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TOWERS OF TORSORS OVER A FIELD

MARCO ANTEI, INDRANIL BISWAS, AND MICHEL EMSALEM

ABSTRACT. Let X be a projective, connected and smooth scheme defined over an algebraically closed field k . In this paper we prove that a tower of finite torsors (i.e., under the action of finite k -group schemes) can be dominated by a single finite torsor. Let G be any finite k -group scheme and Y any G -torsor over X pointed in $y \in Y(k)$; we define over Y , which may not be reduced, in a very natural way the categories of Nori-semistable and essentially finite vector bundles. These categories are proved to be Tannakian. Their Galois k -group schemes $\pi^S(Y, y)$ and $\pi^N(Y, y)$, respectively, thus generalize the S -fundamental and the Nori fundamental group schemes. The latter still classifies all the finite torsors over Y , pointed over y . We also prove that they fit in short exact sequences involving $\pi^S(X, x)$ and $\pi^N(X, x)$ respectively, where x is the image of y .

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1. INTRODUCTION

The existence of a group scheme classifying all the finite torsors over a given scheme X has been first conjectured by Grothendieck in his celebrated work [SGA1, Chapitre X]. However a complete proof was given by Nori in [No1] almost thirty years later when X is a proper, reduced and connected scheme defined over a perfect field k endowed with a section $x \in X(k)$. Then in his PhD thesis [No2] Nori provided a new proof in the more general set-up where X is not necessarily proper and k is any field. In this paper we only consider the case where k is an algebraically closed field and X a smooth, connected and projective scheme over k . In this set-up we first analyze two questions which happen to be tightly linked:

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- (a) Given two finite k -group schemes G_1 and G_2 , a G_1 -torsor $Y_1 \rightarrow X$ and a G_2 -torsor $Y_2 \rightarrow Y_1$, when there is a finite k -group scheme G_3 and a G_3 -torsor $Y_3 \rightarrow X$ together with a faithfully flat morphism $Y_3 \rightarrow Y_2$?
- (b) Given a finite k -group scheme G and a G -torsor $Y \rightarrow X$ when there is a fundamental group scheme $\pi^N(Y, y)$ classifying all the finite torsors over Y pointed over the base point $y \in Y$?

Neither of these two questions is new. Question (a) is well known to have a positive answer when both G_1 and G_2 are étale while in the general case it has first been studied by Garuti in [Ga] with his solution being made *sharp* by Antei and Emsalem in [AE]. Garuti communicated to us that the proof of [Ga] is not correct unfortunately, so the question is still open. Regarding question (b), as a consequence of the aforementioned results in [Ga] and [AE], an answer to it was given, which is thus not correct either. However in [EHS] Esnault, Hai and Sun found a complete answer to question (b) when G is étale.

In the present work we use old techniques combined with new tools in order to find a complete and satisfactory solution to both questions (a) and (b). They happen to be a particular case of a more general aspect which will be described in § 3. More precisely, when X is, as before, a smooth connected and projective k -scheme provided with a section $x \in X(k)$, a complete solution to question (a) is given in Corollary 5.4, while question (b) is answered in Theorem 5.1 and Lemma 6.3. It is now natural to wonder whether there is any canonical relation between the fundamental group scheme $\pi^N(X, x)$ of X and that of Y (the notation of question (b) is being used). A result proved in [EHS] says there exist an exact sequence

$$1 \longrightarrow \pi^N(Y, y) \longrightarrow \pi^N(X, x) \longrightarrow G \longrightarrow 1$$

when G is étale. Here we prove that the same sequence continues to hold when G is any finite k -group scheme (cf. Theorem 5.3). We also prove that a similar short exact sequence for the S -fundamental group scheme holds (cf. Theorem 5.1):

$$1 \longrightarrow \pi^S(Y, y) \longrightarrow \pi^S(X, x) \longrightarrow G \longrightarrow 1,$$

where $\pi^S(Y, y)$ is defined to be the group scheme naturally associated to the category (that we prove to be Tannakian) of generalized Nori-semistable vector bundles over Y , i.e., those vector bundles over Y which are semistable of degree 0 when restricted to any normal and proper curve over k .

Thanks to these two exact sequences we can study further some properties of both Nori's fundamental group scheme and the S -fundamental group scheme. Indeed, Esnault and Mehta proved in [EM] that $\pi^N(X, x)$ trivial implies $\pi^S(X, x)$ trivial; this was further generalized by Langer in [La2] where he proved that $\pi^N(X, x) \simeq \pi^S(X, x)$ whenever the étale fundamental group $\pi^{\text{ét}}(X, x)$ of X is trivial. Using the latter, here we prove (cf. Corollary 5.6) that whenever $\pi^N(X, x)$ is finite then, again, $\pi^N(X, x) \simeq \pi^S(X, x)$. In particular in this case we have that Nori's universal torsor $X^N \rightarrow X$ is both *Nori-simply connected* and *S-simply connected*, meaning that $\pi^N(X^N, x^N) = \pi^S(X^N, x^N) = 0$. A similar statement is true more in general (i.e. not necessarily when $\pi^N(X, x)$ is finite): indeed we prove in Corollary 5.5 that every finite pointed torsor over X^N is trivial.

Whenever $\pi^N(Y, y)$ can be defined, it is natural to study the homomorphism

$$\pi(i) : \pi^N(Y_{\text{red}}, y) \longrightarrow \pi^N(Y, y)$$

induced by the natural inclusion $i : Y_{\text{red}} \rightarrow Y$ with Y_{red} being the reduced subscheme of Y . In the étale case, the analogous homomorphism $\pi^{\text{ét}}(Y_{\text{red}}, y) \rightarrow \pi^{\text{ét}}(Y, y)$ is known to be an isomorphism (cf. [SGA1, I, Théorème 8.3]). We prove that $\pi(i)$ is a closed immersion whenever Y is a finite Galois torsor over a smooth connected and proper curve X defined over k (cf. Theorem 5.7). However $\pi(i)$ is unlikely to be faithfully flat. A similar result holds for $\pi^S(Y, y)$.

2. NORI-SEMISTABLE VECTOR BUNDLES

In this section we recall some well-known facts; the details can be found in [La1], [La2] and [EM].

Let k be any algebraically closed field. Let X be a connected reduced proper scheme, of dimension d , defined over k . The S -fundamental group scheme of X is defined as the group scheme associated to the Tannakian category given by the strongly semistable vector bundles on X of degree zero (see below). Here we need a different description for this category, so from now on let us assume that X is moreover smooth and projective. Let H be a fixed ample line bundle on X ; the degree of torsionfree coherent sheaves on X will be defined using H . A torsion-free coherent sheaf V on X is said to be stable (respectively, semistable) if for every nonzero subsheaf $E \subset V$ with $\text{rank}(E) < \text{rank}(V)$, we have

$$\mu(E) < \mu(V) \quad (\text{respectively, } \mu(E) \leq \mu(V))$$

where $\mu(V)$ denotes the slope of V , i.e., $\mu(V) := \text{deg}(V)/\text{rank}(V)$. We recall that a semistable vector bundle V is said to be strongly semistable if either $\text{char}(k) = 0$ or $\text{char}(k) = p > 0$ and for every $n \in \mathbb{N}$ the Frobenius pullback $(F_X^n)^*V$ is semistable.

A vector bundle V on X is called numerically flat if both V and its dual V^* are nef. It is known that V is numerically flat if and only if it is Nori semistable, i.e., for any morphism $i : C \rightarrow X$ from a smooth projective curve C , the pullback i^*V is semistable of degree zero. Let $\text{Vect}_0^s(X)$ denote the full sub-category of the category of coherent sheaves on X whose objects are strongly semistable reflexive sheaves V with $ch_1(V).H^{d-1} = 0$ and $ch_2(V).H^{d-2} = 0$. A vector bundle V is numerically flat if and only if $V \in \text{Vect}_0^s(X)$ [La1, Proposition 5.1]. Using [La1, Theorem 4.1] we can summarize the previous discussion as follows:

Let X be a connected, smooth and projective scheme of dimension d defined over an algebraically closed field k . Let H be a fixed ample line bundle on X and V a vector bundle on X . Then the following are equivalent:

- (1) $V \in \text{Ns}(X)$;
- (2) V is numerically flat;
- (3) V is strongly semistable with $ch_1(V).H^{d-1} = 0$ and $ch_2(V).H^{d-2} = 0$;
- (4) V is strongly semistable with vanishing (numerical) Chern classes.

The category $\text{Ns}(X)$ of Nori semistable vector bundles is a k -linear abelian rigid tensor category. Fixing a closed point $x \in X$ endows $\text{Ns}(X)$ with a fiber functor, namely $x^* : \text{Ns}(X) \rightarrow k\text{-mod}$ which makes $(\text{Ns}(X), \otimes_{\mathcal{O}_X}, x^*, \mathcal{O}_X)$ into a neutral Tannakian category over k . Then x^* induces an equivalence of categories between the category $\text{Rep}(\pi^S(X, x))$ of finite dimensional representation of the Galois fundamental group $\pi^S(X, x)$ of the Tannakian category $(\text{Ns}(X), \otimes_{\mathcal{O}_X}, x^*, \mathcal{O}_X)$ and $\text{Ns}(X)$. The affine group scheme $\pi^S(X, x)$ is called the

S -fundamental group scheme of X with base point x . This group scheme has been studied in the curve case in [BPS], then in any dimension in [La1] and [Me] independently and later in [La2] and [EM].

3. GENERAL MACHINERY

Let $\theta : X \rightarrow \text{Spec}(k)$ be a scheme over a field k such that $H^0(X, \mathcal{O}_X) = k$ together with a rational point $x \in X(k)$. In this section we state a few general facts about Tannakian categories contained in the category $\text{Coh}(X)$ of coherent sheaves on X ; these will be applied in the next sections.

By a Tannakian category contained in $\text{Coh}(X)$ we mean a full exact k -linear tensor functor $i : \mathcal{T} \hookrightarrow \text{Coh}(X)$, such that \mathcal{T} is Tannakian with the neutral fiber functor x^* . Given two Tannakian categories $\mathcal{T}_1 \hookrightarrow \mathcal{T}_2 \hookrightarrow \text{Coh}(X)$ contained in $\text{Coh}(X)$, we will say that \mathcal{T}_1 is a *full Tannakian sub-category* of \mathcal{T}_2 if the functor $\mathcal{T}_1 \hookrightarrow \mathcal{T}_2$ is an exact tensor fully faithful functor and the category \mathcal{T}_1 is closed in \mathcal{T}_2 by taking sub-objects.

To the data of a Tannakian category contained in $\text{Coh}(X)$ is attached the Galois group scheme $\pi_1(\mathcal{T}, x^*)$ of the Tannakian category, and the universal torsor $\widehat{X} \rightarrow X$, where $\widehat{X} = \underline{\text{Isom}}^\otimes(\theta^*x^*, i)$ is a right torsor under the affine group scheme $\pi_1(\mathcal{T}, x^*)$. The fiber at x of this universal torsor

$$x^*\widehat{X} \simeq \underline{\text{Isom}}^\otimes(x^*\theta^*x^*, x^*) \simeq \pi_1(\mathcal{T}, x^*)$$

is trivial. The universal torsor $\widehat{X} \rightarrow X$ corresponds to a functor

$$F : \text{Rep}(\pi_1(\mathcal{T}, x^*)) \rightarrow \text{Coh}(X)$$

by the formula $\widehat{X} \simeq \underline{\text{Isom}}^\otimes(\theta^*O, F)$, where

$$O : \text{Rep}(\pi_1(\mathcal{T}, x^*)) \rightarrow k\text{-mod}$$

is the forgetful functor with $\text{Rep}(\pi_1(\mathcal{T}, x^*))$ being the category of finite dimensional representations of $\pi_1(\mathcal{T}, x^*)$ and $k\text{-mod}$ the category of finite dimensional k -vector spaces. This formula means that the functor F is the result of twisting θ^*O by the torsor \widehat{X} (see [E]). In particular the two functors θ^*O and F are locally isomorphic for the flat topology.

On the other hand, the functor x^* induces an equivalence of categories

$$\tilde{x} : \mathcal{T} \xrightarrow{\sim} \text{Rep}(\pi_1(\mathcal{T}, x^*))$$

such that $x^* = O \circ \tilde{x}$. The link between the functors F and \tilde{x} is given by the lemma below.

Lemma 3.1. *The functor F and $i \circ \tilde{x}^{-1}$ are equivalent, where \tilde{x}^{-1} denotes a quasi-inverse of \tilde{x} .*

Proof. The following diagram is commutative:

$$\begin{array}{ccccc} \mathcal{T} & \xrightarrow{x^*} & k\text{-mod} & \xrightarrow{\theta^*} & \text{Coh}(X) \\ & \searrow \tilde{x} & \uparrow O & & \uparrow O_X \\ & & \text{Rep}_k(\pi_1(\mathcal{T}, x^*)) & \xrightarrow{\theta^*} & \text{Rep}_X(\pi_1(\mathcal{T}, x^*)) \end{array}$$

where $\text{Rep}_X(\pi_1(\mathcal{T}, x^*))$ denotes the category of \mathcal{O}_X -modules endowed with an action of $\pi_1(\mathcal{T}, x^*)$ and O_X is the forgetful functor. Thus

$$\widehat{X} \simeq \underline{\text{Isom}}^\otimes(\theta^*x^*, i) \simeq \underline{\text{Isom}}^\otimes(\theta^*x^*\tilde{x}^{-1}, i\tilde{x}^{-1}) \simeq \underline{\text{Isom}}^\otimes(\theta^*O, i\tilde{x}^{-1}).$$

Comparing with the formula $\widehat{X} \simeq \underline{\text{Isom}}^\otimes(\theta^*O, F)$ and using the one-to-one correspondence between torsors and functors, one gets the equivalence $F \simeq i\tilde{x}^{-1}$. \square

It follows from Lemma 3.1 that F takes values in the category $\mathcal{T} \subset \text{Coh}(X)$. We will need the following general fact.

Lemma 3.2. *Let $G \rightarrow \text{Spec}(k)$ be an affine group scheme and*

$$j : T \rightarrow X$$

*a G -torsor. Denote by $\mathcal{T} \subset \text{Coh}(X)$ the category of vector bundles V on X such that j^*V is trivial (morphisms in \mathcal{T} are morphisms in $\text{Coh}(X)$). Let*

$$F_T : \text{Rep}(G) \rightarrow \mathcal{T} \subset \text{Coh}(X)$$

be the associated functor. Then F_T is fully faithful if and only if $H^0(T, \mathcal{O}_T) = k$. When this is the case F_T induces an equivalence of categories between $\text{Rep}(G)$ and \mathcal{T} .

Proof. The torsor $j : T \rightarrow X$ defines an equivalence of categories ρ between the category $G\text{-Vect}_T$ of G -vector bundles over T and vector bundles over X . So $F_T = \rho \circ (\theta \circ j)^*$. Thus F_T is fully faithful if and only if $j^* \circ \theta^*$ is fully faithful. Let V_1, V_2 be two objects of $\text{Rep}(G)$. Then $\text{Hom}_{\text{Rep}(G)}(V_1, V_2) = (V_1^\vee \otimes_R V_2)^G$. Analogously if $\mathcal{F}_1, \mathcal{F}_2$ are two objects of $G\text{-Vect}_T$, then $\text{Hom}_{G\text{-Vect}_T}(\mathcal{F}_1, \mathcal{F}_2) = H^0(T, \mathcal{F}_1^\vee \otimes_{\mathcal{O}_T} \mathcal{F}_2)^G$. Thus F_T is fully faithful if and only if for any object $W \in \text{Rep}(G)$, the natural map

$$W^G \rightarrow H^0(T, j^*\theta^*(W))^G \tag{1}$$

is an isomorphism. We have the following sequence of isomorphisms of G -modules (by means of the projection formula):

$$\begin{aligned} H^0(T, j^*\theta^*(W)) &\simeq H^0(\text{Spec}(k), (\theta \circ j)_*(\theta \circ j)^*(W)) \simeq H^0(\text{Spec}(k), (\theta \circ j)_*\mathcal{O}_T \otimes_k W) \\ &\simeq H^0(\text{Spec}(k), (\theta \circ j)_*\mathcal{O}_T) \otimes_k W \simeq H^0(T, \mathcal{O}_T) \otimes_k W. \end{aligned}$$

Then F_T is fully faithful if and only if $(H^0(T, \mathcal{O}_T) \otimes_k W)^G = W^G$ for any object W . This will be the case if $H^0(T, \mathcal{O}_T) = k$. Conversely, suppose that for any representation W of G , we have $(H^0(T, \mathcal{O}_T) \otimes_k W)^G = W^G$. Passing to the limit one gets that $(H^0(T, \mathcal{O}_T) \otimes_k kG)^G = (kG)^G = k$, where kG is the Hopf algebra of G . But if $d = \dim_k H^0(T, \mathcal{O}_T)$, then $(H^0(T, \mathcal{O}_T) \otimes_k kG) \simeq (kG)^{\oplus d}$ and thus $(H^0(T, \mathcal{O}_T) \otimes_k kG)^G \simeq k^{\oplus d}$. Now one concludes that $d = 1$ and $H^0(T, \mathcal{O}_T) = k$.

In order to prove the essential surjectivity, we argue as follows. Take a vector bundle E on X such that j^*E is trivializable. This implies the existence of a finitely generated free k -vector space M such that $\mathcal{E} := j^*E \simeq (\theta \circ j)^*M$. Again applying the projection formula we obtain that

$$(\theta \circ j)_*(\theta \circ j)^*M = (\theta \circ j)_*\mathcal{O}_T \otimes_R M = M.$$

It follows that $\mathcal{E} \simeq (\theta \circ j)^*(\theta \circ j)_*(\theta \circ j)^*M \simeq (\theta \circ j)^*(\theta \circ j)_*\mathcal{E} \simeq (\theta \circ j)^*H^0(T, \mathcal{E})$. We now observe that the previous isomorphism

$$(\theta \circ j)^*(\theta \circ j)_*\mathcal{E} \rightarrow \mathcal{E}$$

is G -equivariant and thus $F_T(\mathbb{H}^0(T, \mathcal{E})) \simeq E$. \square

Applying Lemma 3.2 to the universal torsor $\widehat{X} \rightarrow X$ attached to the Tannakian category $\mathcal{T} \subset \text{Coh}(X)$, we have the following:

Corollary 3.3. $\mathbb{H}^0(\widehat{X}, \mathcal{O}_{\widehat{X}}) = k$, and \mathcal{T} is the category of vector bundles trivialized by the universal torsor $\widehat{X} \rightarrow X$.

Let us introduce the following notion.

Definition 3.4. Let $\mathcal{T} \subset \text{Coh}(X)$ be a Tannakian category (inclusion being an exact tensor full functor). A torsor $g : T \rightarrow X$ under an affine group scheme G will be called a \mathcal{T} -torsor if the essential image of the associated functor $F_T : \text{Rep}(G) \rightarrow \text{Coh}(X)$ lies in \mathcal{T} .

Lemma 3.5. *There is a one-to-one correspondence between the following two:*

- (1) *isomorphism classes of \mathcal{T} -torsors $f : T \rightarrow X$ under affine group schemes G pointed above $x \in X(k)$, and*
- (2) *homomorphisms $\varphi : \pi_1(\mathcal{T}, x^*) \rightarrow G$.*

Given such a torsor and the corresponding morphism $\varphi : \pi_1(\mathcal{T}, x^) \rightarrow G$, the torsor is isomorphic to the contracted product $\widehat{X} \times^{\pi_1(\mathcal{T}, x^*)} G$ through the morphism φ .*

Proof. Let $F_T : \text{Rep}(G) \rightarrow \mathcal{T} \subset \text{Coh}(X)$ be the functor associated to the torsor $f : T \rightarrow X$. As before,

$$F : \text{Rep}(\pi_1(\mathcal{T}, x^*)) \rightarrow \mathcal{T} \subset \text{Coh}(X)$$

is the functor associated to the universal torsor $\widehat{X} \rightarrow X$. From the fact that the torsors are pointed above x , one gets the following 2-commutative diagram:

$$\begin{array}{ccc} \text{Rep}(G) & \xrightarrow{F_T} & \mathcal{T} & \xleftarrow{F} & \text{Rep}(\pi_1(\mathcal{T}, x^*)) \\ & \searrow \scriptstyle O & \downarrow \scriptstyle x^* & & \swarrow \scriptstyle O \\ & & k\text{-mod} & & \end{array}$$

Let F^{-1} be a quasi-inverse of F ; the functor

$$F^{-1} \circ F_T : \text{Rep}(G) \rightarrow \text{Rep}(\pi_1(\mathcal{T}, x^*))$$

induces a homomorphism $\varphi : \pi_1(\mathcal{T}, x^*) \rightarrow G$. The functor associated to the torsor

$$\widehat{X} \times^{\pi_1(\mathcal{T}, x^*)} G \rightarrow X$$

is $F \circ (F^{-1} \circ F_T) \simeq F_T$. Consequently, the torsors $T \rightarrow X$ and $\widehat{X} \times^{\pi_1(\mathcal{T}, x^*)} G \rightarrow X$ are isomorphic. \square

Lemma 3.6. *Let $f : Y \rightarrow X$ be a \mathcal{T} -torsor. Assume that Y is pointed above x . Let*

$$\varphi : \pi_1(\mathcal{T}, x^*) \rightarrow G$$

be the corresponding homomorphism. Then the following are equivalent:

- (1) *φ is faithfully flat.*
- (2) *For all pointed \mathcal{T} -torsors $g : T \rightarrow X$ and morphism of pointed torsors $h : T \rightarrow Y$, h is faithfully flat.*

- (3) For all pointed \mathcal{T} -torsors $g : T \rightarrow X$ and morphism of pointed torsors $h : T \rightarrow Y$ that is a closed immersion, then h is an isomorphism.

Proof. (1) \Rightarrow (2): Consider the homomorphism of affine group schemes

$$\lambda : H \rightarrow G$$

associated to $h : T \rightarrow Y$, and also the homomorphism

$$\psi : \pi_1(\mathcal{T}, x^*) \rightarrow H$$

associated to the torsor $T \rightarrow X$. Then $\varphi = \lambda \circ \psi$, and as φ is supposed to be faithfully flat, the same is true for λ .

(2) \Rightarrow (3): This follows from the fact that a faithfully flat closed immersion is an isomorphism.

(3) \Rightarrow (1): Decompose $\varphi = \lambda \circ \psi$, where λ is a closed immersion and $\psi : \pi_1(\mathcal{T}, x^*) \rightarrow H$ is faithfully flat, and consider $T = \widehat{X} \times^{\pi_1(\mathcal{T}, x^*)} H$ the contracted product through ψ . Then according to (3), λ is an isomorphism, and thus φ is faithfully flat. \square

Definition 3.7. A \mathcal{T} -torsor $f : Y \rightarrow X$ will be called \mathcal{T} -Galois if the equivalent conditions in Lemma 3.6 hold.

Remark 3.8. When X is a proper connected reduced scheme satisfying $H^0(X, \mathcal{O}_X) = k$, and $\mathcal{T} = EF(X)$ the category of essentially finite vector bundles, every torsor under a finite group scheme is a \mathcal{T} -torsor in the sense of Definition 3.4 [No2, Proposition 3.8]. The finite \mathcal{T} -Galois torsors are called “reduced” in [No2].

Remark 3.9. A pointed \mathcal{T} -Galois torsor $g : T \rightarrow X$ under a finite group scheme G defines a fully faithful functor $F_T : \text{Rep}(G) \rightarrow \mathcal{T}$ whose essential image we denote by \mathcal{T}_T . Then the torsor $g : T \rightarrow X$ is isomorphic to the universal torsor associated to the full Tannakian sub-category $\mathcal{T}_T \subset \mathcal{T}$. Let us write $F_T = i \circ \widetilde{F}_T$, where $\widetilde{F}_T : \text{Rep}(G) \rightarrow \mathcal{T}_T$ is an equivalence. Then the corresponding universal torsor is

$$\underline{\text{Isom}}^{\otimes}(\theta^*x^*, i) \simeq \underline{\text{Isom}}^{\otimes}(\theta^*x^*\widetilde{F}_T, i\widetilde{F}_T) \simeq \underline{\text{Isom}}^{\otimes}(\theta^*O, F_T) \simeq T,$$

where O denotes as usual the forgetful functor.

Lemma 3.5 describes the \mathcal{T} -torsors $f : Y \rightarrow X$. More generally, one may consider finite flat morphisms of k -schemes $f : Y \rightarrow X$ such that $f_*\mathcal{O}_Y$ is an object of \mathcal{T} .

Lemma 3.10. *The fiber functor x^* induces an equivalence between the following two categories:*

- (1) *The category of finite flat morphisms of k -schemes $f : Y \rightarrow X$ such that $f_*\mathcal{O}_Y$ is an object of \mathcal{T}*
- (2) *The category of finite k -schemes endowed with an action of $\pi_1(\mathcal{T}, x^*)$.*

Moreover, if $f_i : Y_i \rightarrow X$, $i = 1, 2$, are two objects of the first category, and $g : Y_1 \rightarrow Y_2$ is a morphism such that $f_2 \circ g = f_1$, then g is faithfully flat if and only if the restriction to the fibers of x ,

$$g_x : x^*Y_1 \rightarrow x^*Y_2$$

is surjective.

Proof. The tensor functor $F \simeq i \circ \tilde{x}^{-1}$ induces an equivalence between the category of finite k -algebras endowed with an action of $\pi_1(\mathcal{T}, x^*)$ and the category of finite \mathcal{O}_X -algebras that are objects of \mathcal{T} as \mathcal{O}_X -modules. Thus passing to the spectrum one gets an equivalence between finite k -schemes endowed with an action of $\pi_1(\mathcal{T}, x^*)$ and finite morphisms $f : Y \rightarrow X$ such that $f_*\mathcal{O}_Y$ is an object of \mathcal{T} .

As we noticed that the functors F and $\theta^* \circ O$ are locally isomorphic for the flat topology, the above equivalence of categories transforms faithfully flat morphisms into faithfully flat morphisms. \square

In this context one can define the ‘‘Galois closure’’ of a finite flat morphisms of k -schemes $f : Y \rightarrow X$ such that $f_*\mathcal{O}_Y$ is an object of \mathcal{T} .

Proposition 3.11. *Consider a finite flat morphisms of k -schemes $f : Y \rightarrow X$ such that $f_*\mathcal{O}_Y$ is an object of \mathcal{T} and the full Tannakian sub-category $\mathcal{T}_Y = \langle f_*\mathcal{O}_Y \rangle$ of \mathcal{T} generated by $f_*\mathcal{O}_Y$. Let $g : \widehat{X}_Y \rightarrow X$ its universal torsor under the affine group scheme $\pi_1(\mathcal{T}_Y, x^*)$ pointed at \widehat{x}_Y . If Y has a k -rational point y , there exists a unique morphism $h : \widehat{X}_Y \rightarrow Y$ such that $h(\widehat{x}_Y) = y$. Moreover this morphism h is faithfully flat if and only if $H^0(X, f_*\mathcal{O}_Y) = k$. When this is the case:*

- (1) *The morphism $\widehat{X}_Y \rightarrow Y$ of pointed k -schemes is a pointed torsor under the isotropy subgroup scheme of $y \in x^*Y$ for the action of $\pi_1(\mathcal{T}_Y, x^*)$.*
- (2) *The universal torsor satisfies the following universal property: for any \mathcal{T} -Galois torsor $g' : T \rightarrow X$ pointed at $t \in T(k)$ and any faithfully flat X -morphism of pointed schemes $h' : T \rightarrow Y$ such that $h'(t) = y$, there exists a unique morphism of pointed torsors $j : T \rightarrow \widehat{X}_Y$ making the following diagram commutative*

$$\begin{array}{ccc}
 T & & \\
 \downarrow g' & \searrow h' & \\
 & \widehat{X}_Y & \xrightarrow{h} Y \\
 & \downarrow g & \swarrow f \\
 & X &
 \end{array}$$

Proof. In the equivalence of Lemma 3.10, $\pi_1(\mathcal{T}_Y, x^*) \rightarrow \text{Spec}(k)$ corresponds to the universal torsor $\widehat{X}_Y \rightarrow X$ and $x^*Y \rightarrow \text{Spec}(k)$ to $Y \rightarrow X$. The morphism $\eta : \pi_1(\mathcal{T}_Y, x^*) \rightarrow x^*Y$ given by $g \mapsto g.y$ corresponds thus to a unique pointed morphism $h : \widehat{X}_Y \rightarrow Y$ whose restriction to the fiber at x is η . The quotient $x^*Y/\pi_1(\mathcal{T}_Y, x^*)$ is the spectrum of

$$(\tilde{x}(f_*\mathcal{O}_Y))^{\pi_1(\mathcal{T}_Y, x^*)} \simeq H^0(X, f_*\mathcal{O}_Y).$$

Thus η is surjective if and only if $H^0(X, f_*\mathcal{O}_Y) = k$.

The assertion (1) is the consequence of the fact that the morphism $\eta : G \rightarrow x^*Y$ is a torsor under the isotropy of y .

Finally, let $g' : T \rightarrow X$ be a \mathcal{T} -Galois torsor pointed at t . Then it is the universal torsor attached to the category \mathcal{T}_T (see Remark 3.9). Let $h' : T \rightarrow Y$ be a faithfully flat morphism of pointed schemes. Then $f_*\mathcal{O}_Y \subset \mathcal{T}_T$ and $\langle f_*\mathcal{O}_Y \rangle = \mathcal{T}_{\widehat{X}_Y}$ is a full Tannakian sub-category of \mathcal{T}_T . From this one gets a morphism of universal torsors $T \rightarrow \widehat{X}_Y$. \square

Remark 3.12. Suppose that $\mathcal{T}_1 \subset \mathcal{T}_2 \subset \text{Coh}(X)$ are two Tannakian categories, where \mathcal{T}_1 is a full Tannakian sub-category of \mathcal{T}_2 . One can apply Proposition 3.11 to a finite morphism $f : Y \rightarrow X$ such that $f_*\mathcal{O}_Y$ is an object of \mathcal{T}_1 with respect to \mathcal{T}_1 or \mathcal{T}_2 . With obvious notation, $\langle f_*\mathcal{O}_Y \rangle_{\mathcal{T}_1} = \langle f_*\mathcal{O}_Y \rangle_{\mathcal{T}_2}$ so that the Galois closure $\widehat{X}_Y \rightarrow X$ constructed with \mathcal{T}_1 or \mathcal{T}_2 are the same. This will be the case with the categories $EF(Y) \subset \mathcal{N}(Y)$ introduced in Section 4.

Let $f : Y \rightarrow X$ be a \mathcal{T} -Galois torsor under a finite group scheme G . We will consider the category $\mathcal{T}(Y)$ of coherent sheaves \mathcal{F} on Y such that $f_*\mathcal{F}$ is an object of \mathcal{T} .

Theorem 3.13. *Let X be a k -scheme such that $H^0(X, \mathcal{O}_X) = k$ provided with a rational point $x \in X(k)$, \mathcal{T} a Tannakian category contained in $\text{Coh}(X)$ and G be a finite k -group scheme. Let $f : Y \rightarrow X$ be a pointed \mathcal{T} -Galois torsor under G pointed at $y \in Y(k)$ above x . Then $\mathcal{T}(Y)$ is a Tannakian category whose objects are vector bundles, with fiber functor y^* .*

Moreover the Galois group scheme $\pi_1(\mathcal{T}(Y), y^)$ of this Tannakian category at y^* is the kernel K of the faithfully flat homomorphism $\varphi : \pi_1(\mathcal{T}, x^*) \rightarrow G$ associated to the G -torsor $Y \rightarrow X$, giving rise to the exact sequence*

$$1 \rightarrow \pi_1(\mathcal{T}(Y), y^*) \rightarrow \pi_1(\mathcal{T}, x^*) \rightarrow G \rightarrow 1 \quad (2)$$

and the universal torsor is $\widehat{X} \rightarrow Y$.

Proof. The universal property of universal torsor $p : \widehat{X} \rightarrow X$ implies the existence of a morphism of pointed torsor $p_Y : \widehat{X} \rightarrow Y$ such that $f \circ p_Y = p$ (see Remark 3.9). Observe that as in the proof of [EHS, Proposition 2.7], for any object V of $\mathcal{T}(Y)$, there are two objects W_1, W_2 of \mathcal{T} such that V is the cokernel of a homomorphism $f^*W_1 \rightarrow f^*W_2$. Indeed $W_2 = f_*V$ is by definition an object of \mathcal{T} . Consider the kernel V_1 of the surjection $f^*f_*V \rightarrow V$. As $f^*f_*V \simeq V^{\oplus d}$ is an object of $\mathcal{T}(Y)$, where d is the degree of the torsor $f : Y \rightarrow X$, it follows that $W_1 =: f_*V_1$ is the kernel of the morphism $f_*(f^*f_*V) \rightarrow f_*V$ between objects of \mathcal{T} , and is thus an object of \mathcal{T} . This insures that V_1 is an object of $\mathcal{T}(Y)$. Finally V is the cokernel of $f^*W_1 = f^*f_*V_1 \rightarrow V_1 \subset f^*f_*V = f^*W_2$.

As a consequence, p_Y^*V is the cokernel of $p_Y^*f^*W_1 = p^*W_1 \rightarrow p_Y^*f^*W_2 = p^*W_2$ between trivial vector bundles. The torsor $p_Y : \widehat{X} \rightarrow Y$ is a K -torsor where K is the kernel of the faithfully flat morphism $\varphi : \pi(X) \rightarrow G$ associated to the \mathcal{T} -Galois torsor $f : Y \rightarrow X$. As $H^0(\widehat{X}, \mathcal{O}_{\widehat{X}}) = k$ according to Corollary 3.3, the functor $\eta_Y : \text{Rep}(K) \rightarrow \text{Coh}(Y)$ associated to the K -torsor $p_Y : \widehat{X} \rightarrow Y$ is fully faithful, according to Lemma 3.2. As f^*W_1 and f^*W_2 whose pull back by p_Y are trivial, are objects of the essential image \mathcal{C} of η_Y , V is also an object of \mathcal{C} . One concludes that $\mathcal{T}(Y) \subset \mathcal{C}$.

Conversely, consider an object V of \mathcal{C} . Then

$$p^*f_*V = p_Y^*f^*f_*V \simeq p_Y^*(V^{\oplus d}) \simeq (p_Y^*V)^{\oplus d}.$$

As V is an object of \mathcal{C} , we know that p_Y^*V is trivial, and thus p^*f_*V is trivial. From Lemma 3.2, one concludes that f_*V is an object of \mathcal{T} , and thus V is an object of $\mathcal{T}(Y)$. One concludes that $\mathcal{T}(Y) = \mathcal{C}$ is a Tannakian category, whose Galois group scheme is K . \square

The following statement is an easy generalization of Theorem 3.13 whose proof is left to the reader.

Theorem 3.14. *Let X be a k -scheme such that $H^0(X, \mathcal{O}_X) = k$ provided with a rational point $x \in X(k)$ and \mathcal{T} a Tannakian category contained in $\text{Coh}(X)$. Let $f : Y \rightarrow X$ be a finite faithfully flat morphism pointed at $y \in Y(k)$ above x , such that $f_*\mathcal{O}_Y$ is an object of \mathcal{T} and such that $H^0(X, f_*\mathcal{O}_Y) = k$. Assume that the Galois closure, constructed in Proposition 3.11, $\widehat{X}_Y \rightarrow X$ is finite. Consider the following commutative diagram*

$$\begin{array}{ccccc} \widehat{X} & \xrightarrow{p_Y} & Y & \xrightarrow{f} & X \\ & \searrow j & \uparrow h & \nearrow \pi & \\ & & \widehat{X}_Y & & \end{array}$$

where $\pi : \widehat{X}_Y \rightarrow Y$ is the natural morphism, $\widehat{X} \rightarrow X$ the universal torsor attached to the category \mathcal{T} and x . Then the category $\mathcal{T}(Y)$ of vector bundles V on Y such that π_*h^*V is an object of \mathcal{T} is a Tannakian category; $\mathcal{T}(Y)$ is the category of vector bundles V on Y such that p_Y^*V is trivial; the universal torsor attached to $\mathcal{T}(Y)$ and y is $\widehat{X} \rightarrow Y$, and the Tannakian Galois group $\pi_1(\mathcal{T}(Y), y)$ attached to $\mathcal{T}(Y)$ and y is the isotropy group of y under the action of $\pi_1(\mathcal{T}, x)$ on x^*Y .

4. GENERALIZED NORI-SEMISTABLE BUNDLES

Let us introduce the following definition:

Definition 4.1. A vector bundle \mathcal{F} over a k -scheme T is called *Nori-semistable* if the following holds: for any smooth projective k -curve C and any morphism $i : C \rightarrow T$, the pulled back vector bundle $i^*\mathcal{F}$ is semistable of degree zero. We will denote by $Ns(T)$ the category of Nori-semistable vector bundles.

Let $f : Y \rightarrow T$ be a finite and flat morphism. It is said to be *Nori-semistable* if the direct image $f_*\mathcal{O}_Y$ is a Nori-semistable vector bundle.

Lemma 4.2. *Let T be a k -scheme, where k is an algebraically closed field. The tensor product of two Nori-semistable vector bundles on T is again Nori-semistable.*

Proof. Let (C, i) be as in Definition 4.1 and \mathcal{F}_1 and \mathcal{F}_2 two Nori-semistable vector bundles on T . The statement is the consequence of the facts that $i^*(\mathcal{F}_1 \otimes_{\mathcal{O}_T} \mathcal{F}_2) = i^*\mathcal{F}_1 \otimes_{\mathcal{O}_C} i^*\mathcal{F}_2$ and that $Ns(C)$ is a tensor category. \square

Let X be a smooth projective scheme over an algebraically closed field k , pointed at a rational point $x \in X(k)$. Let G be a finite k -group scheme and $f : Y \rightarrow X$ a Galois G -torsor endowed with a k -rational point $y \in Y_x(k)$. The full sub-category of $\text{Coh}(Y)$ whose objects are those coherent sheaves on Y whose push-forward to X is an essentially finite vector bundle called $\mathcal{F}(Y)$ in [EHS] is a Tannakian category whose objects are vector bundles according to Theorem 3.13 (cf. also [EHS, Proposition 2.7]). Moreover, if G is smooth, then $\mathcal{F}(Y)$ is nothing but the category of essentially finite vector bundles over Y [EHS, Theorem 2.9]. Here we do a similar construction and we call $\mathcal{N}(Y)$ the full sub-category of $\text{Coh}(Y)$ whose objects are those coherent sheaves on Y whose push-forward to X is a Nori-semistable vector bundle.

As a consequence of the following proposition, $Ns(Y)$ is a sub-category of the category $\mathcal{N}(Y)$ of vector bundles \mathcal{F} on Y such that $f_*\mathcal{F}$ is Nori-semistable.

Proposition 4.3. *Let X be a k -scheme over an algebraically closed field k , $f : Y \rightarrow X$ a Nori-semistable morphism and \mathcal{F} a Nori-semistable vector bundle over Y . Then the vector bundle $f_*\mathcal{F}$ on X is Nori-semistable.*

Proof. It suffices to prove the following: for any proper and normal k -curve C and any nonconstant morphism $i : C \rightarrow X$ the vector bundle $i^*f_*\mathcal{F}$ is semistable of degree zero. So take (C, i) as above. Let $C' := Y \times_X C$ be the fiber product; we consider the diagram

$$\begin{array}{ccccc}
 \tilde{C} & & & & \\
 \searrow^s & & & & \\
 & C' & \xrightarrow{i'} & Y & \\
 \swarrow_u & \downarrow f' & & \downarrow f & \\
 & C & \xrightarrow{i} & X &
 \end{array}$$

where \tilde{C} is the normalization of an irreducible component of the reduced curve C'_{red} for C' that surjects to C . The morphism u is finite and faithfully flat (see for instance [Li, § 4, Corollary 3.10]), so in particular $u^*\mathcal{H}$ is a subbundle of $u^*i^*f_*\mathcal{F}$ as long as \mathcal{H} is a subbundle of $i^*f_*\mathcal{F}$. Hence to prove that $i^*f_*\mathcal{F}$ is semistable of degree 0 it is enough to show that $u^*i^*f_*\mathcal{F}$ is semistable of degree 0; indeed, the pullback by u of any subbundle of $i^*f_*\mathcal{F}$ contradicting the semistability condition, contradicts the semistability condition for $u^*i^*f_*\mathcal{F}$ too, while $\text{degree}(u^*i^*(f_*\mathcal{F})) = 0$ implies that $\text{degree}(i^*(f_*\mathcal{F})) = 0$ (cf. for instance [AE, Lemma 3.25]).

To prove that $u^*i^*f_*\mathcal{F}$ is semistable of degree 0, we have

$$u^*i^*f_*\mathcal{F} \simeq u^*f'_*i'^*\mathcal{F} = s^*f'^*f'_*i'^*\mathcal{F} \simeq s^*((i'^*\mathcal{F}) \otimes_{\mathcal{O}_{C'}} (f'^*f'_*\mathcal{O}_{C'}))$$

and the latter is isomorphic to $(s^*i'^*\mathcal{F}) \otimes_{\mathcal{O}_{\tilde{C}}} (u^*f'_*\mathcal{O}_{C'})$. The vector bundle $s^*i'^*\mathcal{F}$ is Nori semistable of degree 0 and the same is true for $u^*f'_*\mathcal{O}_{C'}$ because f' is a Nori-semistable morphism. Therefore, from Lemma 4.2 it follows that $(s^*i'^*\mathcal{F}) \otimes_{\mathcal{O}_{\tilde{C}}} (u^*f'_*\mathcal{O}_{C'})$ is semistable of degree zero. \square

Corollary 4.4. *Let $f : Y \rightarrow X$ a Nori-semistable morphism, then $Ns(Y) = \mathcal{N}(Y)$ where $\mathcal{N}(Y)$ is the category of vector bundles \mathcal{F} on Y such that $f_*\mathcal{F}$ is Nori-semistable.*

Proof. In Proposition 4.3 we have already proved that $Ns(Y) \subseteq \mathcal{N}(Y)$. So let now V be an object of $\mathcal{N}(Y)$, i.e. a vector bundle such that $f_*(V)$ is Nori-semistable. For any morphism $j : C \rightarrow Y$ from a smooth and projective curve C to Y ,

$$(f \circ j)^*f_*V = j^*(f^*f_*V) = j^*(f^*f_*\mathcal{O}_Y \otimes_{\mathcal{O}_Y} V) = (f \circ j)^*(f_*\mathcal{O}_Y) \otimes_{\mathcal{O}_C} j^*V$$

and as $(f \circ j)^*f_*V$ and $(f \circ j)^*(f_*\mathcal{O}_Y)$ are Nori-semistable, j^*V has degree 0. Furthermore we know that V is a quotient bundle of f^*f_*V , which is Nori-semistable, so in particular j^*V remains a quotient bundle of a Nori-semistable vector bundle and this is enough to conclude that j^*V is semistable of degree 0 (cf. [No2, Lemma 3.6]). \square

Corollary 4.5. *Let X be a k -scheme, $f : Y \rightarrow X$ and $g : Z \rightarrow Y$ two Nori-semistable morphisms. Then $g \circ f$ is Nori-semi-stable.*

We now recall that a *Weil-finite* vector bundle over a scheme Y is a vector bundle V such that there exist two polynomials $p(x), q(x) \in \mathbb{N}[x]$, $p(x) \neq q(x)$, with $p(V)$ isomorphic to $q(V)$. In particular if $h : T \rightarrow Y$ is a M -torsor for some finite k -group scheme M , then $h_*\mathcal{O}_T$ is a Weil-finite vector bundle.

Lemma 4.6. *A Weil-finite vector bundle V on a k -scheme Y is Nori-semistable.*

Proof. Let $i : C \rightarrow Y$ a morphism of a smooth projective curve C to Y . Then i^*V is Weil-finite too, and thus semistable of degree 0 ([No1, Proposition 3.4]) \square

Corollary 4.5 and Lemma 4.6 together produce the following:

Corollary 4.7. *Let X be a k -scheme provided with a section $x \in X(k)$, and let G and M be two finite k -group schemes. Let*

$$f : Y \rightarrow X$$

*a Galois G -torsor endowed with a k -rational point $y \in Y_x(k)$ and $h : T \rightarrow Y$ a M -torsor endowed with a k -rational point $t \in T_y(k)$. Then $f_*h_*\mathcal{O}_T$ is Nori-semistable.*

Corollary 4.7 was proved earlier in [EHS, Lemma 2.8] assuming that G is smooth.

Definition 4.8. Let X be a k -scheme such that $Ns(X)$ is a Tannakian sub-category of $\text{Coh}(X)$ (in particular $H^0(X, \mathcal{O}_X) = k$). Consider the full Tannakian sub-category $EF(X)$ of $Ns(X)$ generated by the Weil-finite vector bundles. The objects of $EF(X)$ are called *essentially finite* (this definition applies in particular in the case of a smooth projective scheme over an algebraically closed field, in which case it coincides with Nori's definition [No2]).

Let X be as in Definition 4.8 and $f : Y \rightarrow X$ a torsor under a finite group scheme. Then $f_*\mathcal{O}_Y$ is Weil-finite and one may consider the full Tannakian sub-category $\langle f_*\mathcal{O}_Y \rangle$ of $Ns(X)$ generated by $f_*\mathcal{O}_Y$ (see section 3). Then $\langle f_*\mathcal{O}_Y \rangle$ is a full Tannakian sub-category of $EF(X)$.

With X still as in Definition 4.8, consider the situation of Theorem 3.14 with $\mathcal{T} = Ns(X)$. As the Galois closure $\pi : \widehat{X}_Y \rightarrow X$ of the Nori semistable morphism $f : Y \rightarrow X$ is a torsor under a finite group scheme, $f_*\mathcal{O}_Y$ is essentially finite.

Definition 4.9. Let X be as in Definition 4.8 and $f : Y \rightarrow X$ be a finite flat morphism such that $f_*\mathcal{O}_Y$ is essentially finite. We will say that the morphism f is *essentially finite* (in the case of a smooth projective scheme over an algebraically closed field, this definition agrees with that of [AE]).

Definition 4.10. Let X be as in Definition 4.8 endowed with a rational point $x \in X(k)$. The Galois group of the Tannakian category $EF(X)$ based at x will be denoted $\pi^N(X, x)$ and the corresponding universal torsor $\widehat{X}^N \rightarrow X$. When X is a proper reduced k -scheme such that $H^0(X, \mathcal{O}_X) = k$, these are the Nori fundamental group scheme and its universal torsor.

5. FUNDAMENTAL GROUP SCHEME OF TORSOR

As a consequence of Corollary 4.4 and Theorem 3.13, one gets the following statement.

Theorem 5.1. *Let X be a k -scheme such that $Ns(X)$ is a Tannakian sub-category of $\text{Coh}(X)$, provided with a section $x \in X(k)$ and G be a finite k -group scheme. Let $f : Y \rightarrow X$ be a Galois G -torsor endowed with a k -rational point $y \in Y_x(k)$. Then $\mathcal{N}(Y)$ is a Tannakian category whose objects are vector bundles, with fiber functor y^* .*

Moreover the Galois group scheme $\pi^S(Y, y)$ of this Tannakian category at y^ – also called the S -fundamental group scheme of Y at y – is the kernel of the natural morphism $\pi^S(X, x) \rightarrow G$ associated to the G -torsor $f : Y \rightarrow X$, giving rise to the exact sequence*

$$1 \rightarrow \pi^S(Y, y) \rightarrow \pi^S(X, x) \rightarrow G \rightarrow 1$$

and the universal torsor is $\widehat{X}^S \rightarrow Y$.

*Finally for any object V of $EF(Y)$, f^*f_*V is also an object of $EF(Y)$.*

Proof. The only new statement is the last one. It is clearly true for Weil finite vector bundles as $f^*f_*\mathcal{O}_Y$ is Weil finite and $f^*f_*V \simeq f^*f_*\mathcal{O}_Y \otimes V$. For general elements of $EF(Y)$ one uses the fact that for an affine morphism f , f_* is exact. \square

Lemma 5.2. *With the hypothesis of Theorem 5.1, the k -group scheme $\pi^N(Y, y)$ is profinite and for every finite k -group scheme G and every pointed Galois G -torsor $h : T \rightarrow Y$, there is a unique Y -morphism of pointed torsors $Y^N \rightarrow T$.*

Proof. The second assertion simply follows from the fact that $h_*(\mathcal{O}_T)$ is Weil-finite. For the first assertion first note that over Y the Krull-Schmidt theorem for coherent sheaves holds (properties (a) and (b) of the Corollary to [At, Lemma 3] hold, hence [At, Theorem 1] also holds) so, as in [No1, Lemma 3.1] ((a) \Rightarrow (d)), for a given Weil-finite vector bundle V , the collection $S(V)$ of all indecomposable components of $V^{\otimes r}$, for all non-negative integers r , is finite. Then we argue as in [No1, Lemma 3.9]: Let \mathcal{S} be a finite collection of Weil-finite vector bundles; let W be the direct sum of all the members of \mathcal{S} and their duals. Then W is a Weil-finite vector bundle, and by previous discussion $S(W)$ is finite. The category generated by $S(W)$ has thus a finite number of generators (in the sense of [No1, § 2.1]) and therefore the k -group scheme associated to it is finite too. The category generated by all Weil-finite vector bundles gives thus rise to a profinite k -group scheme. \square

Theorem 5.3. *With notations as in Theorem 5.1, the following sequence of group schemes is exact:*

$$1 \rightarrow \pi^N(Y, y) \rightarrow \pi^N(X, x) \rightarrow G \rightarrow 1. \quad (3)$$

*Moreover $EF(Y)$ is the category of vector bundles V on Y such that f_*V is an object of $EF(X)$.*

Proof. According to Theorem 3.13, one is reduced to prove the last statement of the Theorem. Consider a vector bundle V on Y such that f_*V is essentially finite. Following the proof of Theorem 3.13, there exists two objects W_1 and W_2 of $EF(X)$ such that V is the cokernel of a morphism $f^*W_1 \rightarrow f^*W_2$ between objects of $EF(Y)$. Thus V is itself an object of $EF(Y)$.

On the other direction one has to show that the push-forward by f of an object of $EF(Y)$ is an object of $EF(X)$. As f is affine, f_* is exact and it suffices to prove the statement for Weil finite vector bundles. Denote by \mathcal{C} the category of vector bundles on X trivialized by $f : Y \rightarrow X$ and consider a Weil finite vector bundle V on Y and the full Tannakian

sub-category \mathcal{C}' of $Ns(X)$ generated by \mathcal{C} and f_*V . One has the following commutative diagram

$$\begin{array}{ccccc} \mathcal{C} & \longrightarrow & Ns(X) & \xrightarrow{f^*} & \mathcal{N}(Y) \\ \uparrow = & & \uparrow & & \uparrow \\ \mathcal{C} & \longrightarrow & \mathcal{C}' & \xrightarrow{f^*} & \langle f^*f_*V \rangle \end{array}$$

the vertical maps and the left horizontal maps being fully faithful, where the category $\langle f^*f_*V \rangle$ is a full Tannakian sub-category of $EF(Y)$. As the first line satisfies the condition of Theorem A1 of [EHS] for giving rise to an exact sequence of Galois group schemes, it is easy to check that the second line as well satisfies conditions (i), (ii), (iii) (a) and (iii) (b) of Theorem A1 of [EHS]. Let us verify the condition (iii) (c): for every object U of $\langle f^*f_*V \rangle$ there exists an object W of \mathcal{C}' such that U is a quotient of f^*W . As f^* commutes with tensor product and dual, every object U of $\langle f^*f_*V \rangle$ can be viewed as a sub-object $U \hookrightarrow Q$, where Q is a quotient in $Ns(Y)$ of f^*N for some object N of \mathcal{C}' . As f_* is exact and the image by f_* of objects of $Ns(Y)$ are in $Ns(X)$ according to Proposition 4.3, one gets an embedding $f_*U \hookrightarrow f_*Q$ where f_*Q is a quotient in $Ns(X)$ of f_*f^*N . As $f_*f^*N \simeq N \otimes_{\mathcal{O}_X} f_*\mathcal{O}_Y$ is an object of \mathcal{C}' , f_*Q and thus f_*U are objects of \mathcal{C}' . Finally U is a quotient of f^*f_*U and condition (iii) (c) of Theorem A1 of [EHS] is verified. Thus according to *ibid* the second line of the above diagram gives rise to an exact sequence of Galois group schemes

$$1 \longrightarrow H \longrightarrow G' \longrightarrow G \longrightarrow 1.$$

As $f^*f_*V \simeq f^*f_*\mathcal{O}_Y \otimes_{\mathcal{O}_Y} V$ is Weil finite, the Galois group H of the Tannakian category $\langle f^*f_*V \rangle$ is finite according to Lemma 6.3, and thus the Galois group G' of the Tannakian category \mathcal{C}' is also finite. It follows that $\mathcal{C}' \subset EF(X)$. In particular f_*V is essentially finite. \square

As a consequence of Theorem 5.3 one gets that the Galois closure of a tower of two finite pointed torsors is itself *finite*. More precisely:

Corollary 5.4. *Let X be a k -scheme such that the category $Ns(X)$ is a Tannakian subcategory of $\text{Coh}(X)$, provided with a section $x \in X(k)$, while G and M be two finite k -group schemes, $f : Y \rightarrow X$ a Galois G -torsor endowed with a k -rational point $y \in Y_x(k)$ and $h : T \rightarrow Y$ a M -torsor endowed with a k -rational point $t \in T_y(k)$. Then there exists a finite k -group scheme N , a Galois N -torsor $Z \rightarrow X$, pointed in $z \in Z_x(k)$ and a unique morphism $\lambda : Z \rightarrow T$ with $\lambda(z) = t$. If moreover $H^0(T, \mathcal{O}_T) = k$, then λ is faithfully flat and it has a natural structure of N_t -torsor, N_t being the stabilizer of t , under the action of N on T_x .*

Proof. It follows from Theorem 5.3, (3) that the N -torsor $Z \rightarrow X$ built in Proposition 3.11 is indeed finite. \square

A consequence of Corollary 5.4 is that the universal torsor is “simply connected” in the following sense.

Corollary 5.5. *Let X be k -scheme such that the category $Ns(X)$ is a Tannakian subcategory of $\text{Coh}(X)$, provided with a section $x \in X(k)$, then every pointed torsor under a finite group scheme over the universal torsor \widehat{X} is trivial.*

Proof. We know that $\widehat{X}^N = \lim X_i$, a filtered projective limit, where the $X_i \rightarrow X$ are Galois torsors under finite group schemes. Let $f : Y \rightarrow \widehat{X}$ be a pointed torsor under a finite group scheme G . Using [EGAIV], paragraph 8.8, théorème 8.8.2 and its corollaries, one can easily prove that there exists an index i_0 and a pointed G -torsor $Y_{i_0} \rightarrow X_{i_0}$ such that the following diagram is cartesian :

$$\begin{array}{ccc} Y & \longrightarrow & Y_{i_0} \\ f \downarrow & & \downarrow \\ \widehat{X} & \longrightarrow & X_{i_0} \end{array}$$

(note that the schemes X_i as well as Y_{i_0} are quasi-compact and of finite presentation). According to Corollary 5.4 there exists a pointed torsor $Z \rightarrow X$ under a finite group scheme that dominates the tower of finite torsors $Y_{i_0} \rightarrow X_{i_0} \rightarrow X$. So one may consider $Z' = Z \times_{X_{i_0}} \widehat{X}$ and the following commutative diagram whose squares are cartesian :

$$\begin{array}{ccc} Z' & \longrightarrow & Z \\ h \downarrow & & \downarrow \\ Y & \longrightarrow & Y_{i_0} \\ f \downarrow & & \downarrow \\ \widehat{X} & \longrightarrow & X_{i_0} \\ & \searrow & \downarrow \\ & & X \end{array}$$

and from the universal property of \widehat{X} , one gets a morphism $\widehat{X} \rightarrow Z$ and thus a morphism $g : \widehat{X} \rightarrow Z'$ fitting in the commutative diagram. Thus $h \circ g$ is a section of the given torsor $f : Y \rightarrow \widehat{X}$. \square

Corollary 5.6. *Let X be a smooth projective k -scheme such that $H^0(X, \mathcal{O}_X) = k$, endowed with a k -rational point $x \in X(k)$. If $\pi^N(X, x)$ is finite then $\pi^S(X, x) = \pi^N(X, x)$. Moreover $\pi^N(X^N, x^N) = \pi^S(X^N, x^N) = 1$, where $X^N \rightarrow X$ is the universal $\pi^N(X, x)$ -torsor.*

Proof. Let us denote by $X^{\text{ét}} \rightarrow X$ the universal $\pi^{\text{ét}}(X, x)$ -torsor. Then $\pi^{\text{ét}}(X^{\text{ét}}, x^{\text{ét}}) = 1$ and according to [La2, § 8],

$$\pi^N(X^{\text{ét}}, x^{\text{ét}}) = \pi^S(X^{\text{ét}}, x^{\text{ét}}).$$

But by Theorems 5.1 and 5.3 applied to $X^{\text{ét}} \rightarrow X$ we finally have $\pi^S(X, x) = \pi^N(X, x)$. That $\pi^N(X^N, x^N) = 1$ follows from Corollary 5.5, but in this particular case it is also a consequence of Theorem 5.3 applied to $X^N \rightarrow X$; then by the latter and Theorem 5.1 we obtain that $\pi^S(X^N, x^N) = 1$. \square

Theorem 5.7. *Let X be a reduced, connected and projective k -scheme provided with a section $x \in X(k)$, let G be a finite k -group scheme, $f : Y \rightarrow X$ a Galois G -torsor endowed with a k -rational point $y \in Y_x(k)$. Let us assume that $\dim(Y) = 1$ then the natural morphism*

$$\pi^N(Y_{\text{red}}, y) \rightarrow \pi^N(Y, y)$$

induced by $i : Y_{\text{red}} \rightarrow Y$ is a closed immersion.

Proof. First we observe that Y_{red} satisfies Nori's conditions for defining the Tannakian category $EF(Y_{\text{red}})$ of essentially finite vector bundles, and the Nori's fundamental group scheme $\pi_1^N(Y_{\text{red}}, y)$ [No2]. And secondly that in dimension one every vector bundle over Y_{red} can be deformed to a vector bundle over Y (cf. for instance [Il, Theorem 8.5.3]). Let V be vector bundle over Y . Then f_*V is essentially finite over X if i^*V is essentially finite over Y_{red} . Indeed $i^*f_*f_*V \simeq i^*(f^*f_*\mathcal{O}_Y \otimes V)$ which is essentially finite over Y_{red} . This implies that there exist a finite k -group scheme M and a M -torsor $h : Z \rightarrow Y_{\text{red}}$ such that $h^*i^*f_*f_*V$ is trivial, hence f_*V is essentially finite by [BD, Theorem 2]. According to Theorem 5.3 this implies that V is essentially finite. \square

It is unlikely that the closed immersion of Theorem 5.7 is an isomorphism in general. It is also not known if there exists a statement analogous to Theorem 5.7 for $\dim X > 1$.

6. ESSENTIALLY FINITE MORPHISMS

Let X be k -scheme such that the category $Ns(X)$ is a Tannakian subcategory of $\text{Coh}(X)$, endowed with a k -rational point $x \in X(k)$. Consider a pointed essentially finite morphism $f : Y \rightarrow X$ (Definition 4.9). In this paragraph we will show the existence of a fundamental group scheme $\pi^N(Y, y)$ (Corollary 6.6), generalizing the results of the previous section. The main tool used in the proof is Lemma 6.2 which states that the property of being Nori-semistable is local in a certain sense.

We leave the proof of the following statement to the reader.

Proposition 6.1. *Let $f : Y \rightarrow X$ be a Nori-semistable morphism and consider the Galois closure $g : \widehat{X}_Y \rightarrow X$ defined in Proposition 3.11. The following properties are equivalent:*

- (1) f is essentially finite;
- (2) $g : \widehat{X}_Y \rightarrow X$ is a torsor under a finite group scheme.

Lemma 6.2. *Let $h : Z \rightarrow Y$ be a finite faithfully flat morphism and V a vector bundle on Y . Then the following properties are equivalent:*

- (1) V is Nori-semistable ;
- (2) h^*V is Nori-semistable.

Proof. It is clear that (1) implies (2). Assume (2) and as in the proof of Proposition 4.3 consider a non constant morphism $i : C \rightarrow Y$ from a smooth projective curve C to Y and the following diagram

$$\begin{array}{ccccc}
 & & \tilde{C} & & \\
 & & \searrow^s & & \\
 & & C' & \xrightarrow{i'} & Z \\
 & \searrow^u & \downarrow h' & & \downarrow h \\
 & & C & \xrightarrow{i} & Y
 \end{array}$$

where the square is cartesian and \tilde{C} is the normalization of an irreducible component of C' which surjects onto C . The morphism u is finite and faithfully flat [Li, § 4, Corollary 3.10].

We assume that h^*V is Nori-semistable. Then $s^*i'^*h^*V = u^*i^*V$ is Nori-semistable. We will deduce from this that i^*V is Nori-semistable. So one is reduced to prove the statement for a finite faithfully flat morphism $h : Z \rightarrow Y$, where Z and Y are smooth projective curves. Here h^*V is supposed to be semistable of degree 0 on Z . This implies that V is of degree 0. Suppose that $W \subset V$ is subbundle with $\deg(W) \geq 0$; then $h^*W \subset h^*V$, $\deg(h^*W) \geq 0$ and thus $h^*W = h^*V$. As h is affine, one may suppose that $h : \text{Spec } B \rightarrow \text{Spec } A$, where A and B are k -algebras, h faithfully flat, and V and W free A modules. The fact that $W \otimes_A B = V \otimes_A B$ implies that V and W are of same rank n . Let us choose basis of W and V over A and consider the $n \times n$ matrix M with coefficients in A giving the coordinates of the basis of W in terms of the basis of V . Then by hypothesis $\det M \in A \cap B^\times$. The fact that h is surjective implies that $\det M \in A^\times$ and $W = V$. This proves that V is semis-stable of degree 0. \square

Lemma 6.3. *Let $h : Z \rightarrow Y$ be a Galois torsor under a finite group scheme. Assume that $Ns(Y)$ is a Tannakian sub-category of $\text{Coh}(Y)$ and let V be a vector bundle on Y . Then the following properties are equivalent:*

- (1) V is essentially finite ;
- (2) h^*V is essentially finite.

Proof. Remark first that $Ns(Z)$ is a Tannakian sub-category of $\text{Coh}(Z)$ according to Theorem 5.1, and thus $EF(Z)$ is well defined. It is clear that (1) implies (2). Conversely assume that h^*V is essentially finite. According to Lemma 6.2, V is Nori-semistable. Moreover $V \subset h_*h^*V$ which is essentially finite according to Theorem 5.3, and thus V is essentially finite. \square

Proposition 6.4. *Let X be a k -scheme such that $Ns(X)$ is a Tannakian sub-category of $\text{Coh}(X)$ endowed with a k -rational point $x \in X(k)$, $f : Y \rightarrow X$ a faithfully flat essentially finite morphism pointed at $y \in Y_x(k)$ such that $H^0(Y, \mathcal{O}_Y) = k$, and $g : \widehat{X}_Y \rightarrow X$ the Galois closure of f constructed in Proposition 3.11. With notation of Proposition 3.11 $g = f \circ h$ where $h : \widehat{X}_Y \rightarrow Y$ is a faithfully flat morphism. Then a vector bundle V on Y is Nori-semistable if and only if g_*h^*V is Nori-semistable on X .*

Moreover the category $Ns(Y)$ is a Tannakian sub-category of $\text{Coh}(Y)$.

Proof. According to Lemma 6.2, V is Nori-semistable if and only if h^*V is Nori-semistable, and according to Corollary 4.4 this last condition is equivalent to the fact that g_*h^*V is Nori-semistable. The last statement is an immediate consequence of Theorem 3.14. \square

Corollary 6.5. *Under the hypothesis of Proposition 6.4, V is essentially finite on Y is and only if g_*h^*V is essentially finite on X .*

Proof. If V is essentially finite, h^*V is also essentially finite, and according to Theorem 5.3 g_*h^*V is essentially finite. Conversely if g_*h^*V is essentially finite, h^*V is essentially finite according to Theorem 5.3 and Lemma 6.3 implies that V is essentially finite. \square

Corollary 6.6. *Under the hypothesis of Proposition 6.4, $EF(Y)$ is a Tannakian sub-category of $\text{Coh}(Y)$, and its Galois group based at $y \in Y_x(k)$ is the isotropy group of y in the action of $\pi^N(X, x)$ on Y_x . The universal torsor attached to $EF(Y)$ and y is $\widehat{X}^N \rightarrow Y$.*

Proof. This is an immediate translation of Theorem 3.14 taking in account Corollary 6.5. \square

Corollary 6.7. *Under the hypothesis of Proposition 6.4, every pointed torsor under a finite group scheme over Y is dominated by $\widehat{X}^N \rightarrow Y$.*

Let us finally state another consequence of Corollary 6.5.

Corollary 6.8. *Under the hypothesis of Proposition 6.4, for any essentially finite vector bundle W on Y , f_*W is essentially finite.*

Proof. According to Corollary 6.5, if W is essentially finite, g_*h^*W is essentially finite. Moreover one knows from Proposition 4.3 that f_*W is Nori-semistable. As $f_*W \subset f_*h_*h^*W = g_*h^*W$, f_*W is also essentially finite. \square

REFERENCES

- [AE] ANTEI, M., EMSALEM, M., *Galois closure of essentially finite morphisms*, Jour. Pure Appl. Alg. **215** (2011), 2567–2585.
- [At] ATIYAH, M., *On the Krull-Schmidt theorem with application to sheaves*, Bull. Soc. Math. Fr. **84** (1956), 307–317.
- [BD] BISWAS I., DOS SANTOS, J. P., *Vector bundles trivialized by proper morphisms and the fundamental group scheme*, Jour. Inst. Math. Jussieu **10** (2011), 225–234.
- [BPS] BISWAS, I., PARAMESWARAN, A. J., SUBRAMANIAN, S., *Monodromy group for a strongly semistable principal bundle over a curve*, Duke Math. J. **132** (2006), 1–48.
- [EHS] ESNAULT, H., HAI, P.H., SUN, X., *On Nori’s fundamental group scheme*, Progress in Mathematics 265, 377–398, Birkhäuser, 2007.
- [E] EMSALEM, M., *Twisting by a Torsor*, <http://arxiv.org/abs/1508.02903>
- [EGAIV] GROTHENDIECK, A. AVEC LA COLLABORATION DE J. DIEUDONNÉ, *Éléments de géométrie algébrique, Étude locale des schémas et des morphismes de schémas, Troisième partie*, Publications mathématiques de l’IHÉS, 1966, 5–255.
- [EM] ESNAULT, H., MEHTA, V. B., *Weak density of the fundamental group scheme*, Int. Math. Res. Not. IMRN 2011, 3071–3081.
- [Ga] GARUTI, M., *On the “Galois closure” for torsors*, Proc. Am. Math. Soc. **137** (2009), 3575–3583.
- [SGA1] GROTHENDIECK, A., *Revêtements étales et groupe fondamental*, Séminaire de géométrie algébrique du Bois Marie, (1960-61).
- [HL] HUYBRECHTS, D., LEHN, M., *The geometry of moduli spaces of sheaves*. Aspects of Mathematics, E31. Friedr. Vieweg & Sohn, Braunschweig, 1997.
- [Il] ILLUSIE, L., *Grothendieck’s existence theorem in formal geometry*. With a letter of J.-P. Serre. Math. Surveys Monogr., 123, Fundamental algebraic geometry, 179-233, Amer. Math. Soc., Providence, RI, 2005.
- [La1] LANGER, A., *On the S -fundamental group scheme*, Ann. Inst. Fourier **61** (2011), 2077–2119.
- [La2] LANGER, A., *On the S -fundamental group scheme II*, Jour. Inst. Math. Jussieu **11** (2012), 835–854.
- [Li] LIU, Q., *Algebraic geometry and arithmetic curves*, Oxford Science Publications (2002).
- [Me] MEHTA, V. B., *Some remarks on the local fundamental group scheme and the big fundamental group scheme*, unpublished manuscript.
- [No1] NORI, M. V., *On the Representations of the Fundamental Group*, Compositio Math. **33** (1976). 29–42.
- [No2] NORI, M. V., *The Fundamental Group-Scheme*, Proc. Indian Acad. Sci. (Math. Sci.) **91** (1982), 73–122.
- [RR] RAMANAN, S., RAMANATHAN, A., *Some remarks on the instability flag*, Tohoku Math. J. **36** (1984), 269–291.

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