362\,880} + \ldots,$$
$$\nu_1 = \frac{1}{3} + \frac{1}{48} + \frac{7}{1080} + \frac{17}{5760} + \frac{41}{25\,200} + \frac{731}{725\,760} + \ldots,$$
$$\nu_2 = \frac{1}{4} + \frac{1}{80} + \frac{1}{270} + \frac{11}{6720} + \frac{89}{100\,800} + \frac{5849}{10\,886\,400} + \ldots,$$
$$\nu_3 = \frac{1}{5} + \frac{1}{120} + \frac{1}{420} + \frac{83}{80\,640} + \frac{59}{108\,000} + \frac{397}{1\,209\,600} + \ldots,$$
$$\nu_4 = \frac{1}{6} + \frac{1}{168} + \frac{5}{3024} + \frac{17}{24\,192} + \frac{557}{1\,512\,000} + \frac{5249}{23\,950\,080} + \ldots,$$
$$\nu_5 = \frac{1}{7} + \frac{1}{224} + \frac{11}{9072} + \frac{41}{80\,640} + \frac{439}{1\,663\,200} + \frac{311}{1\,995\,840} + \ldots,$$
$$\nu_6 = \frac{1}{8} + \frac{1}{288} + \frac{1}{1080} + \frac{73}{190\,080} + \frac{47}{237\,600} + \frac{2581}{22\,239\,360} + \ldots,$$

Proof. In order to prove Proposition 2 above, we use the following lemma (cf. [Xu] Eq. (2.27) and (2.28)):

Lemma 1. For all integers $m \geq 1$ and $p \geq 0$, one has

$$\int_0^1 \frac{\ln^m(1-x)\,\ln^p(x)}{x}\,dx = (-1)^{m+p}\,m!\,p!\,\zeta(m+1, \underbrace{1, \ldots, 1}_{p}).$$

Then, we can write the following equalities:

$$\begin{aligned}
\sum_{j=2}^{\infty}\frac{(-1)^j}{j+k}\zeta(j,\underbrace{1,\dots,1}_{p}) &= \sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{m+k+1}\zeta(m+1,\underbrace{1,\dots,1}_{p}) \\
&= \frac{(-1)^{p+1}}{p!}\sum_{m=1}^{\infty}\frac{1}{m+k+1}\int_0^1 \frac{\ln^m(1-x)}{m!}\frac{\ln^p(x)}{x}\,dx \\
&= \frac{(-1)^{p+1}}{p!}\sum_{m=1}^{\infty}\frac{(-1)^m}{m+k+1}\int_0^1\left(\sum_{n=1}^{\infty}|S_1(n,m)|\frac{x^n}{n!}\right)\frac{\ln^p(x)}{x}\,dx \\
&= -\sum_{n=1}^{\infty}\frac{1}{n!}\sum_{m=1}^{n}\frac{(-1)^m}{m+k+1}|S_1(n,m)|\,\frac{(-1)^p}{p!}\int_0^1 x^{n-1}\ln^p(x)\,dx \\
&= \sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{1}{n!}\sum_{m=1}^{n}\frac{S_1(n,m)}{m+k+1}\right)\frac{1}{n^{p+1}} \\
&= \sum_{n=1}^{\infty}(-1)^{n+1}\frac{G_n^{(k+1)}}{n^{p+1}} = \sum_{n=1}^{\infty}\frac{|G_n^{(k+1)}|}{n^{p+1}}\,.
\end{aligned}$$

□

Example 3.

$$\sum_{j=2}^{\infty}\frac{(-1)^j}{j}\sum_{n=1}^{\infty}\frac{H_n}{n^j} = \nu_{0,1}-\nu_{-1}+\zeta(2) = \sum_{n=1}^{\infty}\frac{|G_n|}{n^2}-\sum_{n=1}^{\infty}\frac{|G_n^{(0)}|}{n}+\frac{\pi^2}{6}\,.$$

3 Integral representations

In this section, we study the extension to the case of a complex parameter k and give two beautiful integral representations for the series ν_k.

Proposition 3. For any $k \in \mathbb{C}$ such that $\mathrm{Re}(k) > -3/2$, then

$$\nu_k = \frac{1}{2}\int_{-\infty}^{+\infty}\frac{\zeta\left(3/2 \pm ix\right)}{\left(3/2 \pm ix + k\right)\cosh \pi x}\,dx\,. \tag{10}$$

For any $k \in \mathbb{C}$ such that $\mathrm{Re}(k) > -1/2$, then

$$\nu_k = \frac{\gamma}{k+1} - \frac{1}{(k+1)^2} - \frac{1}{2}\int_{-\infty}^{+\infty}\frac{\zeta\left(1/2 \pm ix\right)}{\left(1/2 \pm ix + k\right)\cosh \pi x}\,dx \tag{11}$$

By identification with formula (2), we deduce the following corollary:

Corollary 1. For any positive integer k, we have

$$\frac{1}{2}\int\limits_{-\infty}^{+\infty}\frac{\zeta\left(1/2\pm ix\right)}{\left(1/2\pm ix+k\right)\cosh\pi x}\,dx=\sum_{j=0}^{k-1}(-1)^{j+1}\binom{k}{j}\zeta'(-j)-C_k-\frac{1}{(k+1)^2}\,,$$

where C_k is the rational constant defined by (9).

Example 4.

$$\frac{1}{2}\int\limits_{-\infty}^{+\infty}\frac{\zeta\left(3/2\pm ix\right)}{\left(3/2\pm ix\right)\cosh\pi x}\,dx=\gamma\,,$$

$$\frac{1}{2}\int\limits_{-\infty}^{+\infty}\frac{\zeta\left(3/2\pm ix\right)}{\left(1/2\pm ix\right)\cosh\pi x}\,dx=\nu_{-1}\,,$$

$$\frac{1}{2}\int\limits_{-\infty}^{+\infty}\frac{\zeta\left(1/2\pm ix\right)}{\left(1/2\pm ix\right)\cosh\pi x}\,dx=-1\,,$$

$$\frac{1}{2}\int_{-\infty}^{\infty}\frac{\zeta(1/2+ix)}{(3/2+ix)\cosh(\pi x)}\,dx=\frac{1}{2}\ln 2\pi-\frac{5}{4}\,,$$

$$\frac{1}{2}\int_{-\infty}^{\infty}\frac{\zeta(1/2+ix)}{(5/2+ix)\cosh(\pi x)}\,dx=\frac{1}{2}\ln 2\pi-2\ln(A)-\frac{11}{18}\,,$$

where $A:=\exp\left\{\frac{1}{12}-\zeta'(-1)\right\}$ is the Glaisher-Kinkelin constant.

Proof. In order to prove Proposition 3, we resort to the contour integration method. Consider the following line integral taken along a contour C consisting of the interval $[-R,+R]$, $R\in\mathbb{N}$, on the real axis, and a semicircle of the radius R in the upper half-plane, denoted C_R,

$$\oint\limits_{C}\frac{\zeta\left(1/2-iz\right)}{\left(1/2-iz+k\right)\cosh\pi z}\,dz=\int\limits_{-R}^{+R}\frac{\zeta\left(1/2-ix\right)}{\left(1/2-ix+k\right)\cosh\pi x}\,dx+\int\limits_{C_R}\frac{\zeta\left(1/2-iz\right)}{\left(1/2-iz+k\right)\cosh\pi z}\,dz\,. \tag{12}$$

On the contour C_R the last integral may be bounded as follows:

$$\left|\int\limits_{C_R}\frac{\zeta\left(1/2-iz\right)}{\left(1/2-iz+k\right)\cosh\pi z}\,dz\right|=R\left|\int\limits_{0}^{\pi}\frac{\zeta\left(1/2-iRe^{i\varphi}\right)e^{i\varphi}}{\left(1/2-iRe^{i\varphi}+k\right)\cosh\left(\pi Re^{i\varphi}\right)}\,d\varphi\right|\leqslant$$

$$\leqslant R\max_{\varphi\in[0,\pi]}\left|\frac{\zeta\left(1/2-iRe^{i\varphi}\right)}{1/2-iRe^{i\varphi}+k}\right|\cdot I_R\leqslant\max_{\varphi\in[0,\pi]}\left|\zeta\left(1/2-iRe^{i\varphi}\right)\right|\cdot I_R \tag{13}$$

where we denoted

$$I_R := \int_0^\pi \frac{d\varphi}{\left|\cosh\left(\pi R e^{i\varphi}\right)\right|}\,, \qquad R>0\,,$$

for the purpose of brevity. Now, in the half–plane $\sigma > 1$, the absolute value of $\zeta(\sigma+it)$ may be always bounded by a constant $C=\zeta(\sigma)$, which decreases and tends to 1 as $\sigma\to\infty$. In contrast, in the strip $0\leqslant\sigma\leqslant 1$ the upper bound of the function $\left|\zeta(\sigma+it)\right|$ grows, and presently it is not known which is its exact rate of grow. However, it follows from the general theory of Dirichlet series that it cannot be faster than $O(t^{1/4})$. Hence, since $\sin\varphi\geqslant 0$ and if R is large enough, this rough estimate gives us

$$\left|\zeta\left(1/2 - iRe^{i\varphi}\right)\right| = \left|\zeta\left(1/2 + R\sin\varphi - iR\cos\varphi\right)\right| = O\left(\sqrt[4]{R}\right)$$

in the interval $\varphi\in[0,\pi]$. On the other hand, as R tends to infinity and remains integer the integral I_R tends to zero as $O(1/R)$. To show this, it is sufficient to remark that

$$\frac{1}{\left|\cosh\left(\pi R e^{i\varphi}\right)\right|} = \frac{\sqrt{2}}{\sqrt{\cosh(2\pi R\cos\varphi)+\cos(2\pi R\sin\varphi)}} = O\left(e^{-\pi R|\cos\varphi|}\right), \qquad R\to\infty\,,$$

since $0\leqslant\varphi\leqslant\pi$ and R is integer. Thus, accounting for the symmetry of $\left|\cosh\left(\pi R e^{i\varphi}\right)\right|^{-1}$ about $\varphi=\frac{1}{2}\pi$, we deduce that

$$\begin{aligned}
I_R &= \int_0^\pi \frac{\sqrt{2}}{\sqrt{\cosh(2\pi R\cos\varphi)+\cos(2\pi R\sin\varphi)}}\,d\varphi \\
&= \int_0^{\frac{\pi}{2}} \frac{2\sqrt{2}}{\sqrt{\cosh(2\pi R\cos\varphi)+\cos(2\pi R\sin\varphi)}}\,d\varphi \\
&= O\left(\int_0^{\frac{\pi}{2}} e^{-\pi R\cos\varphi}\,d\varphi\right) = O\left(\int_0^{\frac{\pi}{2}} e^{-\pi R\sin\vartheta}\,d\vartheta\right), \qquad R\to\infty\,.
\end{aligned} \tag{14}$$

From the well–known inequality

$$\frac{2\vartheta}{\pi}\leqslant\sin\vartheta\leqslant\vartheta\,, \qquad \vartheta\in\left[0,\tfrac{1}{2}\pi\right]$$

it finally follows that

$$\frac{1-e^{-\frac{1}{2}\pi^2 R}}{\pi R} \leqslant \int_0^{\frac{\pi}{2}} e^{-\pi R\sin\vartheta}\, d\vartheta \leqslant \frac{1-e^{-\pi R}}{2R}\,, \tag{15}$$

and since R is large, exponential terms on both sides may be neglected. Thus $I_R = O(1/R)$ at $R\to\infty$.[2] Inserting both latter results into (13), we obtain

$$\left|\int_{C_R} \frac{\zeta\left(1/2 - iz\right)}{\left(1/2 - iz + k\right)\cosh\pi z}\, dz\right| = O\left(R^{-3/4}\right) \to 0 \qquad \text{as} \quad R\to\infty\,,\ R\in\mathbb{N}\,,$$

Hence, making $R\to\infty$, equality (12) becomes

$$\int_{-\infty}^{+\infty} \frac{\left(a-ix\right)^{1-s}}{\cosh^2\pi x}\, dx = \oint_C \frac{\left(a-iz\right)^{1-s}}{\cosh^2\pi z}\, dz \tag{17}$$

where the latter integral is taken around an infinitely large semicircle in the upper half-plane. The integrand is not a holomorphic function: it has the poles of the second order at $z = z_n \equiv i\left(n+\frac{1}{2}\right)$, $n\in\mathbb{N}_0$, due to the hyperbolic secant, and a branch point at $z=-ia$ due to the term in the numerator. If $\operatorname{Re}(a)>0$, the branch point lies outside the integration contour and we may use the Cauchy residue theorem:

$$\begin{aligned}
\oint_C \frac{\left(a-iz\right)^{1-s}}{\cosh^2\pi z}\, dz &= 2\pi i \sum_{n=0}^{\infty} \operatorname*{res}_{z=z_n} \frac{\left(a-iz\right)^{1-s}}{\cosh^2\pi z} = \\
&= -\frac{2i}{\pi}\sum_{n=0}^{\infty} \frac{\partial}{\partial z}\left(a-iz\right)^{1-s}\Bigg|_{z=i\left(n+\frac{1}{2}\right)} = \\
&= \frac{2(s-1)}{\pi}\sum_{n=0}^{\infty}\left(a+\tfrac{1}{2}+n\right)^{-s} = \frac{2(s-1)}{\pi}\,\zeta\left(s, a+\tfrac{1}{2}\right).
\end{aligned} \tag{18}$$

[2]Another way to obtain the same result is to recall that the integral (15) may be evaluated in terms of the modified Bessel function $I_n(z)$ of the first kind and the modified Struve function $L_n(z)$. Using the asymptotic expansions of these special functions we obtain even a more exact result, namely

$$\int_0^{\frac{\pi}{2}} e^{-\pi R\sin\vartheta}\, d\vartheta = \frac{\pi}{2}\Big\{I_0(\pi R) - L_0(\pi R)\Big\} \sim \frac{1}{\pi R}\,, \qquad R\to\infty\,, \tag{16}$$

i.e. the integral asymptotically tends to the left bound (15).

Equating (17) with the last result yields

$$\zeta\Big(s, a+\tfrac{1}{2}\Big) \;=\; \frac{\pi}{2(s-1)} \int_{-\infty}^{+\infty} \frac{\big(a-ix\big)^{1-s}}{\cosh^2 \pi x}\, dx\,, \qquad \mathrm{Re}(a) > 0\,. \tag{19}$$

Splitting the interval of integration in two parts $(-\infty, 0]$ and $[0, +\infty]$ and recalling that

$$\big(a+ix\big)^s + \big(a-ix\big)^s \;=\; 2\big(a^2+x^2\big)^{\frac{s}{2}} \cos\Big(s \,\mathrm{arctg}\, \frac{x}{a}\Big) \tag{20}$$

the latter expression may also be written as

$$\zeta\Big(s, a+\tfrac{1}{2}\Big) \;=\; \frac{\pi}{2(s-1)} \int_0^{\infty} \frac{\big(a+ix\big)^{1-s} + \big(a-ix\big)^{1-s}}{\cosh^2 \pi x}\, dx \tag{21}$$

$$= \frac{\pi}{s-1} \int_0^{\infty} \frac{\cos\Big[(s-1)\,\mathrm{arctg}\,\frac{x}{a}\Big]}{\big(a^2+x^2\big)^{\frac{1}{2}(s-1)} \cosh^2 \pi x}\, dx\,, \qquad \mathrm{Re}(a) > 0\,. \tag{22}$$

□

Appendix

Let us consider the forward shifted Mascheroni series which are defined by

$$\sigma_r := \sum_{n=1}^{\infty} \frac{|G_{n+r}|}{n}\,, \quad \text{for } r = 0, 1, 2, \cdots .$$

These series were the main subject of a recent article ([CY]). We have among other things established the following decomposition of $\zeta'(-j)$ (cf. [CY], Proposition 3):

$$\zeta'(-j) = \sum_{r=2}^{j+1} (-1)^{j-r}(r-1)!S_2(j, r-1)\sigma_r - \frac{B_{j+1}}{j+1}\gamma - \frac{B_{j+1}}{(j+1)^2}\,, \quad \text{for } j = 1, 2, 3, \cdots$$

where $S_2(j, r)$ are the Stirling numbers of the second kind. Then, substituting this relation in (2) enables to write the series ν_k as an integer linear combination of $\sigma_1, \sigma_2, \cdots, \sigma_k$ and a rational constant D_k linked to C_k (the coefficient of $\gamma = \sigma_0$ vanishes by a well-known relation between the Bernoulli numbers[3]). More precisely, we obtain the following nice rewriting of formula (2):

[3]The coefficient of γ is $\frac{1}{k+1}\sum_{j=0}^{k}\binom{k+1}{j}B_j$ which is equal to zero.

$$\nu_k = D_k + \sum_{r=1}^{k}(-1)^r(r-1)!\left(\sum_{j=r-1}^{k-1}\binom{k}{j}S_2(j,r-1)\right)\sigma_r, \quad \text{for } k = 1,2,3,\cdots$$

with

$$D_k = C_k - \frac{1}{2} + \sum_{r=1}^{[\frac{k}{2}]}\binom{k}{2r}\frac{B_{2r}}{2r(k+1-2r)} = \frac{1}{k} - \frac{1}{2} + \sum_{r=1}^{[\frac{k}{2}]}\binom{k}{2r}\frac{B_{2r}H_{2r}}{k+1-2r}\,. \qquad (23)$$

Example 5. For the first six values of k, we obtain the following relations:

$$\begin{aligned}
\nu_1 &= \frac{1}{2} - \sigma_1\,,\\
\nu_2 &= \frac{1}{4} - \sigma_1 + 2\sigma_2\,,\\
\nu_3 &= \frac{5}{24} - \sigma_1 + 6\sigma_2 - 6\sigma_3\,,\\
\nu_4 &= \frac{13}{72} - \sigma_1 + 14\sigma_2 - 36\sigma_3 + 24\sigma_4\,,\\
\nu_5 &= \frac{109}{720} - \sigma_1 + 30\sigma_2 - 150\sigma_3 + 240\sigma_4 - 120\sigma_5\,,\\
\nu_6 &= \frac{23}{180} - \sigma_1 + 62\sigma_2 - 420\sigma_3 + 1560\sigma_4 - 1800\sigma_5 + 720\sigma_6\,.
\end{aligned}$$

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