A note on some constants related to the zeta–function and their relationship with the Gregory coefficients

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Abstract

In this article, new series for the first and second Stieltjes constants (also known as generalized Euler's constant), as well as for some closely related constants are obtained. These series contain rational terms only and involve the so–called Gregory coefficients, which are also known as (reciprocal) logarithmic numbers, Cauchy numbers of the first kind and Bernoulli numbers of the second kind. In addition, two interesting series with rational terms for Euler's constant γ and the constant $\ln 2\pi$ are given, and yet another generalization of Euler's constant is proposed and various formulas for the calculation of these constants are obtained. Finally, we mention in the paper that almost all the constants considered in this work admit simple representations via the Ramanujan summation.

Keywords: Stieltjes constants, Generalized Euler's constants, Series expansions, Ramanujan summation, Harmonic product of sequences, Gregory's coefficients, Logarithmic numbers, Cauchy numbers, Bernoulli numbers of the second kind, Stirling numbers of the first kind, Harmonic numbers.

I. Introduction and definitions

The zeta-function

$$\zeta(s) \equiv \sum_{n=1}^{\infty} n^{-s} = \prod_{n=1}^{\infty} \left(1 - p_n^{-s}\right)^{-1}, \qquad \begin{array}{l} \operatorname{Re} s > 1 \\ p_n \in \mathbb{P} \equiv \{2, 3, 5, 7, 11, \ldots\} \end{array}$$

is of fundamental and long-standing importance in analytic number theory, modern analysis, theory of L–functions, prime number theory and in a variety of other fields. The ζ–function is a meromorphic function on the entire complex plane, except at point $s = 1$ at which it has one simple pole with residue 1. The coefficients of the regular part of its Laurent series, denoted γ_m,

$$\zeta(s) = \frac{1}{s-1} + \gamma + \sum_{m=1}^{\infty} \frac{(-1)^m (s-1)^m}{m!} \gamma_m, \qquad s \neq 1. \tag{1}$$

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where γ is Euler's constant[1], and those of the Maclaurin series δ_m

$$\zeta(s) \;=\; \frac{1}{s-1} + \frac{1}{2} + \sum_{m=1}^{\infty} \frac{(-1)^m s^m}{m!}\delta_m\,, \qquad\qquad s \neq 1. \tag{2}$$

are of special interest and have been widely studied in literature, see e.g. [25], [1, vol. I, letter 71 and following], [23, p. 166 *et seq.*], [28, 29, 24, 26, 2, 11, 5, 33, 27, 34, 20, 6, 32, 37, 19]. Coefficients γ_m are usually called *Stieltjes constants* or *generalized Euler's constants* (both names being in use), while δ_m do not possess a special name.[2] It may be shown with the aid of the Euler–MacLaurin summation that γ_m and δ_m may be also given by the following asymptotic representations

$$\gamma_m = \lim_{n\to\infty}\left\{\sum_{k=1}^{n}\frac{\ln^m k}{k} - \frac{\ln^{m+1} n}{m+1}\right\}, \quad m = 1,2,3,\ldots\,, \quad ^3 \tag{3}$$

and

$$\delta_m \;=\lim_{n\to\infty}\left\{\sum_{k=1}^{n}\ln^m k \;-\; n\,m!\sum_{k=0}^{m}(-1)^{m+k}\frac{\ln^k n}{k!} \;-\; \frac{\ln^m n}{2}\right\}, \quad m = 1,2,3,\ldots\,, \quad ^4 \tag{4}$$

These representations may be translated into these simple expressions

$$\gamma_m \;=\sum_{k\geqslant 1}^{\mathcal{R}}\frac{\ln^m k}{k}\,, \qquad \delta_m \;=\sum_{k\geqslant 1}^{\mathcal{R}}\ln^m k\,, \qquad m = 1,2,3,\ldots$$

where $\overset{\mathcal{R}}{\sum}$ stands for the sum of the series in the sense of the Ramanujan summation of divergent series [5]. Due to the reflection formula for the zeta-function $\zeta(1-s) = 2\zeta(s)\Gamma(s)(2\pi)^{-s}\cos\frac{1}{2}\pi s$, numbers δ_m and γ_m are related to each other polynomially and also involve Euler's constant γ and the values of the ζ–function at naturals. For the first values of m, this gives

$$\delta_1 \;=\; \tfrac{1}{2}\ln 2\pi - 1 \;=\; -0.08106146679\ldots$$

$$\delta_2 \;=\; \gamma_1 + \tfrac{1}{2}\gamma^2 - \tfrac{1}{2}\ln^2 2\pi - \tfrac{1}{24}\pi^2 + 2 \;=\; -0.006356455908\ldots \quad ^6$$

$$\delta_3 \;=\; -\tfrac{3}{2}\gamma_2 - 3\gamma_1\gamma - \gamma^3 - \left(3\gamma_1 + \tfrac{3}{2}\gamma^2 - \tfrac{1}{8}\pi^2\right)\ln 2\pi + \zeta(3) + \tfrac{1}{2}\ln^3 2\pi - 6 \;=\; +0.004711166862\ldots$$

and conversely

$$\gamma_1 \;=\; \delta_2 + 2\delta_1^2 + 4\delta_1 - \tfrac{1}{2}\gamma + \tfrac{1}{24}\pi^2 \;=\; -0.07281584548\ldots$$

$$\begin{aligned}\gamma_2 \;&=\; -\tfrac{2}{3}\delta_3 - 2\delta_2(\gamma+2) - 4\delta_1\delta_2 - \tfrac{16}{3}\delta_1^3 - 4\delta_1^2(\gamma+4) - 8\delta_1(\gamma+1) - \tfrac{1}{12}\gamma\pi^2 + \tfrac{1}{3}\gamma^3 + \tfrac{2}{3}\zeta(3) - \tfrac{4}{3}\\ &=\; -0.009690363192\ldots\end{aligned}$$

Relationships between higher–order coefficients become very cumbersome, but may be found via a semi–recursive procedure described in [3]. Altough there exist numerous representations for γ_m

[1]We recall that $\gamma = \lim_{n\to\infty}(H_n - \ln n) = -\Gamma'(1) = 0.5772156649\ldots$, where H_n is the harmonic number.

[2]It follows from (2) that $\delta_m = (-1)^m\{\zeta^{(m)}(0) + m!\}$

[3]This representation is very old and was already known to Adolf Pilz, Stieltjes, Hermite and Jensen [6, p. 366].

[4]A slightly different expression for δ_m was given earlier by and by Lehmer [32, Eq. (5), p. 266], Sitaramachandrarao [37, Theorem 1], Finch [23, p. 168 *et seq.*] and Connon [19, Eqs. (2.15), (2.19)]. The formula given by these writers differ from our (4) by the presence of the definite integral of $\ln^m x$ taken over $[1, n]$, which in fact may be reduced to a finite combination of logarithms and factorials.

[5]For more details on the *Ramanujan summation*, see [4, Ch. 6], [15, 13, 16, 12].

[6]This expression for δ_2 was found by Ramanujan, see e.g. [4, (18.2)].

and δ_m, no convergent series with rational terms only are known for them (unlike for Euler's constant γ, see e.g. [6, Sect. 3], or for various expressions containing it [8, p. 413, Eqs. (41), (45)–(47)]). Recently, divergent envelopping series for γ_m containing rational terms only have been obtained in [6, Eqs. (46)–(47)]. In this paper, by continuing the same line of investigation, we derive convergent series representations with rational coefficients for $\gamma_1, \delta_1, \gamma_2$ and δ_2, and also find two new series of the same type for Euler's constant γ and $\ln 2\pi$ respectively. These series are not simple and involve a product of *Gregory coefficients* G_n, which are also known as *(reciprocal) logarithmic numbers*, *Bernoulli numbers of the second kind* b_n, and normalized *Cauchy numbers of the first kind* $C_{1,n}$. Similar expressions for higher–order constants γ_m and δ_m may be obtained by the same procedure, using the harmonic product of sequences introduced in [14], but are quite cumbersome. Since the Stieltjes constants γ_m generalize Euler's constant γ and since our series contain the product of G_n, these new series may also be seen as the generalization of the famous Fontana–Mascheroni series

$$\gamma = \sum_{n=1}^{\infty} \frac{|G_n|}{n} = \frac{1}{2} + \frac{1}{24} + \frac{1}{72} + \frac{19}{2880} + \frac{3}{800} + \frac{863}{362\,880} + \frac{275}{169\,344} + \ldots \tag{5}$$

which is the first known series representation for Euler's constant having rational terms only, see [8, pp. 406, 413, 429], [6, p. 379]. In Appendix A, we introduce yet another set of constants $\kappa_m = \sum_{n\geqslant 1} |G_n|\, n^{-m-1}$, which also generalize Euler's constant γ. These numbers, similarly to γ_m, coincide with Euler's constant at $m = 0$ and have various interesting series and integral representations, none of them being reducible to the "classical" mathematical constants.

II. Series expansions

II.1. Preliminaries

Since the results, that we come to present here, are essentially based on the Gregory coefficients and Stirling numbers, it may be useful to briefly recall their definition and properties. Gregory numbers, denoted below G_n, are rational alternating $G_1 = {+1/2}$, $G_2 = {-1/12}$, $G_3 = {+1/24}$, $G_4 = {-19/720}$, $G_5 = {+3/160}$, $G_6 = {-863/60\,480}, \ldots$, decreasing in absolute value, and are also closely related to the theory of finite differences; they behave as $\left(n \ln^2 n\right)^{-1}$ at $n \to \infty$ and may be bounded from below and above accordingly to [8, Eqs. (55)–(56)]. They may be defined either via their generating function

$$\frac{z}{\ln(1+z)} = 1 + \sum_{n=1}^{\infty} G_n\, z^n, \qquad |z| < 1\,, \tag{6}$$

or recursively

$$G_n = \frac{(-1)^{n+1}}{n+1} + \sum_{l=1}^{n-1} \frac{(-1)^{n-l} G_l}{n+1-l}\,, \qquad G_1 = \frac{1}{2}\,, \quad n = 2,3,4,\ldots \tag{7}$$

or explicitly[7]

$$G_n = \frac{1}{n!} \int_0^1 x\,(x-1)\,(x-2)\cdots(x-n+1)\,dx\,, \qquad n = 1,2,3,\ldots \tag{8}$$

[7] For more information about G_n, see [8, pp. 410–415], [6, p. 379], [9], and the literature given therein (nearly 50 references).

Throughout the paper, we also make use of the Stirling numbers of the first kind, which we denote below by $S_1(n,l)$. Since there are different definitions and notations for them, we specify that in our definition they are simply the coefficients in the expansion of falling factorial

$$x\,(x-1)\,(x-2)\cdots(x-n+1) \;=\; \sum_{l=1}^{n} S_1(n,l)\cdot x^l\,, \qquad n=1,2,3,\ldots \tag{9}$$

and may equally be defined via the generating function

$$\frac{\ln^l(1+z)}{l!} \;=\sum_{n=l}^{\infty}\frac{S_1(n,l)}{n!}z^n \;=\sum_{n=0}^{\infty}\frac{S_1(n,l)}{n!}z^n\,, \qquad |z|<1\,,\quad l=0,1,2,\ldots \tag{10}$$

It is important to note that $\operatorname{sgn}\left[S_1(n,l)\right] \;=\; (-1)^{n\pm l}$. We also recall that the Stirling numbers of the first kind and the Gregory coefficients are linked by the following relation[8]

$$G_n \;=\; \frac{1}{n!}\sum_{l=1}^{n}\frac{S_1(n,l)}{l+1}, \qquad n=1,2,3,\ldots \tag{11}$$

II.2. Some auxiliary lemmas

Before we proceed with the series expansions for δ_m and γ_m, we need to prove several useful lemmas.

Lemma 1. *For each natural number k, let*

$$\sigma_k \equiv \sum_{n=1}^{\infty}\frac{|\,G_n|}{n+k}\,,$$

the following equality holds

$$\sigma_k \;=\; \frac{1}{k}+\sum_{m=1}^{k}(-1)^m\binom{k}{m}\ln(m+1)\,, \qquad k=1,2,3,\ldots \tag{12}$$

Proof. By using (11) and by making use of the generating equation for the Stirling numbers of the first kind (10), we obtain

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{|\,G_n|}{n+k} \;&=\sum_{n=1}^{\infty}\frac{1}{n!}\sum_{l=1}^{n}\frac{(-1)^{l+1}\big|S_1(n,l)\big|}{l+1}\cdot\underbrace{\int_0^1 x^{n+k-1}dx}_{1/(n+k)} \;=\; -\sum_{l=1}^{\infty}\frac{1}{(l+1)!}\int_0^1 x^{k-1}\ln^l(1-x)\,dx \\
&=\sum_{l=1}^{\infty}\frac{(-1)^{l+1}}{(l+1)!}\int_0^{\infty}\big(1-e^{-t}\big)^{k-1}t^l e^{-t}\,dt \;=\sum_{l=1}^{\infty}\frac{(-1)^{l+1}}{(l+1)!}\sum_{m=0}^{k-1}(-1)^m\binom{k-1}{m}\underbrace{\int_0^{\infty}e^{-t(m+1)}t^l\,dt}_{l!\,(m+1)^{-l-1}} \\
&=\sum_{l=1}^{\infty}\frac{(-1)^{l+1}}{l+1}\sum_{m=0}^{k-1}(-1)^m\binom{k-1}{m}\frac{1}{(m+1)^{l+1}} \;=\sum_{m=0}^{k-1}(-1)^m\binom{k-1}{m}\left\{\frac{1}{m+1}-\ln\frac{m+2}{m+1}\right\}
\end{aligned} \tag{13}$$

[8] More information and references (more than 60) on the Stirling numbers of the first kind may be found in [8, Sect. 2.1] and [6, Sect. 1.2]. We also note that our definitions for the Stirling numbers agree with those adopted by MAPLE or MATHEMATICA: our $S_1(n,l)$ equals to `Stirling1(n,l)` from the former and to `StirlingS1[n,l]` from the latter.

where at the last stage we made a change of variable $x = 1 - e^{-t}$ and used the well-known formula for the Γ–function. But since

$$\sum_{m=0}^{k-1} \frac{(-1)^m}{m+1} \binom{k-1}{m} = \frac{1}{k}, \qquad \text{and} \qquad \binom{k-1}{m} + \binom{k-1}{m-1} = \binom{k}{m},$$

the last finite sum in (13) reduces to (12)[9]. □

Remark 1. *One may show* [10]*that σ_k may also be written in terms of the Ramanujan summation:*

$$\sigma_k = \sum_{n\geqslant 1}^{\mathcal{R}} \frac{\Gamma(k+1)\Gamma(n)}{\Gamma(n+k+1)} = \sum_{n\geqslant 1}^{\mathcal{R}} B(k+1, n). \tag{14}$$

where B stands for the Euler beta-function.

Lemma 2. *Let $a = (a(1), a(2), \ldots, a(n), \ldots)$ be a sequence of complex numbers. The following identity is true for all nonnegative integers n:*

$$\sum_{l=0}^{n} (-1)^l \binom{n}{l} \frac{a(l+1)}{l+1} = \frac{1}{n+1} \sum_{k=0}^{n} \sum_{l=0}^{k} (-1)^l \binom{k}{l} a(l+1) \tag{15}$$

In particular, if $a = \ln^m$ for any natural m, then this identity reduces to

$$\sum_{l=0}^{n} (-1)^l \binom{n}{l} \frac{\ln^m(l+1)}{l+1} = \frac{1}{n+1} \sum_{k=1}^{n} \sum_{l=1}^{k} (-1)^l \binom{k}{l} \ln^m(l+1) \tag{16}$$

Proof. Formula (15) is an explicit translation of [14, Proposition 7]. □

Lemma 3. *For all natural m*

$$\gamma_m = \sum_{n=1}^{\infty} \frac{|G_{n+1}|}{n+1} \sum_{k=1}^{n} \sum_{l=1}^{k} (-1)^l \binom{k}{l} \ln^m(l+1) \tag{17}$$

$$\delta_m = \sum_{n=1}^{\infty} |G_{n+1}| \sum_{l=1}^{n} (-1)^l \binom{n}{l} \ln^m(l+1) \tag{18}$$

Proof. Using this representation for the ζ–function

$$\zeta(s) = \frac{1}{s-1} + \sum_{n=0}^{\infty} |G_{n+1}| \sum_{k=0}^{n} (-1)^k \binom{n}{k} (k+1)^{-s}, \qquad s \neq 1,$$

see e.g. [6, pp. 382–383], [7], we first have

$$\gamma_m = \sum_{n=0}^{\infty} |G_{n+1}| \sum_{l=0}^{n} (-1)^l \binom{n}{l} \frac{\ln^m(l+1)}{l+1}$$

and

$$\delta_m = \sum_{n=0}^{\infty} |G_{n+1}| \sum_{l=0}^{n} (-1)^l \binom{n}{l} \ln^m(l+1).$$

Then formula (17) follows from property (16).

□

[9]For a different proof and a slightly more general result, see [21, Proposition 1].
[10]See [12] Eq. (4.31).

II.3. Series with rational terms for the first Stieltjes constant γ_1 and for the coefficient δ_1

Theorem 1. *The first Stieltjes constant γ_1 may be given by the following series*

$$\begin{aligned}\gamma_1 &= \frac{3}{2}-\frac{\pi^2}{6}+\sum_{n=2}^{\infty}\left[\frac{|G_n|}{n^2}+\sum_{k=1}^{n-1}\frac{|G_k\,G_{n+1-k}|\cdot(H_n-H_k)}{n+1-k}\right]\\ &= \frac{3}{2}-\frac{\pi^2}{6}+\frac{1}{32}+\frac{5}{432}+\frac{1313}{207\,360}+\frac{42\,169}{10\,368\,000}+\frac{137\,969}{48\,384\,000}+\frac{1\,128\,119}{5\,334\,336\,00}+\ldots \end{aligned}\tag{19}$$

containing π^2 and positive rational coefficients only. Using Euler's formula $\pi^2=6\sum n^{-2}$, the latter may be reduced to a series with rational terms only.

Proof. By (17) with $m=1$, one has

$$\gamma_1=\sum_{n=1}^{\infty}\frac{|G_{n+1}|}{n+1}\sum_{k=1}^{n}\sum_{m=1}^{k}(-1)^m\binom{k}{m}\ln(m+1)$$

and by (12),

$$\frac{1}{n+1}\sum_{k=1}^{n}\sum_{m=1}^{k}(-1)^m\binom{k}{m}\ln(m+1)=\frac{1}{n+1}\sum_{k=1}^{n}\sigma_k-\frac{H_{n+1}}{n+1}+\frac{1}{(n+1)^2}\,.$$

Thus

$$\begin{aligned}\gamma_1 &= \sum_{n=1}^{\infty}\frac{|G_{n+1}|}{n+1}\sum_{k=1}^{n}\sigma_k-\sum_{n=0}^{\infty}\frac{|G_{n+1}|\,H_{n+1}}{n+1}+\sum_{n=0}^{\infty}\frac{|G_{n+1}|}{(n+1)^2}\\ &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{|G_{n+1}\,G_m|(H_{m+n}-H_m)}{n+1}-\zeta(2)+1+\sum_{n=1}^{\infty}\frac{|G_n|}{n^2}\end{aligned}$$

since

$$\sum_{n=1}^{\infty}\frac{|G_n|\cdot H_n}{n}=\zeta(2)-1$$

see e.g. [41, p. 2952, Eq. (1.3)], [17, p. 20,Eq. (3.6)], [13, p. 307, Eq. for $F_0(2)$], [8, p. 413, Eq. (44)]. Rearranging the double absolutely convergent series as follows

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{|G_{n+1}\,G_m|\cdot(H_{m+n}-H_m)}{n+1}=\sum_{n=2}^{\infty}\sum_{k=1}^{n-1}\frac{|G_k\,G_{n+1-k}|\cdot(H_n-H_k)}{n+1-k}$$

we finally arrive at (19). □

Remark 2. *It seems that the sum $\kappa_1\equiv\sum|G_n|\,n^{-2}=0.5290529699\ldots$ cannot be reduced to the "standard" mathematical constants. However, it admits several interesting representations, which we give in Appendix A.*

Theorem 2. *The first MacLaurin coefficient $\delta_1=\frac{1}{2}\ln 2\pi-1$ admits a series representation similar to that for γ_1, namely*

$$\delta_1=\sum_{n=1}^{\infty}\frac{1}{n}\sum_{k=1}^{n}|G_k\,G_{n+1-k}|+\frac{1-\ln 2\pi}{2}\tag{20}$$

Proof. Proceeding analogously to the previous case and recalling that

$$\sum_{n=2}^{\infty}\frac{|G_n|}{n-1} \;=\; -\frac{\gamma+1-\ln 2\pi}{2}$$

see e.g. [8, p. 413, Eq. (41)], [42, Corollary 9], we have

$$\begin{aligned}
\delta_1 &= \sum_{n=0}^{\infty}\big|G_{n+1}\big|\sum_{l=0}^{n}(-1)^l\binom{n}{l}\ln(l+1) \\
&= \sum_{n=1}^{\infty}\big|G_{n+1}\big|\left(\sigma_n-\frac{1}{n}\right) = \sum_{n=1}^{\infty}\big|G_{n+1}\big|\,\sigma_n-\sum_{n=1}^{\infty}\frac{\big|G_{n+1}\big|}{n} \\
&= \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{\big|G_{n+1}\,G_k\big|}{n+k}+\frac{\gamma+1-\ln 2\pi}{2} \;=\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{\big|G_n\,G_k\big|}{n+k-1}+\frac{1-\ln 2\pi}{2} \qquad (21)\\
&= \sum_{n=1}^{\infty}\frac{1}{n}\sum_{k=1}^{n}\big|G_k\,G_{n+1-k}\big|+\frac{1-\ln 2\pi}{2}
\end{aligned}$$

where in (21) we could eliminate γ thanks to the fact that $G_1 = 1/2$ and that the sum of $|G_n|/n$ over all natural n equals precisely Euler's constant, see (5). □

Corollary 1. *The constant* $\ln 2\pi$ *has the following beautiful series representation with rational terms only and containing a product of Gregory coefficients*

$$\ln 2\pi \;=\; \frac{3}{2}+\sum_{n=1}^{\infty}\frac{1}{n}\sum_{k=1}^{n}\big|G_k\,G_{n+1-k}\big| \;=\; \frac{3}{2}+\frac{1}{4}+\frac{1}{24}+\frac{7}{432}+\frac{1}{120}+\frac{43}{8640}+\frac{79}{24\,192}+\ldots \qquad (22)$$

which directly follows from (20)*. From the latter, one can also readily derive a series with rational coefficients only for* $\ln\pi$ *(for instance, with the aids of the Mercator series).*

Corollary 2. *Euler's constant* γ *admits the following series representation with rational terms*

$$\begin{aligned}
\gamma \;&=\; 2\ln 2\pi-3-2\sum_{n=1}^{\infty}\frac{1}{n+1}\sum_{k=1}^{n}\big|G_k\,G_{n+2-k}\big| \;=\; 2\ln 2\pi-3-\frac{1}{24}-\frac{1}{54}-\frac{29}{2880}- \\
&-\frac{67}{10\,800}-\frac{1507}{362\,880}-\frac{3121}{1\,058\,400}-\frac{12\,703}{15\,806\,080}-\frac{164\,551}{97\,977\,600}-\ldots \qquad (23)
\end{aligned}$$

which seems to be undiscovered yet. This curious series straightforwardly follows from (21)*, from the transformation*

$$\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\frac{\big|G_{k+1}\,G_n\big|}{n+k} \;=\sum_{n=1}^{\infty}\frac{1}{n+1}\sum_{k=1}^{n}\big|G_k\,G_{n+2-k}\big|$$

and from Eq. (5)*.*

II.4. Generalizations to the second–order coefficients δ_2 *and* γ_2 *via an application of the harmonic product*

We recall the main properties of the harmonic product of sequences which are stated and proved in [14]). If $a = (a(1), a(2), \ldots)$ and $b = (b(1), b(2), \ldots)$ are two sequences in $\mathbb{C}^{\mathbb{N}^*}$, then the harmonic product $a \bowtie b$ admits the explicit expression:

$$(a \bowtie b)(m+1) = \sum_{0\leqslant l\leqslant k\leqslant m}(-1)^{k-l}\binom{m}{k}\binom{k}{l}\,a(k+1)b(m+1-l)\,, \qquad m=0,1,2,\ldots \qquad (24)$$

For small values of m, this gives:

$$
\begin{aligned}
&(a \bowtie b)(1) = a(1)b(1)\,,\\
&(a \bowtie b)(2) = a(2)b(1) + a(1)b(2) - a(2)b(2)\,,\\
&(a \bowtie b)(3) = a(3)b(1) + a(1)b(3) + 2a(2)b(2) - 2a(3)b(2) - 2a(2)b(3) + a(3)b(3)\\
&\qquad \text{etc.}
\end{aligned}
$$

The harmonic product $\bowtie$ is associative and commutative.

Let D be the operator defined by

$$D(a)(m+1) = \sum_{j=0}^{m} (-1)^j \binom{m}{j} a(j+1)\,, \qquad m = 0, 1, 2, \ldots$$

then $D = D^{-1}$ and the harmonic product satisfies the following property:

$$D(ab) = D(a) \bowtie D(b) \tag{25}$$

In particular, if $a(m) = \ln m$, then $D(a)(1) = \ln 1 = 0$, and by (12),

$$D(a)(m+1) = \sum_{j=1}^{m} (-1)^j \binom{m}{j} \ln(j+1) = \sigma_m - \frac{1}{m}\,, \qquad m = 1, 2, 3, \ldots \tag{26}$$

Therefore, if $a = \ln$ then, by (24), (25), and (26), the following identity holds

$$D(\ln^2)(m+1) = \sum_{\substack{0 \leqslant l \leqslant k \leqslant m \\ k \neq 0 \\ l \neq m}} (-1)^{k-l} \binom{m}{k} \binom{k}{l} \left(\sigma_k - \frac{1}{k}\right)\left(\sigma_{m-l} - \frac{1}{m-l}\right) \tag{27}$$

From this identity results the following theorem:

Theorem 3. *The second coefficients γ_2 and δ_2 may be given by the following series*

$$\gamma_2 = \sum_{n=1}^{\infty} \frac{|G_{n+1}|}{n+1} \sum_{\substack{0 \leqslant l \leqslant k \leqslant m \leqslant n \\ k \neq 0 \\ l \neq m}} (-1)^{k-l} \binom{m}{k} \binom{k}{l} \left(\sigma_k - \frac{1}{k}\right)\left(\sigma_{m-l} - \frac{1}{m-l}\right)$$

and

$$\delta_2 = \sum_{m=1}^{\infty} |G_{m+1}| \sum_{\substack{0 \leqslant l \leqslant k \leqslant m \\ k \neq 0 \\ l \neq m}} (-1)^{k-l} \binom{m}{k} \binom{k}{l} \left(\sigma_k - \frac{1}{k}\right)\left(\sigma_{m-l} - \frac{1}{m-l}\right),$$

respectively.

Proof. Applying (17) with $m = 2$, and using equation (27), we can write the following equalities:

$$\begin{aligned}
\gamma_2 &= \sum_{n=1}^{\infty} \frac{|G_{n+1}|}{n+1} \sum_{m=1}^{n} \sum_{j=1}^{m} (-1)^j \binom{m}{j} \ln^2(j+1) \\
&= \sum_{n=1}^{\infty} \frac{|G_{n+1}|}{n+1} \sum_{m=1}^{n} D(\ln^2)(m+1) \\
&= \sum_{n=1}^{\infty} \frac{|G_{n+1}|}{n+1} \sum_{m=1}^{n} \sum_{\substack{0\leqslant l\leqslant k\leqslant m \\ k\neq 0 \\ l\neq m}} (-1)^{k-l} \binom{m}{k} \binom{k}{l} \left(\sigma_k - \frac{1}{k}\right)\left(\sigma_{m-l} - \frac{1}{m-l}\right),
\end{aligned}$$

and for δ_2,

$$\begin{aligned}
\delta_2 &= \sum_{n=0}^{\infty} |G_{n+1}| \sum_{j=0}^{n} (-1)^j \binom{n}{j} \ln^2(j+1) \\
&= \sum_{n=1}^{\infty} |G_{n+1}| D(\ln^2)(n+1) \\
&= \sum_{m=1}^{\infty} |G_{m+1}| \sum_{\substack{0\leqslant l\leqslant k\leqslant m \\ k\neq 0 \\ l\neq m}} (-1)^{k-l} \binom{m}{k} \binom{k}{l} \left(\sigma_k - \frac{1}{k}\right)\left(\sigma_{m-l} - \frac{1}{m-l}\right).
\end{aligned}$$

□

By following the same method, one may also obtain expressions for higher–order constants γ_m and δ_m. However, the resulting expressions are more theoretical than practical.

Appendice A. Yet another generalization of Euler's constant

The numbers $\kappa_p \equiv \sum |G_n|\, n^{-p-1}$, where the summation extends over $n = [1, \infty)$, may also be regarded as one of the possible generalizations of Euler's constant (since $\kappa_0 = \gamma_0 = \gamma$ and $\kappa_{-1} = \gamma_{-1} = 1$).[11,12] These constants, which do not seem to be reducible to the "classical mathematical constants", admit several interesting representations as stated in the following proposition.

Proposition 1. *Generalized Euler's constants $\kappa_p \equiv \sum |G_n|\, n^{-p-1}$, where the summation extends over $n =$*

[11]Numbers κ_0 and κ_{-1} are found for the values to which Fontana–Mascheroni and Fontana series converge respectively [8, pp. 406, 410].

[12]Other possible generalizations of Euler's constant were proposed by Briggs, Lehmer, Dilcher and some other authors [10, 31, 38, 35, 39, 22].

$[1,\infty)$, *admit the following representations:*

$$\kappa_p \;=\; \frac{(-1)^p}{\Gamma(p+1)}\int\limits_0^1 \left\{\frac{1}{\ln(1-x)}+\frac{1}{x}\right\}\ln^p x\,dx\,,\quad \operatorname{Re} p > -1\,. \tag{28}$$

$$= \underbrace{\int\limits_0^1 \cdots \int\limits_0^1}_{p\text{-fold}} \left\{\operatorname{li}\left(1-\prod_{k=1}^p x_k\right)+\gamma+\sum_{k=1}^p \ln x_k\right\}\frac{dx_1\cdots dx_p}{x_1\cdots x_p}\,,\quad p=1,2,3,\ldots \tag{29}$$

$$= \sum_{k=2}^\infty \frac{(-1)^k}{k}\sum_{n=p+1}^\infty \frac{\left|S_1(n,p+1)\right|}{n!\,n^{k-1}}\,,\quad p=0,1,2,\ldots \tag{30}$$

$$= \sum_{k=2}^\infty \frac{(-1)^k}{k}\sum_{n=p}^\infty \frac{P_p\big(H_n^{(1)},-H_n^{(2)},\ldots,(-1)^{p-1}H_n^{(p)}\big)}{(n+1)^k}\,,\quad p=0,1,2,\ldots \tag{31}$$

$$= \sum_{n\geqslant 1}^{\mathcal{R}} \frac{1}{n}\sum_{n\geqslant n_1\geqslant\ldots\geqslant n_p\geqslant 1}\frac{1}{n_1\ldots n_p} = \sum_{n\geqslant 1}^{\mathcal{R}} \frac{P_p\big(H_n^{(1)},H_n^{(2)},\ldots,H_n^{(p)}\big)}{n}\,,\quad p=1,2,3,\ldots \tag{32}$$

where li *is the integral logarithm function,* $H_n^{(m)} \equiv \sum_{k=1}^n k^{-m}$ *stands for the generalized harmonic number and* P_m *denotes the modified Bell polynomials*

$$P_0=1,\quad P_1(x_1)=x_1,\quad P_2(x_1,x_2)=\tfrac{1}{2}\left(x_1^2+x_2\right),$$
$$P_3(x_1,x_2,x_3)=\tfrac{1}{6}\left(x_1^3+3x_1x_2+2x_3\right),\quad \ldots$$ [13]

In particular, for the series κ_1 *which we encountered in Theorem 1 and Remark 2, this gives*

$$\kappa_1 = \sum_{n=1}^\infty \frac{|G_n|}{n^2} = -\int\limits_0^1 \left\{\frac{1}{\ln(1-x)}+\frac{1}{x}\right\}\ln x\,dx \tag{33}$$

$$= \int\limits_0^1 \frac{-\operatorname{li}(1-x)+\gamma+\ln x}{x}\,dx = \int\limits_0^\infty \left\{-\operatorname{li}\left(1-e^{-x}\right)+\gamma-x\right\}dx \tag{34}$$

$$= \sum_{k=2}^\infty \frac{(-1)^k}{k}\sum_{n=2}^\infty \frac{H_{n-1}}{n^k} = \sum_{n\geqslant 1}^{\mathcal{R}} \frac{H_n}{n}\,. \tag{35}$$

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[13] More generally, these polynomials are defined by the generating function: $\exp\left(\sum_{n=1}^\infty x_n \frac{t^n}{n}\right) = \sum_{m=0}^\infty P_m(x_1,\cdots,x_m)\,t^m$.

Moreover, we also have

$$\kappa_1 = \gamma_1 + \frac{\gamma^2}{2} - \frac{\pi^2}{12} + \int_0^1 \frac{\Psi(x+1)+\gamma}{x}\,dx \tag{36}$$

$$= \frac{\gamma^2}{2} + \frac{\pi^2}{12} - \frac{1}{2} + \frac{1}{2}\int_0^1 \Psi^2(x+1)\,dx \tag{37}$$

where Ψ *denotes the digamma function (logarithmic derivative of the* Γ*–function).*

Proof of formula (28)

We first write the generating equation for Gregory's coefficients, Eq. (6), in the following form

$$\frac{1}{\ln(1-x)} + \frac{1}{x} = \sum_{n=1}^{\infty} |G_n|\, x^{n-1}, \qquad |x| < 1. \tag{38}$$

Multiplying both sides by $\ln^p x$, integrating over the unit interval and changing the order of summation and integration[14] yields:

$$\int_0^1 \left\{ \frac{1}{\ln(1-x)} + \frac{1}{x} \right\} \ln^p x\, dx = \sum_{n=1}^{\infty} |G_n| \int_0^1 x^{n-1} \ln^p x\, dx, \qquad \operatorname{Re} p > -1. \tag{39}$$

The last integral may be evaluated as follows. Considering Legendre's integral $\Gamma(p+1) = \int t^p e^{-t} dt$ taken over $[0,\infty)$ and making a change of variable $t = -(s+1)\ln x$, we have

$$\int_0^1 x^s \ln^p x\, dx = (-1)^p \frac{\Gamma(p+1)}{(s+1)^{p+1}}, \qquad \begin{array}{l} \operatorname{Re} s > -1 \\ \operatorname{Re} p > -1 \end{array}. \tag{40}$$

Inserting this formula into (39) and setting $n-1$ instead of s, yields (28).

Proof of formula (29)

Putting in (38) $x = x_1 x_2 \cdots x_{p+1}$ and integrating over the volume $[0,1]^{p+1}$, where $p \in \mathbb{N}$, on the one hand, we have

$$\underbrace{\int_0^1 \cdots \int_0^1}_{(p+1)\text{-fold}} \sum_{n=1}^{\infty} |G_n| \big(x_1 x_2 \cdots x_{p+1}\big)^{n-1} dx_1 \cdots dx_{p+1} = \sum_{n=1}^{\infty} \frac{|G_n|}{n^{p+1}} \tag{41}$$

On the other hand

$$\int_0^1 \left\{ \frac{1}{\ln(1-xy)} + \frac{1}{xy} \right\} dx = -\frac{\operatorname{li}(1-y) - \gamma - \ln y}{y}$$

Taking instead of y the product $x_1 x_2 \cdots x_p$ and setting $x = x_{p+1}$, and then integrating p times over the unit hypercube and equating the result with (41) yields (29).

[14] The series being uniformly convergent.

Proof of formula (36)–(37)

Using Eqs. (3.21)–(3.23) of [12] we obtain (36)–(37).

Proof of formulas (30)–(31)

Writing in the generating equation (10) x instead of z, multiplying it by $\ln^m x/x$ and integrating over the unit interval, we obtain the following relation[15]

$$\Omega(k,m) = (-1)^{m+k} m!\, k! \sum_{n=k}^{\infty} \frac{\big|S_1(n,k)\big|}{n!\, n^{m+1}}$$

where

$$\Omega(k,m) \equiv \int_0^1 \frac{\ln^k(1-x)\,\ln^m x}{x}\,dx\,, \qquad \begin{array}{l} k\in\mathbb{N} \\ m\in\mathbb{N} \end{array}$$

By integration by parts, it may be readily shown that

$$\Omega(k,m) = \frac{k}{m+1}\Omega(m+1,k-1)$$

and thus, we deduce the duality formula:

$$\sum_{n=k}^{\infty} \frac{\big|S_1(n,k)\big|}{n!\, n^{m+1}} = \sum_{n=m+1}^{\infty} \frac{\big|S_1(n,m+1)\big|}{n!\, n^{k}}\,.$$

Now, writing

$$x + \ln(1-x) = -\sum_{k=1}^{\infty} \frac{\ln^{k+1}(1-x)}{(k+1)!}$$

we obtain

$$\begin{aligned} \int_0^1 \left\{ \frac{1}{\ln(1-x)} + \frac{1}{x} \right\} \ln^m x\, dx &= -\int_0^1 \sum_{k=1}^{\infty} \frac{\ln^{k+1}(1-x)}{(k+1)!} \cdot \frac{\ln^m x}{\ln(1-x)} \cdot \frac{dx}{x} \\ &= -\sum_{k=1}^{\infty} \frac{\Omega(k,m)}{(k+1)!} = (-1)^m m! \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(k+1)} \sum_{n=m+1}^{\infty} \frac{\big|S_1(n,m+1)\big|}{n!\, n^k} \end{aligned}$$

which is identical with (30) if setting $m = p$. Furthermore, it is well known that

$$\frac{\big|S_1(n+1,m+1)\big|}{n!} = P_m\big(H_n^{(1)}, -H_n^{(2)}, \ldots, (-1)^{m-1} H_n^{(m)}\big)\,.$$

see [18, p. 217], [36, p. 1395], [30, p. 425, Eq. (43)], [6, Eq. (16)], which immediately gives (31) and completes the proof.

Proof of formula (32)

This formula straightforwardly follows form the fact that $\kappa_p = F_p(1)$, see [13, p. 307, 318 *et seq.*], where $F_p(s)$ is the special function introduced in the above–cited reference.

[15] See also [40, Theorem 2.7].

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