A note on shifted Mascheroni series and their relations with certain alternating series involving zeta values

Marc-Antoine Coppo, Iaroslav Blagouchine

A note on shifted Mascheroni series and their relations with certain alternating series involving zeta values

Marc-Antoine Coppo*

March 14, 2018

Introduction

This short article is devoted to the alternating series S_k defined by

$$S_k := \sum_{p=2}^{\infty} (-1)^p \frac{\zeta(p)}{p+k}, \quad \text{for } k = 0, 1, 2, \cdots$$

By a classical result due to Euler, it is well-known (cf. [SC], Eq. (23), p. 272) that

$$S_0 = \sum_{p=2}^{\infty} (-1)^p \frac{\zeta(p)}{p} = \gamma,$$

where γ denotes the Euler-Mascheroni constant. It is also fairly well-known that $S_1 = \frac{\gamma}{2} - \frac{1}{2}\ln(2\pi) + 1$ ([SC], Eq. (483), p. 312, [SV], Eq. (1.5)), and for $k = 2$, $S_2 = \frac{\gamma}{3} + \ln(2^{-\frac{1}{2}}\pi^{-\frac{1}{2}}A^2) + \frac{1}{2}$, where A is the Glaisher-Kinkelin constant (cf. [SC], Eq. (529), p. 318), which may be rewritten as follows (cf. [C], p. 93)

$$S_2 = \frac{\gamma}{3} - \frac{1}{2}\ln(2\pi) - 2\zeta'(-1) + \frac{2}{3}.$$

More generally, for $k \geq 2$, S_k admits the following explicit evaluation:

$$S_k = \frac{\gamma}{k+1} - \frac{1}{2}\ln(2\pi) + \sum_{j=1}^{k-1} (-1)^j \binom{k}{j} \zeta'(-j) + \frac{H_k}{k+1} - \sum_{j=1}^{k} \frac{B_j}{j(k+1-j)}, \quad (1)$$

*Université Côte d'Azur, CNRS, LJAD (UMR 7351), France. *Email:* coppo@unice.fr

where H_k is the kth harmonic number

$$H_k = \sum_{j=1}^{k} \frac{1}{j} \, ,$$

and the B_j $(j = 1, \cdots, k)$ are Bernoulli numbers defined by the generating function:

$$\frac{t}{e^t - 1} = \sum_{k=0}^{\infty} \frac{B_k}{k!} t^k \, ,$$

or by the equivalent recursion:

$$B_0 = 1 \quad \text{and} \quad \sum_{j=0}^{k} \frac{B_j}{j!(k+1-j)!} = 0 \quad \text{for } k \geq 1.$$

This expression of S_k may be deduced from a certain relation between generating series given in [C] (see Section 1 below).

Let us introduce now the forward shifted Mascheroni series studied in [CY] which are defined by

$$\sigma_r := \sum_{n=1}^{\infty} \frac{|b_{n+r}|}{n} \, , \quad \text{for } r = 0, 1, 2, \cdots \, ,$$

where (b_n) denotes the sequence of Bernoulli numbers of the second kind which are determined by the generating function:

$$\frac{t}{\ln(1+t)} = \sum_{n=0}^{\infty} b_n t^n \, ,$$

or recursively by

$$b_0 = 1 \quad \text{and} \quad \sum_{j=0}^{n} \frac{(-1)^j b_j}{n+1-j} = 0 \quad \text{for } n \geq 1 \, ,$$

the first values of the sequence being

$$b_1 = \frac{1}{2} \, , b_2 = -\frac{1}{12} \, , b_3 = \frac{1}{24} \, , b_4 = -\frac{19}{720} \, , b_5 = \frac{3}{160} \, , \cdots$$

One has the classical identity (originally due to Mascheroni)

$$\sum_{n=1}^{\infty} \frac{|b_n|}{n} = \gamma \, ,$$

which translates here into $S_0 = \sigma_0$. Furthermore, we have recently obtained (cf. [CY], Proposition 3) the following decomposition

$$\zeta'(-k) = \sum_{r=2}^{k+1} (-1)^{k-r}(r-1)! S_2(k, r-1)\sigma_r - \frac{B_{k+1}}{k+1}\gamma - \frac{B_{k+1}}{(k+1)^2} \,,$$

where $S_2(k, r)$ are Stirling numbers of the second kind, and this relation enables us to write an interesting expression of S_k as an integer linear combination of $\sigma_1, \sigma_2, \cdots, \sigma_k$ and a rational constant (the coefficient of γ vanishing by a well-known relation between the Bernoulli numbers). More precisely, we prove (see Section 2 below) the following relations:

$$S_1 = \frac{1}{2} - \sigma_1 \,,$$

and for $k \geq 2$,

$$S_k = D_k - \sigma_1 + \sum_{r=2}^{k} (-1)^r (r-1)! \left(\sum_{j=r-1}^{k-1} \binom{k}{j} S_2(j, r-1) \right) \sigma_r \,, \tag{2}$$

where D_k is the rational number

$$D_k = \frac{H_k}{k+1} + \frac{1}{2k} - \sum_{r=1}^{[\frac{k}{2}]} \frac{B_{2r}}{2r(k+1-2r)} + \sum_{r=1}^{[\frac{k}{2}]} \binom{k}{2r-1} \frac{B_{2r}}{(2r)^2} - \frac{1}{2} \,. \tag{3}$$

We give in Section 3 an alternative expression of D_k deduced from a formula of Blagouchine (cf. Eq. (8)).

1 Proof of formula (1)

This formula may be deduced by expanding in powers of z a relation between generating series given in [C], p. 93. More precisely, we expand the following relation:

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{(-1)^k z^k}{k!} \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{j} \zeta^{\mathcal{R}}(j-k) &= (1-e^z) \sum_{k=0}^{\infty} \frac{(-1)^k z^k}{k!} \zeta'(-k) \\ &+ (1-e^z) \sum_{k=0}^{\infty} \frac{(-1)^k z^k}{k!} \frac{1}{(k+1)^2} \\ &+ \int_0^1 \ln(t+1) e^{-zt} \, dt \,, \end{aligned}$$

with

$$\zeta^{\mathcal{R}}(j-k) = \begin{cases} \gamma & \text{if } j = k+1 \\ \zeta(j-k) - \frac{1}{j-k-1} & \text{otherwise.} \end{cases}$$

Rewriting the series S_k under the following form:

$$S_k = \sum_{j=k+2}^{\infty} \frac{(-1)^{j-k}}{j} \zeta(j-k)\,,$$

and using

$$\zeta'(0) = -\frac{1}{2}\ln(2\pi)\,,$$

the identification of the coefficient of $\dfrac{z^k}{k!}$ in the previous development leads to the relation

$$S_k = C_k + \frac{\gamma}{k+1} - \frac{1}{2}\ln(2\pi) + \sum_{j=1}^{k-1}(-1)^j \binom{k}{j} \zeta'(-j)$$

where C_k is the rational

$$\begin{aligned} C_k = \sum_{j=0}^{k-1}(-1)^j \binom{k}{j} \frac{1}{(j+1)^2} & \\ & - \frac{1}{2k} - \sum_{j=1}^{k-1} \frac{(-1)^j}{k-j} \left(\frac{1-B_{j+1}}{j+1} \right) \\ & \qquad + \left(1-(-1)^k\right) \frac{1}{k+1} \sum_{j=1}^{k+1} \frac{(-1)^{j-1}}{j}\,. \end{aligned}$$

However, since

$$\sum_{j=0}^{k-1}(-1)^j \binom{k}{j} \frac{1}{(j+1)^2} = \frac{H_{k+1}}{k+1} - \frac{(-1)^k}{(k+1)^2}\,,$$

this expression of C_k may be simplified as

$$C_k = \frac{H_k}{k+1} + \frac{1}{2k} + \sum_{j=1}^{k-1} \frac{(-1)^j B_{j+1}}{(k-j)(j+1)}\,.$$

Since $B_1 = -\frac{1}{2}$ and $B_{2r+1} = 0$ for $r \geq 1$, the constant C_k may also be rewritten

$$\begin{aligned} C_k &= \frac{H_k}{k+1} - \sum_{j=1}^{k} \frac{B_j}{j(k+1-j)} \\ &= \frac{H_k}{k+1} + \frac{1}{2k} - \sum_{r=1}^{[\frac{k}{2}]} \frac{B_{2r}}{2r(k+1-2r)}\,. \end{aligned} \tag{4}$$

Hence, formula (1) is established.

2 Proofs of formulae (2) and (3)

We have seen above that

$$S_k = \frac{\gamma}{k+1} - \frac{1}{2}\ln(2\pi) + \sum_{j=1}^{k-1}(-1)^j \binom{k}{j}\zeta'(-j) + C_k \tag{5}$$

where the expression of C_k is given by (4). Furthermore, using [CY], Proposition 2 and Proposition 3, we can write the relations

$$\frac{1}{2}\ln(2\pi) = \sigma_1 + \frac{\gamma}{2} + \frac{1}{2}\,,$$

and

$$\zeta'(-j) = \sum_{r=2}^{j+1}(-1)^{j-r}(r-1)!S_2(j,r-1)\sigma_r - \frac{B_{j+1}}{j+1}\gamma - \frac{B_{j+1}}{(j+1)^2} \quad \text{for } j \geq 1 \ .$$

Then, substituting these relations in (5) gives

$$S_k = \frac{\gamma}{k+1}\sum_{j=0}^{k}\binom{k+1}{j}B_j - \sigma_1 + \sum_{j=1}^{k-1}\sum_{r=2}^{j+1}\binom{k}{j}(-1)^r(r-1)!S_2(j,r-1)\sigma_r$$
$$- \frac{1}{2} + \sum_{j=1}^{k-1}(-1)^{j+1}\binom{k}{j}\frac{B_{j+1}}{(j+1)^2} + C_k\,.$$

The coefficient of γ vanishes since

$$\sum_{j=0}^{k}\binom{k+1}{j}B_j = 0\,,$$

and moreover we may write

$$\sum_{j=1}^{k-1}(-1)^{j+1}\binom{k}{j}\frac{B_{j+1}}{(j+1)^2} = \sum_{r=1}^{[\frac{k}{2}]}\binom{k}{2r-1}\frac{B_{2r}}{(2r)^2}\,.$$

Finally, interchanging the symbols Σ leads to

$$S_k = D_k - \sigma_1 + \sum_{r=2}^{k}(-1)^r(r-1)!\left(\sum_{j=r-1}^{k-1}\binom{k}{j}S_2(j,r-1)\right)\sigma_r$$

with

$$D_k = C_k + \sum_{r=1}^{[\frac{k}{2}]}\binom{k}{2r-1}\frac{B_{2r}}{(2r)^2} - \frac{1}{2}\,,$$

hence, after substitution of C_k by its value given by (4), formulae (2) and (3) are now established.

3 Another expressions of the constants C_k and D_k

Some time ago, Blagouchine ([B]) gave another formula for S_k (cf. [B], Eq. (38), p. 413):

$$S_k = \frac{1}{k} - \frac{\ln(2\pi)}{k+1} + \frac{\gamma}{2} + \\ + \sum_{r=1}^{[\frac{k}{2}]} (-1)^r \binom{k}{2r-1} \frac{(2r)!}{r(2\pi)^{2r}} \zeta'(2r) + \sum_{r=1}^{[\frac{k+1}{2}]-1} (-1)^r \binom{k}{2r} \frac{(2r)!}{2(2\pi)^{2r}} \zeta(2r+1) \quad (6)$$

A differentiation of the functional equation

$$\zeta(s) = 2(2\pi)^{s-1}\Gamma(1-s)\zeta(1-s)\sin(\frac{\pi s}{2}),$$

leads to the relations

$$(-1)^r \frac{(2r)!}{2(2\pi)^{2r}} \zeta(2r+1) = \zeta'(-2r)$$

and

$$(-1)^r \frac{(2r)!}{r(2\pi)^{2r}} \zeta'(2r) = -\zeta'(1-2r) + \frac{B_{2r}}{2r} \left(H_{2r-1} - \gamma - \ln(2\pi)\right).$$

Substituting these relations in (6), a comparison with (5) gives

$$C_k = \frac{1}{k} + \sum_{r=1}^{[\frac{k}{2}]} \binom{k}{2r-1} \frac{B_{2r}}{2r} H_{2r-1}. \quad (7)$$

Yet, the constant D_k in (3) is equal to

$$C_k + \sum_{r=1}^{[\frac{k}{2}]} \binom{k}{2r-1} \frac{B_{2r}}{(2r)^2} - \frac{1}{2},$$

hence, this constant also admits the following new expression:

$$D_k = \frac{1}{k} - \frac{1}{2} + \sum_{r=1}^{[\frac{k}{2}]} \binom{k}{2r-1} \frac{B_{2r}}{2r} H_{2r}. \quad (8)$$

4 Examples

For the first values of k, we compute the following evaluations

$$
\begin{aligned}
S_1 &= \frac{\gamma}{2} - \frac{1}{2}\ln(2\pi) + 1\,, \\
S_2 &= \frac{\gamma}{3} - \frac{1}{2}\ln(2\pi) - 2\zeta'(-1) + \frac{2}{3} \\
S_3 &= \frac{\gamma}{4} - \frac{1}{2}\ln(2\pi) - 3\zeta'(-1) + 3\zeta'(-2) + \frac{7}{12} \\
S_4 &= \frac{\gamma}{5} - \frac{1}{2}\ln(2\pi) - 4\zeta'(-1) + 6\zeta'(-2) - 4\zeta'(-3) + \frac{47}{90} \\
S_5 &= \frac{\gamma}{6} - \frac{1}{2}\ln(2\pi) - 5\zeta'(-1) + 10\zeta'(-2) - 10\zeta'(-3) + 5\zeta'(-4) + \frac{167}{360}\,,
\end{aligned}
$$

and the relations

$$
\begin{aligned}
S_1 &= \frac{1}{2} - \sigma_1\,, \\
S_2 &= \frac{1}{4} - \sigma_1 + 2\sigma_2\,, \\
S_3 &= \frac{5}{24} - \sigma_1 + 6\sigma_2 - 6\sigma_3\,, \\
S_4 &= \frac{13}{72} - \sigma_1 + 14\sigma_2 - 36\sigma_3 + 24\sigma_4\,, \\
S_5 &= \frac{109}{720} - \sigma_1 + 30\sigma_2 - 150\sigma_3 + 240\sigma_4 - 120\sigma_5.
\end{aligned}
$$

5 Final remark

Recently, we have shown (cf. [BC], Eq. (36)) that

$$\kappa_1 := \sum_{n=1}^{\infty} \frac{|b_n|}{n^2} = \sum_{p=2}^{\infty} \frac{(-1)^p}{p} \sum_{n=2}^{\infty} \frac{H_{n-1}}{n^p} = \sum_{p=2}^{\infty} \frac{(-1)^p}{p}\zeta(p,1)\,,$$

where $\zeta(p,q)$ is the double zeta value

$$\zeta(p,q) = \sum_{n>m} \frac{1}{n^p m^q}\,.$$

It would be interesting to investigate the existence of similar relations for the series

$$S_k(q) := \sum_{p=2}^{\infty} (-1)^p \frac{\zeta(p,q)}{p+k} \qquad \text{for } q = 1, 2, \cdots, \text{ and } k = 0, 1, 2, \cdots$$

References

[B] I. V. Blagouchine, Two series expansions for the logarithm of the gamma function involving Stirling numbers and containing only rational coefficients for certain arguments related to π^{-1}, *J. Math. Anal. App*, **44** (2016), 404-434.

[BC] I. V. Blagouchine and M-A. Coppo, A note on some constants related to the zeta function, *Ramanujan J.* (2018), to appear.

[C] B. Candelpergher, Ramanujan summation of divergent series, Lecture Notes in Math. Series, vol. 2185, Springer, 2017.

[CY] M-A. Coppo and P. T. Young, Shifted Mascheroni series and hyperharmonic numbers, *J. Number Theory*, **169** (2016), 1-20.

[SC] H.M. Srivastava and Junesong Choi, Zeta and q-Zeta functions and associated series and integrals, Elsevier, 2012.

[SV] R.J. Singh and V.P. Verma, Some series involving Riemann zeta function, *Yokohama Math. J.*, **31** (1983), 1-4.