On certain alternating series involving zeta values

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A note on shifted Mascheroni series and their relations with certain alternating series involving zeta values

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Introduction

This short article is devoted to the alternating series ν_k defined by

$$\nu_k := \sum_{p=2}^{\infty} (-1)^p \frac{\zeta(p)}{p+k}, \quad \text{for } k = 0, 1, 2, \cdots$$

By a classical result due to Euler, it is well-known (cf. [SC], p. 272, Eq. (23)) that

$$\nu_0 = \sum_{p=2}^{\infty} (-1)^p \frac{\zeta(p)}{p} = \gamma\,,$$

where γ denotes the Euler-Mascheroni constant. It is also fairly well-known that $\nu_1 = \frac{\gamma}{2} - \frac{1}{2}\ln(2\pi) + 1$ (cf. [SC], p. 312, Eq. (483), [SV], Eq. (1.5), [Ca], p. 93), and $\nu_2 = \frac{\gamma}{3} + \ln(2^{-\frac{1}{2}}\pi^{-\frac{1}{2}}A^2) + \frac{1}{2}$, where A is the Glaisher-Kinkelin constant (cf. [SC], p. 318, Eq. (529)), or equivalently (cf. [Ca], p. 93)

$$\nu_2 = \frac{\gamma}{3} - \frac{1}{2}\ln(2\pi) - 2\zeta'(-1) + \frac{2}{3}\,.$$

More generally, we show that for $k \geq 2$, the series ν_k admits the following explicit evaluation:

$$\nu_k = \frac{\gamma}{k+1} - \frac{1}{2}\ln(2\pi) + \sum_{j=1}^{k-1} (-1)^j \binom{k}{j} \zeta'(-j) + C_k\,, \tag{1}$$

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with

$$C_k = \frac{H_k}{k+1} + \frac{1}{2k} - \sum_{r=1}^{[\frac{k}{2}]} \frac{B_{2r}}{2r(k+1-2r)}\,, \tag{2}$$

where H_k is the kth harmonic number

$$H_k = \sum_{j=1}^{k} \frac{1}{j}\,,$$

and B_{2r} are the Bernoulli numbers defined by the generating function

$$\frac{z}{e^z - 1} = 1 - \frac{z}{2} + \sum_{r=1}^{\infty} \frac{B_{2r}}{(2r)!} z^{2r}\,.$$

This expression of ν_k may be deduced from a certain relation between generating series given in [Ca] (see Section 1 below).

Let us introduce now the forward shifted Mascheroni series studied in [CY] which are defined by

$$\sigma_r := \sum_{n=1}^{\infty} \frac{|G_{n+r}|}{n}\,, \quad \text{for } r = 0, 1, 2, \cdots\,,$$

where G_n denotes the Bernoulli numbers of the second kind[1] (also called the Gregory coefficients) determined by the generating function:

$$\frac{z}{\ln(1+z)} = 1 + \sum_{n=1}^{\infty} G_n z^n\,,$$

the first values being

$$G_1 = \frac{1}{2}\,, G_2 = -\frac{1}{12}\,, G_3 = \frac{1}{24}\,, G_4 = -\frac{19}{720}\,, G_5 = \frac{3}{160}\,, \cdots$$

A classical identity (originally due to Mascheroni) leads to the relation[2]:

$$\gamma = \sum_{n=1}^{\infty} \frac{|G_n|}{n} = \sum_{p=2}^{\infty} \frac{(-1)^p}{p} \zeta(p)\,, \quad \text{i.e. } \sigma_0 = \nu_0.$$

[1]In [CY], these numbers are quoted b_n.

[2]More generally, it may be shown (cf. [BC], Eq. (36)) that

$$\kappa_1 := \sum_{n=1}^{\infty} \frac{|G_n|}{n^2} = \sum_{p=2}^{\infty} \frac{(-1)^p}{p} \zeta(p, 1)\,,$$

where $\zeta(p, q)$ is the double zeta series

$$\sum_{n>m} \frac{1}{n^p m^q}\,.$$

Furthermore, the following decomposition (cf. [CY], Proposition 3):

$$\zeta'(-k) = \sum_{r=2}^{k+1} (-1)^{k-r}(r-1)! S_2(k, r-1)\sigma_r - \frac{B_{k+1}}{k+1}\gamma - \frac{B_{k+1}}{(k+1)^2} \,,$$

where $S_2(k,r)$ are Stirling numbers of the second kind, enables to write an interesting expression of ν_k as an integer linear combination of $\sigma_1, \sigma_2, \cdots, \sigma_k$ and a rational constant (the coefficient of γ vanishing by a well-known relation between the Bernoulli numbers). More precisely, we prove (see Section 2 below) the following relations:

$$\nu_1 = \frac{1}{2} - \sigma_1 \,,$$

and for $k \geq 2$,

$$\nu_k = D_k - \sigma_1 + \sum_{r=2}^{k} (-1)^r (r-1)! \left(\sum_{j=r-1}^{k-1} \binom{k}{j} S_2(j, r-1) \right) \sigma_r \,, \tag{3}$$

where D_k is the rational number

$$D_k = C_k - \frac{1}{2} + \sum_{r=1}^{[\frac{k}{2}]} \binom{k}{2r-1} \frac{B_{2r}}{(2r)^2} \,. \tag{4}$$

In Section 4, we give an alternative expression of the constants C_k and D_k (cf. Eq. (6) and (7)) deduced from a formula of Blagouchine, and write an amazing relation between the harmonic numbers and the Bernoulli numbers (cf. Eq. (8)).

Finally, in Section 5, we write another interesting expression of the series ν_k in terms of a series involving the Gregory coefficients of higher order $G_n^{(k)}$ (cf. Eq. (9)) and state a conjecture for a natural extension of ν_k (cf. Conjecture 1).

1 Proof of formulae (1) and (2)

These formulae may be deduced by expanding in powers of z the following relation between generating series (cf. [Ca], p. 93):

$$\begin{aligned}\sum_{k=0}^{\infty} \frac{(-1)^k z^k}{k!} \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{j} \zeta^{\mathcal{R}}(j-k) = (1-e^z) &\sum_{k=0}^{\infty} \frac{(-1)^k z^k}{k!} \zeta'(-k) \\ &+ (1-e^z) \sum_{k=0}^{\infty} \frac{(-1)^k z^k}{k!} \frac{1}{(k+1)^2} \\ &+ \int_0^1 \ln(t+1) e^{-zt} \, dt \,,\end{aligned}$$

with

$$\zeta^{\mathcal{R}}(j-k) = \begin{cases} \gamma & \text{if } j = k+1 \\ \zeta(j-k) - \frac{1}{j-k-1} & \text{otherwise.} \end{cases}$$

Rewriting the series ν_k under the following form:

$$\nu_k = \sum_{j=k+2}^{\infty} \frac{(-1)^{j-k}}{j} \zeta(j-k)\,,$$

and using the well-known relation

$$\zeta'(0) = -\frac{1}{2}\ln(2\pi)\,,$$

the identification of the coefficient of $\dfrac{z^k}{k!}$ in the previous development leads to the relation

$$\nu_k = C_k + \frac{\gamma}{k+1} - \frac{1}{2}\ln(2\pi) + \sum_{j=1}^{k-1}(-1)^j \binom{k}{j}\zeta'(-j)$$

where C_k is the rational

$$C_k = \sum_{j=0}^{k-1}(-1)^j \binom{k}{j} \frac{1}{(j+1)^2} - \frac{1}{2k} - \sum_{j=1}^{k-1} \frac{(-1)^j}{k-j}\left(\frac{1-B_{j+1}}{j+1}\right) + \left(1-(-1)^k\right)\frac{1}{k+1}\sum_{j=1}^{k+1}\frac{(-1)^{j-1}}{j}\,.$$

However, since

$$\sum_{j=0}^{k-1}(-1)^j \binom{k}{j} \frac{1}{(j+1)^2} = \frac{H_{k+1}}{k+1} - \frac{(-1)^k}{(k+1)^2}\,,$$

this expression of C_k may be (higly) simplified as

$$C_k = \frac{H_k}{k+1} + \frac{1}{2k} + \sum_{j=1}^{k-1} \frac{(-1)^j B_{j+1}}{(k-j)(j+1)}\,.$$

Since $B_1 = -\frac{1}{2}$ and $B_{2r+1} = 0$ for $r \geq 1$, the constant C_k may also be rewritten

$$\begin{aligned} C_k &= \frac{H_k}{k+1} - \sum_{j=1}^{k} \frac{B_j}{j(k+1-j)} \\ &= \frac{H_k}{k+1} + \frac{1}{2k} - \sum_{r=1}^{[\frac{k}{2}]} \frac{B_{2r}}{2r(k+1-2r)}\,, \end{aligned}$$

and therefore (1) and (2) are established.

2 Proof of formulae (3) and (4)

We have shown above that

$$\nu_k = \frac{\gamma}{k+1} - \frac{1}{2}\ln(2\pi) + \sum_{j=1}^{k-1}(-1)^j\binom{k}{j}\zeta'(-j) + C_k$$

where the expression of C_k is given by (2). Furthermore, Propositions 2 and 3 of [CY] enable to write the relations

$$\frac{1}{2}\ln(2\pi) = \sigma_1 + \frac{\gamma}{2} + \frac{1}{2}\,,$$

and

$$\zeta'(-j) = \sum_{r=2}^{j+1}(-1)^{j-r}(r-1)!S_2(j,r-1)\sigma_r - \frac{B_{j+1}}{j+1}\gamma - \frac{B_{j+1}}{(j+1)^2} \qquad \text{for } j \geq 1 \ .$$

Then, substituting these relations in (1) gives

$$\nu_k = \frac{\gamma}{k+1}\sum_{j=0}^{k}\binom{k+1}{j}B_j - \sigma_1 + \sum_{j=1}^{k-1}\sum_{r=2}^{j+1}\binom{k}{j}(-1)^r(r-1)!S_2(j,r-1)\sigma_r$$

$$- \frac{1}{2} + \sum_{j=1}^{k-1}(-1)^{j+1}\binom{k}{j}\frac{B_{j+1}}{(j+1)^2} + C_k\,.$$

The coefficient of γ vanishes since

$$\sum_{j=0}^{k}\binom{k+1}{j}B_j = 0\,,$$

and moreover we may write

$$\sum_{j=1}^{k-1}(-1)^{j+1}\binom{k}{j}\frac{B_{j+1}}{(j+1)^2} = \sum_{r=1}^{[\frac{k}{2}]}\binom{k}{2r-1}\frac{B_{2r}}{(2r)^2}\,.$$

Finally, interchanging the symbols Σ leads to

$$\nu_k = D_k - \sigma_1 + \sum_{r=2}^{k}(-1)^r(r-1)!\left(\sum_{j=r-1}^{k-1}\binom{k}{j}S_2(j,r-1)\right)\sigma_r$$

with

$$D_k = C_k + \sum_{r=1}^{[\frac{k}{2}]}\binom{k}{2r-1}\frac{B_{2r}}{(2r)^2} - \frac{1}{2}\,,$$

hence, formulae (3) and (4) are now established.

3 Examples

For the first values of k, formulae (1) to (4) give the evaluations

$$\begin{aligned}
\nu_1 &= \frac{\gamma}{2} - \frac{1}{2}\ln(2\pi) + 1\,, \\
\nu_2 &= \frac{\gamma}{3} - \frac{1}{2}\ln(2\pi) - 2\zeta'(-1) + \frac{2}{3} \\
\nu_3 &= \frac{\gamma}{4} - \frac{1}{2}\ln(2\pi) - 3\zeta'(-1) + 3\zeta'(-2) + \frac{7}{12} \\
\nu_4 &= \frac{\gamma}{5} - \frac{1}{2}\ln(2\pi) - 4\zeta'(-1) + 6\zeta'(-2) - 4\zeta'(-3) + \frac{47}{90} \\
\nu_5 &= \frac{\gamma}{6} - \frac{1}{2}\ln(2\pi) - 5\zeta'(-1) + 10\zeta'(-2) - 10\zeta'(-3) + 5\zeta'(-4) + \frac{167}{360}\,,
\end{aligned}$$

and the following relations between ν_k and σ_k:

$$\begin{aligned}
\nu_1 &= \frac{1}{2} - \sigma_1\,, \\
\nu_2 &= \frac{1}{4} - \sigma_1 + 2\sigma_2\,, \\
\nu_3 &= \frac{5}{24} - \sigma_1 + 6\sigma_2 - 6\sigma_3\,, \\
\nu_4 &= \frac{13}{72} - \sigma_1 + 14\sigma_2 - 36\sigma_3 + 24\sigma_4\,, \\
\nu_5 &= \frac{109}{720} - \sigma_1 + 30\sigma_2 - 150\sigma_3 + 240\sigma_4 - 120\sigma_5.
\end{aligned}$$

4 Other expressions of the constants C_k and D_k

Recently, Blagouchine ([B1], p. 413, Eq. (38)) found the following alternative formula for ν_k:

$$\begin{aligned}
\nu_k = \frac{1}{k} - \frac{\ln(2\pi)}{k+1} + \frac{\gamma}{2} + \\
+ \sum_{r=1}^{[\frac{k}{2}]} (-1)^r \binom{k}{2r-1} \frac{(2r)!}{r(2\pi)^{2r}}\zeta'(2r) + \sum_{r=1}^{[\frac{k+1}{2}]-1} (-1)^r \binom{k}{2r} \frac{(2r)!}{2(2\pi)^{2r}}\zeta(2r+1)\,. \quad (5)
\end{aligned}$$

A differentiation of the functional equation

$$\zeta(s) = 2(2\pi)^{s-1}\Gamma(1-s)\zeta(1-s)\sin(\frac{\pi s}{2})\,,$$

leads to the relations

$$(-1)^r \frac{(2r)!}{2(2\pi)^{2r}} \zeta(2r+1) = \zeta'(-2r)$$

and

$$(-1)^r \frac{(2r)!}{r(2\pi)^{2r}} \zeta'(2r) = -\zeta'(1-2r) + \frac{B_{2r}}{2r} \left(H_{2r-1} - \gamma - \ln(2\pi) \right) .$$

Substituting these relations in (5) and identifying with (1) gives the following alternating expression for the constant C_k:

$$C_k = \frac{1}{k} + \sum_{r=1}^{[\frac{k}{2}]} \binom{k}{2r-1} \frac{B_{2r}}{2r} H_{2r-1} , \tag{6}$$

and from (4), we also deduce this expression of D_k:

$$D_k = \frac{1}{k} - \frac{1}{2} + \sum_{r=1}^{[\frac{k}{2}]} \binom{k}{2r-1} \frac{B_{2r}}{2r} H_{2r} . \tag{7}$$

Moreover, a comparison of (2) and (6) leads to the following amazing relation:

$$\frac{H_k}{k+1} = \frac{1}{2k} + \sum_{r=1}^{[\frac{k}{2}]} \frac{B_{2r}}{2r} \left[\frac{1}{k+1-2r} + \binom{k}{2r-1} H_{2r-1} \right] . \tag{8}$$

5 Yet another expression of ν_k

Another interesting expression of ν_k is given by the following formula (cf. [B1], p. 413, Eq. (38)): let

$$G_n^{(k)} := \frac{1}{n!} \sum_{l=1}^{n} \frac{S_1(n,l)}{l+k} ,$$

where $S_1(n,l)$ are the Stirling numbers of the first kind, then

$$\nu_{k-1} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{G_n^{(k)}}{n} , \quad k = 1, 2, 3, \cdots \tag{9}$$

Note that $G_n^{(1)} = G_n$ and the numbers $G_n^{(k)}$ may also be determined by the generating function (cf. [B2], p. 20, Eq. (68)):

$$(-1)^{k+1} \frac{(k-1)!z}{\ln^k(z+1)} + (1+z) \sum_{l=1}^{k-1} (-1)^{l+1} \frac{(k-l+1)_{l-1}}{\ln^l(z+1)} = \frac{1}{k} + \sum_{n=1}^{\infty} G_n^{(k)} z^n .$$

Finally, let

$$\zeta(s_1, s_2, \cdots, s_k) = \sum_{n_1 > n_2 > \cdots > n_k \geq 1} \frac{1}{n_1^{s_1} n_2^{s_2} \cdots n_k^{s_k}},$$

then we state the following conjecture (already checked in the case $k = 0$):

Conjecture 1.

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{G_n^{(k+1)}}{n^{m+1}} = \sum_{p=2}^{\infty} \frac{(-1)^p}{p+k} \zeta(p, \underbrace{1, \ldots, 1}_{m}).$$

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