FURTHER REMARKS ON THE FUNDAMENTAL
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ABSTRACT. Let X be any scheme defined over a Dedekind scheme S with a given section $x \in X(S)$. We prove the existence of a pro-finite S-group scheme $\aleph(X, x)$ and a universal $\aleph(X, x)$-torsor dominating all the pro-finite pointed torsors over X. When $\aleph(X, x)$ and $\aleph(X, x)'$ are two distinct group schemes with the same universal property then there exist two faithfully flat morphisms $\aleph(X, x) \to \aleph(X, x)'$ and $\aleph(X, x)' \to \aleph(X, x)$ whose compositions may not be isomorphisms. In a similar way we prove the existence of a pro-algebraic S-group scheme $\aleph^{\mathrm{alg}}(X, x)$ and a $\aleph^{\mathrm{alg}}(X, x)$-torsor dominating all the pro-algebraic and affine pointed torsors over X. The case where $X \to S$ has no sections is also considered.

CONTENTS

1. INTRODUCTION

The existence of a group scheme classifying all finite torsors over a given scheme X was first conjectured by Grothendieck in [9, Chapitre X], but it was first proved almost thirty years later by Nori (cf. [10]

and [11]) who named it the *fundamental group scheme*. Nori first constructed it for a connected, proper and reduced scheme X over a perfect[1] field k with a given section $x \in X(k)$, as the k-group scheme $\pi(X, x)$ naturally associated to a neutral tannakian category over k whose objects are essentially finite vector bundles over X. He also gave a different proof for the existence of his fundamental group scheme $\pi(X, x)$ for a reduced and connected scheme X (not necessarily proper) over any field k endowed with a k-rational point x, showing that the category of all finite pointed torsors over X is cofiltered. This method has been generalized for schemes defined over Dedekind schemes, first in [7] and then in [4]. Though it is known that the category of all finite torsors over a reduced scheme is not cofiltered, in [2] the first author claimed that one can construct a fundamental group scheme for a non reduced schemes by showing that the category of finite pointed *Galois* torsors (i.e. those finite torsors which are targeted only by faithfully flat morphisms, see also Definition 2.1) is cofiltered. Unfortunately that proof contains a mistake, and here we actually provide a counterexample to this claim (cf. Example 2.3). In this paper we keep the idea of only considering Galois torsors instead of taking all of them. However a more refined notion will be needed; we will not consider Galois objects in the category of all finite torsors but in the category of pro-finite torsors: only in this new environment we are able to show that there exists a Galois torsor (which will be called *universal*) dominating all the pro-finite Galois torsors (cf. Theorem 3.4). The structural group scheme of this universal torsor will be denoted by $\aleph(X, x)$ and called the *pseudo-fundamental group scheme* of X at the point x. Though in general this new object need not be unique. if there exists any other $\aleph(X, x)'$-universal torsor (i.e. dominating all the pro-finite torsors) then, of course, there exist (at least) two faithfully flat morphisms $\aleph(X, x) \twoheadrightarrow \aleph(X, x)'$ and $\aleph(X, x)' \twoheadrightarrow \aleph(X, x)$ whose compositions need not be isomorphisms. So any two distinct pseudo-fundamental group schemes are tightly related. Again let S be a Dedekind scheme and X a connected S-scheme of finite type with a given section $x \in X(S)$. Let X_{red} be its reduced part. It is natural to wonder how the natural morphism $\pi(X_{\mathrm{red}}, x) \to \aleph(X, x)$ behaves, provided $\pi(X_{\mathrm{red}}, x)$ exists[2]. We

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[1]In [3] it has been pointed out that the perfectness assumption for the field k was only needed to ensure that $H^0(X, \mathcal{O}_X) = k$, so instead of considering only perfect fields one can take *any* field k with the additional assumption on the scheme X that $H^0(X, \mathcal{O}_X) = k$

[2]When S has dimension 0 then we do not need any extra assumptions, but if $dim(S) = 1$ then few extra assumptions may be needed, on X_{red}, if we want to construct $\pi(X_{\mathrm{red}}, x)$, cf. [4].

will discuss this at the end of §3.1. The techniques used in this paper work in a very general setting, in particular this allows us to construct a universal torsor dominating all the Galois objects in the category of pointed pro-algebraic torsors (i.e. those torsors whose structural group scheme is affine and of finite type), implying the existence of a *algebraic pseudo-fundamental group scheme* $\aleph(X,x)^{\mathrm{alg}}$ satisfying similar properties. The importance of introducing $\aleph(X,x)^{\mathrm{alg}}$ is the possibility to have a finer invariant, even for a reduced scheme X, than $\pi(X,x)$ as stressed in Remark 3.5. Moreover again the same techniques are used in §3.2 to define a *non pointed* version of pseudo-fundamental group schemes in the pro-finite and the pro-algebraic environment: in the classical case Nori used pointed torsors in order to avoid the fact that product of two torsors over a third one is not empty and in order to prove uniqueness of $\pi(X_{\mathrm{red}},x)$ (up to a unique isomorphism). Here we do not put these two restrictions so we can define a *global pseudo-fundamental group scheme* $\aleph(X)^{\mathrm{alg}}$ (resp. *global algebraic pseudo-fundamental group scheme* $\aleph(X)^{\mathrm{alg}}$) classifying all the pro-finite (resp. pro-algebraic), not necessarily pointed, torsors over X: this becomes important when k is not algebraically closed and X does not have a k-rational point. This last construction also represents an alternative to the *fundamental groupoid schemes* and *fundamental gerbes* already considered in [6] and [5] respectively.

2. Preliminaries

Let S be any Dedekind scheme (e.g. the spectrum of a field or a discrete valuation ring) and $\eta = Spec(K)$ be its generic point. Let X be a scheme over S endowed with a S-valued point $x : Spec(S) \to X$. A triple (Y,G,y) over X is a *fpqc*-torsor $Y \to X$, under the (right) action of a flat S-group scheme G together with a S-valued point $y \in Y_x(S)$. The morphism between two triples $(Y,G,y) \to (Y',G',y')$ are morphisms of S-schemes $\alpha : G \to G'$, $\beta : Y \to Y'$ such that $\beta(y) = y'$ and the following diagram commutes;

$$
\begin{array}{ccc}
Y \times G & \xrightarrow{\beta \times \alpha} & Y' \times G' \\
{\scriptstyle G\text{-}action}\Big\downarrow & & \Big\downarrow{\scriptstyle G'\text{-}action} \\
Y & \xrightarrow{\beta} & Y'
\end{array}
\qquad .
$$

The category whose objects are triples (Y,G,y) with the additional assumption that G is finite and flat is denoted by $\mathcal{P}(X)$. We denote by $Pro-\mathcal{P}(X)$ the category whose objects are projective limits of objects

in $\mathcal{P}(X)$. For any two objects $(Y, G, y) = \varprojlim_i (Y_i, G_i, y_i)$, $(T, M, t) = \varprojlim_j (T_j, M_j, t_j)$ in $Pro - \mathcal{P}(X))$, morphisms between them is given by

$$Hom\left((Y, G, y), (T, M, t)\right) = \varprojlim_j \varinjlim_i Hom_{\mathcal{P}(X)}\left((Y_i, G_i, y_i), (T_j, M_j, t_j)\right)$$

Definition 2.1. We say that an object (Y, G, y) of $\mathcal{P}(X)$ (resp. of $Pro-\mathcal{P}(X)$) over X is Galois relatively to $\mathcal{P}(X)$ (resp. to $Pro-\mathcal{P}(X)$) if for every triple (Y', G', y') of $\mathcal{P}(X)$ (resp. of $Pro-\mathcal{P}(X)$) and every morphism $(Y', G', y') \to (Y, G, y)$ the group scheme morphism $G' \to G$ is faithfully flat (or, equivalently the morphism $Y' \to Y$ is faithfully flat). The full subcategory of $\mathcal{P}(X)$ (resp. of $Pro - \mathcal{P}(X)$) whose objects are Galois triples is denoted by $\mathcal{G}(X)$ (resp. $\mathcal{G}(X)'$)

Remark 2.2. A projective limit of objects of $\mathcal{G}(X)$ denoted as $Pro-\mathcal{G}(X)$ is an object of $\mathcal{G}(X)'$ [[8], section 8.3.8]. We can restate this saying that $Pro - \mathcal{G}(X)$ is a full subcategory of $\mathcal{G}(X)'$. It is not clear to us if the inclusion functor is essentially surjective.

If the category $\mathcal{G}(X)$ was cofiltered we could easily deduce the existence of a universal torsor projective limit of all the objects in $\mathcal{G}(X)$ (unique up to a unique isomorphism). Unfortunately it is not true in general when X is not reduced. Indeed we provide an example where an object of $\mathcal{G}(X)$ where $X = Spec(k[x]/x^2)$ has a non trivial automorphism, which implies that $\mathcal{G}(X)$ is not cofiltered:

Example 2.3. Here we show that if $X = Spec(k[x]/x^2)$, where k is a field of characteristic 2, the category $\mathcal{G}(X)$ is not cofiltered. It is sufficient to find a k-group scheme G and a pointed G-torsor Y in $\mathcal{G}(X)$ and an automorphism (in $\mathcal{G}(X)$) different from the identity. We choose $G := \alpha_2 = Spec(k[x]/x^2)$ and $Y := Spec(k[x, y]/(x^2, y^2 + x))$ is a G-torsor pointed in the origin. It is not trivial as for all $a \in k[x]/x^2$, $x \neq a^2$. A non-trivial pointed $\alpha_2$torsor is therefore necessarily Galois. The right action of G on Y can be described as a coaction as follows:

$$
\begin{array}{rcl}
\rho : k[x, y]/(x^2, y^2 + x) & \to & k[x]/x^2 \otimes_k k[x, y]/(x^2, y^2 + x) \\
x & \mapsto & 1 \otimes x \\
y & \mapsto & x \otimes 1 + 1 \otimes y
\end{array}
$$

and this action is giving the following isomorphism making Y a G-torsor:

$$\begin{array}{ccc}
\frac{k[x,y]}{x^2,y^2+x} \otimes_{k[x]/x^2} \frac{k[x,y]}{x^2,y^2+x} & \to & \frac{k[x]}{x^2} \otimes_k \frac{k[x,y]}{x^2,y^2+x} \\
1 \otimes x & \mapsto & 1 \otimes x \\
1 \otimes y & \mapsto & 1 \otimes y \\
y \otimes 1 & \mapsto & x \otimes 1 + 1 \otimes y.
\end{array}$$

Now we consider the morphism of $k[x]/x^2$-algebras

$$\varphi^\# : k[x,y]/(x^2, y^2 + x) \to k[x,y]/(x^2, y^2 + x), \qquad y \mapsto x + y$$

and we observe that it commutes with the coaction as $(id \otimes \varphi^\#)\rho = \rho\varphi^\#$. The induced morphism of X-schemes $\varphi : Y \to Y$ and the identity morphism on G give a morphism in $\mathcal{G}(X)$, different from the identity. Hence $\mathcal{G}(X)$ is not cofiltered.

We recall the following well known definition from category theory which will be used crucially in this paper.

Definition 2.4. A skeleton of a category $\mathcal{C}$ is a full subcategory $\mathrm{Sk}(\mathcal{C})$ in which every object in $\mathcal{C}$ is isomorphic to an object in $\mathrm{Sk}(\mathcal{C})$ and no distinct objects in $\mathrm{Sk}(\mathcal{C})$ are isomorphic in $\mathcal{C}$.

Theorem 2.5. *A skeleton of $\mathcal{C}$ always exists. Every skeleton of $\mathcal{C}$ is equivalent to $\mathcal{C}$.*

Proof. Cf. [1], Proposition 4.14. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

Definition 2.6. A S-morphism $i : Y \to Z$ is said to be a generically closed immersion if its restriction $i_\eta : Y_\eta \to Z_\eta$ to η is a closed immersion. When S is the spectrum of a field that simply means that i is a closed immersion.

Remark 2.7. A S-morphism of group schemes $G \to G'$ can be factored into a faithfully flat morphism $G \to Q$, a *model map* (i.e. generically an isomorphism) $Q \to M$ and a closed immersion $M \to G'$. When S has dimension 0 then $Q \to M$ is an isomorphism.

Proposition 2.8. *Given an object (Y, G, y) of $Pro-\mathcal{P}(X)$, there exists an object (T, H, t) of $\mathcal{G}(X)'$ and a morphism $(T, H, t) \to (Y, G, y)$ where $T \to Y$ (or equivalently $H \to G$) is a generically closed immersion. We say in this case that (T, H, t) is contained in (Y, G, y).*

Proof. Let C_Y denote the category whose objects are (A, H, a, f) where (A, H, a) is an object in $Pro - \mathcal{P}(X)$ and $f : A \longrightarrow Y$ is a generically closed immersion in $Pro - \mathcal{P}(X)$ which takes the point a to y. By abuse of notation we denote an object in C_Y by (A, f). The morphisms between two objects (A, f) and (B, g) are morphisms (which turn out

to be generically closed immersions) $h : A \longrightarrow B$ such that following diagram commutes:

$$
\begin{array}{ccc}
A & \xrightarrow{\;f\;} & Y \\
{\scriptstyle h}\Big\downarrow & \nearrow{\scriptstyle g} & \\
B & &
\end{array}
$$

Note that the only endomorphism of any object in C_Y is the identity: it is easy to verify over the generic point η of S then we observe that since A is flat, then the same holds globally.

Let $\mathrm{Sk}(C_Y)$ denote a skeleton of C_Y. We still denote objects of $\mathrm{Sk}(C_Y)$ as (A, f). We define an ordering "$\leq$" in $\mathrm{Sk}(C_Y)$ as following: $(A, f) \leq (B, g)$ if there exists a morphism $h \in \mathrm{Hom}_{C_Y}((A, f), (B, g))$. Since composition of generically closed immersions are generically closed immersions and two isomorphic objects in $\mathrm{Sk}(C_Y)$ are identical then "$\leq$" is a partial order in $\mathrm{Sk}(C_Y)$. Let $\{(Y_i, f_i)\}_{i \in I}$ be a totally ordered subclass of $Ob(Sk(C_Y))$, i.e. a chain in $\mathrm{Sk}(C_Y)$ with distinct objects. The projective limit $\varprojlim_{i \in I} Y_i$ exists and is an object in $Pro - \mathcal{P}(X)$. It is naturally provided with a morphism $\varprojlim_{i \in I} f_i : \varprojlim_{i \in I} Y_i \to Y$ which makes it an object of C_Y; it is thus isomorphic to an object (Y_I, f_I) in $\mathrm{Sk}(C_Y)$. This (Y_I, f_I) is a lower bound for the given chain $\{(Y_i, f_i)\}_{i \in I}$. By applying Zorn's lemma $\mathrm{Sk}(C_Y)$ has a minimal element. We denote it by $(Y_{\min}, f_{\min})$. We claim that the corresponding object $(Y_{\min}, G_{\min}, y_{\min})$ in $Pro - \mathcal{P}(X)$ is Galois. If not, then there exists (as recalled in Remark 2.7) a triple (U, M, u) in $Pro - \mathcal{P}(X)$ and a morphism $(U, M, u) \to (Y_{min}, G_{min}, y_{min})$ which is not faithfully flat. Hence it will factor through a faithfully flat morphism $(U, M, u) \to (U', M', u')$ and a generically closed immersion $(U', M', u') \to (Y_{min}, G_{min}, y_{min})$ contradicting the minimality of (Y_{min}, f_{min}) in $\mathrm{Sk}(C_Y)$, so we can set $(T, H, t) := (Y_{min}, G_{min}, y_{min})$. $\qquad \square$

Corollary 2.9. *Given two objects (Y_i, G_i, y_i) in $\mathcal{G}(X)'$, $i = 1, 2$, there exists an object $(Y_3, G_3, y_3) \in \mathcal{G}(X)'$ with morphisms $(Y_3, G_3, y_3) \to (Y_i, G_i, y_i)$, for $i = 1, 2$.*

Proof. It is sufficient to consider $(Y_1 \times_X Y_2, G_1 \times_k G_2, y_1 \times_k y_2)$ and then to take a Galois triple contained in it following Proposition 2.8. $\qquad \square$

Remark 2.10. Note that in the proof of Proposition 2.8 we have not used any property of finite group schemes. Therefore instead of finite group scheme torsors we can work with finite type group scheme torsors which is in fact a much larger category. We denote the corresponding categories as $\mathcal{P}^{\mathrm{alg}}(X)$, $Pro - \mathcal{P}^{\mathrm{alg}}(X)$, $\mathcal{G}^{\mathrm{alg}}(X)$, $\mathcal{G}^{\mathrm{alg}}(X)'$ respectively.

Under these definitions the proof of Proposition 2.8 goes through exactly in similar fashion, hence the Corollary 2.9.

3. The pseudo-fundamental group scheme

We first study the problem of the existence of a group scheme classifying all finite (resp. affine and of finite type) torsors in the "classical" case of pointed schemes, considering maps between torsors sending the marked point of the source torsor to the marked point of the target. For this reason at the end of §3.1 we will be able to compare $\aleph(X, x)$ to $\pi(X_{\mathrm{red}}, x)$, the latter being Nori's fundamental group scheme. In §3.2 we will provide a short overview on the category of non pointed torsors.

3.1. **The case of pointed torsors.**

Definition 3.1. X has a *pseudo-fundamental group scheme* (PFGS) $\aleph(X, x)$ if there is a triple $(\widehat{X}, \aleph(X, x), \widehat{x})$ in the category $\mathcal{G}(X)'$ such that for each object (Y, G, y) of $\mathcal{G}(X)'$ there is a morphism $(\widehat{X}, \aleph(X, x), \widehat{x}) \to (Y, G, y)$. In this case $\widehat{X}$ is called the universal $\aleph(X, x)$-torsor over X pointed in $\widehat{x}$.

Remark 3.2. It follows from the definition that if $(\widehat{X}, \aleph(X, x), \widehat{x})$ and $(\widehat{X}', \aleph(X, x)', \widehat{x}')$ are two PFGS triples for X then there are faithfully flat morphisms $(\widehat{X}, \aleph(X, x), \widehat{x}) \twoheadrightarrow (\widehat{X}', \aleph(X, x)', \widehat{x}')$
and $(\widehat{X}', \aleph(X, x)', \widehat{x}') \twoheadrightarrow (\widehat{X}, \aleph(X, x), \widehat{x})$ but they need not be isomorphic in $\mathcal{G}'(X)$. However if the PFGS of X is known to be finite then they are all isomorphic (but the isomorphism may not be unique unlike in Nori's case.) For finite type torsors one can similarly define the algebraic pseudo-fundamental group scheme (APFGS) $\aleph^{\mathrm{alg}}(X, x)$ as an object $(\widehat{X}^{\mathrm{alg}}, \aleph^{\mathrm{alg}}(X, x), \widehat{x})$ in $\mathcal{G}^{\mathrm{alg}}(X)'$

Definition 3.3. For two triples (Y_1, G_1, y_1) and (Y_2, G_2, y_2) in $\mathcal{G}(X)'$ (resp. $\mathcal{G}^{\mathrm{alg}}(X)'$) we say that (Y_1, G_1, y_1) dominates (Y_2, G_2, y_2) if there exists a (maybe not unique) morphism $(Y_1, G_1, y_1) \to (Y_2, G_2, y_2)$.

Theorem 3.4. *Let X be a scheme over S with a S-valued point x. Then X has a PFGS (resp. APFGS) $\aleph(X, x)$ (resp. $\aleph^{alg}(X, x)$)).*

Proof. The proof for the existence of PFGS and APFGS are exactly similar. Here we give a proof for the existence of PFGS. We first consider the quotient category $q\mathcal{G}(X)'$ of $\mathcal{G}(X)'$, whose objects are identical with $\mathcal{G}(X)'$ and cardinality of any morphism set in $q\mathcal{G}(X)'$ is at most one and is nonempty if and only if it is nonempty in $\mathcal{G}(X)'$. Then we consider the category $Sk(q\mathcal{G}(X)')$ of $q\mathcal{G}(X)'$, and define the

following relation "$\geq$" in the class $Ob(Sk(q\mathcal{G}(X)'))$, $A \geq B$ if there exists a morphism $A \to B$. The relation "$\geq$" is clearly reflexive and transitive. It is also antisymmetric since we are in $Sk(q\mathcal{G}(X)')$. Hence $(Ob(Sk(U\mathcal{G}'(X))), \geq)$ is a partially ordered class. Let us consider any totally ordered subclass of $Ob(Sk(q\mathcal{G}(X)')$, i.e. any chain in $Sk(q\mathcal{G}(X)')$

$$(1) \qquad \ldots \xrightarrow{d_i} A_i \xrightarrow{d_{i-1}} A_{i-1} \xrightarrow{d_{i-2}} \ldots \xrightarrow{d_1} A_1 \; ,$$

where each A_i's are distinct. If for any i we choose a lift morphism $d_i' : A_{i+1} \to A_i$ in $\mathcal{G}(X)'$, this gives a chain (clearly cofiltered since all the involved objects are distinct) in $\mathcal{G}(X)'$

$$(2) \qquad \ldots \xrightarrow{d_i'} A_i \xrightarrow{d_{i-1}'} A_{i-1} \xrightarrow{d_{i-2}'} \ldots \xrightarrow{d_1'} A_1$$

Let A be the projective limit $\varprojlim_i (A_i, d_i)$ in $\mathcal{G}(X)'$ and A' be the object in $Sk(q\mathcal{G}(X)')$ representing the isomorphism class $[A]$ in $q\mathcal{G}(X)'$. Note that A' may not be isomorphic to A in $\mathcal{G}(X)'$. Clearly A' is an upper bound for (1) in $Sk(q\mathcal{G}(X)')$. By Zorn's lemma there is a maximal element in $Ob(Sk(q\mathcal{G}(X)'))$, we call it M. We claim that M dominates all the objects of $\mathcal{G}(X)'$, i.e. for any $B \in Ob(\mathcal{G}(X)')$ there exists a morphism (in $\mathcal{G}(X)'$) $M \to B$. Indeed by Corollary 2.9 we can build an object M' in $\mathcal{G}(X)'$ dominating both M and B; by the maximality of M in $Sk(q\mathcal{G}(X)')$ we deduce that M and M' are isomorphic in $q\mathcal{G}(X)'$, whence the existence of a morphism $M \to M'$ in $\mathcal{G}(X)'$: thus M dominates any object B. If we now set $(\widehat{X}, \aleph(X, x), \widehat{x}) := M$ then $(\widehat{X}, \aleph(X, x), \widehat{x})$ is a universal torsor and $\aleph(X, x)$ a pseudo-fundamental group scheme. $\qquad \square$

Let now X be a connected S-scheme of finite type with a given section $x \in X(S)$. Let X_{red} be its reduced part. As precised in §1 we assume that for X_{red} we are able to build the fundamental group scheme $\pi(X_{\mathrm{red}}, x)$; this is always possible when $dim(S) = 0$. We choose a PFGS $\aleph(X, x)$ and a universal $\aleph(X, x)$-torsor $\widehat{X} \to X$, pointed in $\widehat{x} \in \widehat{X}_x(k)$. We consider its pullback $\widehat{X}_X$ over X and the unique morphism of torsors

$$\varphi_{\mathrm{red}} : X^N \to \widehat{X}$$

where $X^N \to X$ is the ("N" stands for Nori) universal $\pi(X_{\mathrm{red}}, x)$-fundamental group scheme. Though in characteristic 0 this morphism is known to be an isomorphism, in positive characteristic this is no longer true: for instance when $S = Spec(k)$, k being a field with $char(k) = 2$, $X = Spec(k[x]/x^2)$, in Example 2.3 we recalled that over $X = Spec(k[x]/x^2)$ there are non trivial Galois pointed torsors

while over $X_{\mathrm{red}} = Spec(k)$ there are only trivial pointed torsors. So in this case φ_{red} is trivially a closed immersion. This remains true for any X affine and this follows essentially from [2], §3.2. In a similar way one can study the morphisms between $\aleph^{\mathrm{alg}}(X_{\mathrm{red}}, x)$ and $\aleph^{\mathrm{alg}}(X, x)$ and a similar conclusion when X is affine still holds.

Remark 3.5. It is not difficult to prove that if X is a smooth projective scheme over a field k then $\aleph^{\mathrm{alg}}(X, x)$ is trivial if and only if $X = Spec(k)$. This is false and well known for $\pi(X, x)$.

3.2. **The case of non pointed torsors.** In §1 we made clear that we first defined the pseudo fundamental group scheme giving a S-valued point x on X in order to compare it to Nori's fundamental group scheme whose constructions (both the tannakian and the pro-finite) always need a given point. However when we work over non algebraically closed fields or Dedekind schemes it can be useful to have a similar object even when such a point does not exist. The reader certainly observed that the proofs of §3.1 still holds if the base scheme X and torsors are not pointed. Without repeating the proofs we only introduce new definitions and recall the main properties following same arguments of §3.1. Here $\mathcal{T}(X)$ and $Pro - \mathcal{T}(X)$ will denote respectively the category of finite torsors over X and that of pro-finite torsors over X.

Definition 3.6. We say that an object (Y, G) of $Pro - \mathcal{T}(X)$ is Galois if for every object (Y', G') of $Pro - \mathcal{T}(X)$ and every morphism $(Y', G') \to (Y, G)$ the group scheme morphism $G' \to G$ is faithfully flat (or, equivalently the morphism $Y' \to Y$ is faithfully flat). The full subcategory of $Pro - \mathcal{T}(X)$ whose objects are Galois is denoted by $\mathcal{F}(X)$.

Definition 3.7. X has a *global pseudo-fundamental group scheme* $\aleph(X)$ if there is a pair $(\widehat{X}, \aleph(X))$ in the category $\mathcal{F}(X)$ such that for each object (Y, G) of $\mathcal{F}(X)$ there is a morphism $(\widehat{X}, \aleph(X)) \to (Y, G)$. In this case $\widehat{X}$ is called the universal $\aleph(X)$-torsor over X.

Again it is clear by this definition that whenever $\aleph(X)$ and $\aleph(X)'$ are two distinct global pseudo-fundamental group schemes then we have two (maybe not unique) faithfully flat morphisms $\aleph(X)' \to \aleph(X)$ and $\aleph(X) \to \aleph(X)'$ whose compositions are not necessarily automorphisms.

Theorem 3.8. *Let X be a scheme over a Dedekind scheme S. Then X has a global pseudo-fundamental group scheme $\aleph(X)$.*

In a similar way one can define the *global algebraic pseudo-fundamental group scheme* $\aleph^{\mathrm{alg}}(X)$ (with a obvious meaning) of a scheme X without specifying the existence of a section $x \in X(S)$ and verify that the statements just recalled still hold. The following remark will conclude the paper:

Remark 3.9. When S is the spectrum of an algebraically closed field, $x \in X$ any point and we assume that X has a fundamental group scheme $\pi(X, x)$ then it is not difficult to prove that $\pi(X, x)$ and $\aleph(X)$ are isomorphic.

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