# STABILITY OF THE OPTIMAL FILTER IN CONTINUOUS TIME: BEYOND THE BENEŠ FILTER

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Abstract. We are interested in the optimal filter in a continuous time setting. We want to show that the optimal filter is stable with respect to its initial condition. We reduce the problem to a discrete time setting and apply truncation techniques coming from [OR05]. Due to the continuous time setting, we need a new technique to solve the problem. In the end, we show that the forgetting rate is at least a power of the time $t$. The results can be re-used to prove the stability in time of a numerical approximation of the optimal filter.

## 1. Introduction

We are given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We are interested in the processes $(X_t)_{t\geq 0}$, $(Y_t)_{t\geq 0}$ solutions of the following SDE's in $\mathbb{R}$

$$X_t = X_0 + \int_0^t f(X_s)ds + V_t\,,$$

$$Y_t = \int_0^t h \times X_s ds + W_t\,,$$

where $V$, $W$ are two independent standard Brownian motions, $X_0$ is a random variable in $\mathbb{R}$, of law $\pi_0$. We set $(\mathcal{F}_t)_{t\geq 0}$ to be the filtration associated to $(V_t, W_t)$. For $t \geq 0$, we call optimal filter at time $t$ the law of $X_t$ knowing $(Y_s)_{0\leq s\leq t}$, and we denote it by $\pi_t$. Let $\tau > 0$, this parameter will be adjusted later. For any $t > 0$, we set $Q_t$ to be the transition kernel of the Markov chain $(X_{kt})_{k\geq 0}$. We set $Q = Q_\tau$.

**Hypothesis 1.** *We suppose that $f$ is $C^1$ and that $\|f\|_\infty$, $\|f'\|_\infty$ are bounded by a constant $M$. We suppose $h \geq 1$ and $\tau > 1$.*

*Remark* 1.1. We make the assumption that $h \geq 1$ in order to simplify bounds in the following computations. All the results still hold with any $h > 0$.

We are interested in the stability of $(\pi_t)_{t\geq 0}$ with respect to its initial condition. As explained below in Equation (1.2), for all $t$, $\pi_t$ can be written as a functional of $(Y_s)_{0\leq s\leq t}$ and $\pi_0$. Suppose now we plug into this functional a probability $\pi'_0$ instead of $\pi_0$, we obtain then what is called a "wrongly initialized filter" $\pi'_t$ (Equation (1.3)). One natural question is to ask wether $\pi_t - \pi'_t \underset{t\to+\infty}{\longrightarrow} 0$ in any sense. We would then say that the filter $(\pi_t)$ is stable with respect to its initial condition. This question has been answered for more general processes $(X_t)$ and $(Y_t)$ evolving in continuous time, in the cases where $(X_t)$ stays in a compact space (see, for example [AZ97]), or not (see, for example, [OP96], [Ata98], [Sta05, Sta06, Sta08], [CR11]). We can further classify these results as to wether the rate of convergence is exponential or not; in the case of an exponential rate, the filter would be called "exponentially stable" (with respect to its initial condition). The widespread idea is that exponential stability induces that a numerical approximation of the optimal filter would not deteriorate in time. Such an approximation is usually based on a time-recursive computation and it is believed that exponential stability will prevent an accumulation of errors. In order to use a stability result in a proof concerning a numerical scheme, we also need that the distance between $\pi_t$ and $\pi'_t$ to be expressed in term of the distance between $\pi_0$ and $\pi'_0$, and there is no such result, at least when the time is continuous.

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Our aim in this paper is to show exponential stability in such a way that the results can be used in a proof that a numerical scheme remains good uniformly in time. We follow [OR05] by introducing a "robust filter" restricted to compact spaces. We show that this filter remains close to the optimal filter uniformly in time and this is enough to prove the stability of the optimal filter with respect to its initial condition. As in [OR05], we do not show that the optimal filter is exponentially stable, nor can we write the dependency in $\pi_0$, $\pi_0'$ in the stability result. However, in a future work, we will use the stability properties of the robust filter to show that there exists an numerical approximation that remains uniformly good in time.

In the case where $f$ satisfies a particular differential equation, then $\pi_t$ is called the Beneš (see [Ben81], [BC09]) and there exists an explicit formula for the density of $\pi_t$, for all $t$. The study of the Beneš filter is developed in [Oco99]. What we present here is a case in the neighborhood of the Beneš filter.

The outline of the paper is the following. In Sections 1 and 2, we reduce the problem to a filtering problem in discrete time in which we have a handle on the likelihoods. In Section 3, we recall useful notions on filtering. In Section 4 and 5, we introduce the robust filter and its properties. At the beginning of Section 5, we explain our strategy. In Section 6, we prove the two main results: that the optimal filter can be approximated by robust filters uniformly in time (Proposition 6.3), and that the optimal filter is stable with respect to its initial condition (Theorem 6.4).

1.1. **Estimation of the transition density.** Following [BC09] (Chapter 6, Section 6.1), we introduce the process

$$\widehat{V}_t = V_t + \int_0^t f(X_s)ds\,,\ t \geq 0\,.$$

We introduce a new probability $\widetilde{\mathbb{P}}$ defined by

$$\left.\frac{d\mathbb{P}}{d\widetilde{\mathbb{P}}}\right|_{\mathcal{F}_t} = \exp\left(\int_0^t f(X_s)d\widehat{V}_s - \frac{1}{2}\int_0^t f(X_s)^2 ds\right)\,.$$

By Girsanov's theorem, $\widehat{V}$ is a standard Brownian motion under $\widetilde{\mathbb{P}}$. We set $F$ to be a primitive of $f$. We have, for all $t \geq 0$,

$$\begin{aligned}\int_0^t f(X_s)d\widehat{V}_s - \frac{1}{2}\int_0^t f(X_s)^2 ds &= \int_0^t f(X_s)dX_s - \frac{1}{2}\int_0^t f(X_s)^2 ds \\ &= F(X_t) - F(X_0) - \frac{1}{2}\int_0^t f'(X_s)ds - \frac{1}{2}\int_0^t f(X_s)^2 ds \\ &\geq -M|X_t - X_0| - \frac{Mt}{2} - \frac{M^2 t}{2}\,.\end{aligned}$$

So, for any test function $\varphi$ in $\mathcal{C}_b^+(\mathbb{R})$ (the set of bounded continuous functions on $\mathbb{R}$), $t \geq 0$

$$\begin{aligned}\mathbb{E}(\varphi(X_t)) &= \mathbb{E}^{\mathbb{P}}(\varphi(X_t)) \\ &= \mathbb{E}^{\widetilde{\mathbb{P}}}\left(\varphi(X_t)\left.\frac{d\mathbb{P}}{d\widetilde{\mathbb{P}}}\right|_{\mathcal{F}_t}\right) \\ &\geq \mathbb{E}^{\widetilde{\mathbb{P}}}\left(\varphi(X_t)\exp\left(-M|X_t - X_0| - \frac{Mt}{2} - \frac{M^2 t}{2}\right)\right)\,.\end{aligned}$$

Similarly:

$$\mathbb{E}(\varphi(X_t)) \leq \mathbb{E}^{\widetilde{\mathbb{P}}}\left(\varphi(X_t)\exp\left(M|X_t - X_0| + \frac{Mt}{2}\right)\right)$$

So we have the following Lemma.

**Lemma 1.2.** *For all $x, y \in \mathbb{R}$, $Q(x, dy)$ has a density $Q(x, y)$ with respect to the Lebesgue measure and*

$$\frac{e^{-\frac{(y-x)^2}{2\tau}}}{\sqrt{2\pi\tau}} e^{-M|y-x| - \tau\left(\frac{M}{2} + \frac{M^2}{2}\right)} \leq Q(x, y) \leq \frac{\exp^{-\frac{(y-x)^2}{2\tau}}}{\sqrt{2\pi\tau}} e^{M|y-x| + \frac{M\tau}{2}}\,.$$

1.2. **Estimation of the likelihood.** Following [BC09] (Chapter 6, Section 6.1), we define a new probability $\widehat{\mathbb{P}}$ by (for all $t \geq 0$)

$$\left.\frac{d\mathbb{P}}{d\widehat{\mathbb{P}}}\right|_{\mathcal{F}_t} = \widehat{Z}_t = \exp\left(\int_0^t f(X_s)d\widehat{V}_s - \frac{1}{2}\int_0^t f(X_s)^2 ds + \int_0^t hX_s dY_s - \frac{1}{2}\int_0^t h^2 X_s^2 ds\right)$$

We define, for all $0 \leq s \leq t$,

$$Y_{s:t} = (Y_u)_{s \leq u \leq t}\,.$$

For any test function $\varphi$ in $\mathcal{C}_b^+([0,t])$ and any $t \geq 0$,

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}}(\varphi(Y_{0:t})|X_0, X_t) &= \frac{\mathbb{E}^{\widehat{\mathbb{P}}}\left(\varphi(Y_{0:t})\left.\frac{d\mathbb{P}}{d\widehat{\mathbb{P}}}\right|_{\mathcal{F}_t}\middle| X_0, X_t\right)}{\mathbb{E}^{\widehat{\mathbb{P}}}\left(\left.\frac{d\mathbb{P}}{d\widehat{\mathbb{P}}}\right|_{\mathcal{F}_t}\middle| X_0, X_t\right)} \\
&= \frac{\mathbb{E}^{\widehat{\mathbb{P}}}\left(\varphi(Y_{0:t})\mathbb{E}^{\widehat{\mathbb{P}}}\left(\left.\frac{d\mathbb{P}}{d\widehat{\mathbb{P}}}\right|_{\mathcal{F}_t}\middle| X_0, X_t, Y_{0:t}\right)\middle| X_0, X_t\right)}{\mathbb{E}^{\widehat{\mathbb{P}}}\left(\left.\frac{d\mathbb{P}}{d\widehat{\mathbb{P}}}\right|_{\mathcal{F}_t}\middle| X_0, X_t\right)}\,.
\end{aligned}$$

By Girsanov's theorem, $(\widehat{V}, Y)$ is a standard two-dimensional Brownian motion under $\widehat{\mathbb{P}}$. So, conditionally on $X_0$, $X_t$, the law of $Y_{0:t}$ under $\mathbb{P}$ has the following density with respect to the Wiener measure:

$$y_{0:t} \mapsto \psi_t(Y_{0:t}, X_0, X_t) = \frac{\mathbb{E}^{\widehat{\mathbb{P}}}\left(\left.\frac{d\mathbb{P}}{d\widehat{\mathbb{P}}}\right|_{\mathcal{F}_t}\middle| X_0, X_t, Y_{0:t}\right)}{\mathbb{E}^{\widehat{\mathbb{P}}}\left(\left.\frac{d\mathbb{P}}{d\widehat{\mathbb{P}}}\right|_{\mathcal{F}_t}\middle| X_0, X_t\right)}$$

We have

$$\begin{aligned}
\left.\frac{d\mathbb{P}}{d\widehat{\mathbb{P}}}\right|_{\mathcal{F}_t} = \exp\Bigg(F(X_1) - F(X_0) - \frac{1}{2}\int_0^t f'(X_s)ds - \frac{1}{2}\int_0^1 f(X_s)^2 ds \\
+ \int_0^t hX_s dY_s - \frac{1}{2}\int_0^t h^2 X_s^2 ds\Bigg)
\end{aligned}$$

$$\mathbb{E}^{\widehat{\mathbb{P}}}\left(\left.\frac{d\mathbb{P}}{d\widehat{\mathbb{P}}}\right|_{\mathcal{F}_t}\middle| X_0, X_t\right) = \mathbb{E}^{\widehat{\mathbb{P}}}\left(\exp\left(F(X_1) - F(X_0) - \frac{1}{2}\int_0^t f'(X_s)ds - \frac{1}{2}\int_0^t f(X_s)^2 ds\right)\middle| X_0, X_1\right),$$

so

$$\exp\left(-M|X_t - X_0| - \frac{t(M+M^2)}{2}\right) \leq \mathbb{E}^{\widehat{\mathbb{P}}}\left(\left.\frac{d\mathbb{P}}{d\widehat{\mathbb{P}}}\right|_{\mathcal{F}_t}\middle| X_0, X_t\right) \leq \exp\left(M|X_t - X_0| + \frac{tM}{2}\right).$$

We set

(1.1)

$$\widehat{\psi}_t(y_{0:t}, x_0, x_1) = \mathbb{E}^{\widehat{\mathbb{P}}}\left(\exp\left(\int_0^t hX_s dY_s - \frac{1}{2}\int_0^t (hX_s)^2 ds\right)\middle| X_0 = x_0, X_t = x_1, Y_{0:t} = y_{0:t}\right).$$

Using the above calculations, we can write:

$$\begin{aligned}
\exp\left(-M|X_t - X_0| - \frac{t(M+M^2)}{2}\right)\widehat{\psi}_t(Y_{0:t}, x_0, x_1) \leq \\
\mathbb{E}^{\widehat{\mathbb{P}}}\left(\left.\frac{d\mathbb{P}}{d\widehat{\mathbb{P}}}\right|_{\mathcal{F}_t}\middle| X_0, X_t, Y_{0:t}\right) \leq \exp\left(M|X_t - X_0| + \frac{tM}{2}\right)\widehat{\psi}(Y_{0:t}, x_0, x_1)\,.
\end{aligned}$$

So we have the following Lemma.

**Lemma 1.3.** *For all $t>0$, the law of $Y_{0:t}$ under $\mathbb{P}$ and conditionally on $X_0$, $X_t$ has a density denoted by $y_{0:t}\mapsto\psi_t(y_{0:t},X_0,X_\tau)$ with respect to the Wiener measure. This density satisfies, for all $x$, $z\in\mathbb{R}$ and any continuous trajectory $y_{0:t}$*

$$\widehat{\psi}_t(y_{0:t},x,z)e^{-2M|z-x|-\tau\left(M+\frac{M^2}{2}\right)}\leq\psi_t(y_{0:t},x,z)\leq e^{2M|z-x|+\tau\left(M+\frac{M^2}{2}\right)}\widehat{\psi}_t(y_{0:t},x,z)\,.$$

We set

$$\psi=\psi_\tau\,,\ \widehat{\psi}=\widehat{\psi}_\tau\,.$$

The Kallianpur-Striebel formula (see [BC09], p. 57) gives us the following result.

**Lemma 1.4.** *For all $t>0$ and all bounded continuous $\varphi$,*

$$\pi_t(\varphi)=\frac{\int_{\mathbb{R}}\varphi(y)Q_t(x,dy)\psi_t(Y_{0:t},x,y)\pi_0(dx)}{\int_{\mathbb{R}}Q_t(x,dy)\psi_t(Y_{0:t},x,y)\pi_0(dx)}\,.\tag{1.2}$$

*Proof.* We define a new probability $\check{\mathbb{P}}$ by

$$\left.\frac{d\mathbb{P}}{d\check{\mathbb{P}}}\right|_{\mathcal{F}_t}=\exp\left(\int_o^t hX_sdY_s-\frac{1}{2}\int_0^t h^2X_s^2ds\right)\,,\ \forall t\geq 0\,.$$

For all bounded continuous $\varphi$ and all $t\geq 0$, we have (Kallianpur-Striebel)

$$\mathbb{E}(\varphi(X_t)|Y_{0:t})=\frac{\mathbb{E}^{\check{\mathbb{P}}}\left(\varphi(X_t)\left.\frac{d\mathbb{P}}{d\check{\mathbb{P}}}\right|_{\mathcal{F}_t}|Y_{0:t}\right)}{\mathbb{E}^{\check{\mathbb{P}}}\left(\left.\frac{d\mathbb{P}}{d\check{\mathbb{P}}}\right||Y_{0:t}\right)}\,,$$

and

$$\mathbb{E}^{\check{\mathbb{P}}}\left(\varphi(X_t)\left.\frac{d\mathbb{P}}{d\check{\mathbb{P}}}\right|_{\mathcal{F}_t}|Y_{0:t}\right)=\mathbb{E}^{\check{\mathbb{P}}}\left(\varphi(X_t)\mathbb{E}^{\check{\mathbb{P}}}\left(\left.\frac{d\mathbb{P}}{d\check{\mathbb{P}}}\right|_{\mathcal{F}_t}|Y_{0:t},X_0,X_t\right)|Y_{0:t}\right)\,,$$

and

$$\begin{aligned}
\mathbb{E}^{\check{\mathbb{P}}}\left(\left.\frac{d\mathbb{P}}{d\check{\mathbb{P}}}\right|_{\mathcal{F}_t}|Y_{0:t},X_0,X_t\right) &= \frac{\mathbb{E}^{\widehat{\mathbb{P}}}\left(\left.\frac{d\mathbb{P}}{d\check{\mathbb{P}}}\right|_{\mathcal{F}_t}\left.\frac{d\check{\mathbb{P}}}{d\widehat{\mathbb{P}}}\right|_{\mathcal{F}_t}|Y_{0:t},X_0,X_t\right)}{\mathbb{E}^{\widehat{\mathbb{P}}}\left(\left.\frac{d\check{\mathbb{P}}}{d\widehat{\mathbb{P}}}\right|_{\mathcal{F}_t}|Y_{0:t},X_0,X_t\right)}\\
&= \frac{\psi_t(Y_{0:t},X_0,X_t)\mathbb{E}^{\widehat{\mathbb{P}}}\left(\left.\frac{d\mathbb{P}}{d\widehat{\mathbb{P}}}\right|_{\mathcal{F}_t}|X_0,X_t\right)}{\mathbb{E}^{\widehat{\mathbb{P}}}\left(\left.\frac{d\mathbb{P}}{d\widehat{\mathbb{P}}}\right|_{\mathcal{F}_t}|Y_{0:t},X_0,X_t\right)}\\
&= \psi_t(Y_{0:t},X_0,X_t)\times\frac{\mathbb{E}^{\widehat{\mathbb{P}}}\left(\left.\frac{d\mathbb{P}}{d\widehat{\mathbb{P}}}\right|_{\mathcal{F}_t}|X_0,X_t\right)}{\mathbb{E}^{\widehat{\mathbb{P}}}\left(\left.\frac{d\mathbb{P}}{d\widehat{\mathbb{P}}}\right|_{\mathcal{F}_t}|X_0,X_t\right)}\\
&= \psi_t(Y_{0:t},X_0,X_t)\,.
\end{aligned}$$

As the law of $(X_s)_{s\geq 0}$ is the same under $\mathbb{P}$ or $\check{\mathbb{P}}$, we get the desired result. □

For any probability law $\pi'_0$ on $\mathbb{R}$, we define the wrongly initialized filter (with initial condition $\pi'_0$) by, for any $t>0$,

$$\pi'_t(\varphi)=\frac{\int_{\mathbb{R}}\varphi(y)Q_t(x,dy)\psi_t(Y_{0:t},x,y)\pi'_0(dx)}{\int_{\mathbb{R}}Q_t(x,dy)\psi_t(Y_{0:t},x,y)\pi'_0(dx)}\,.\tag{1.3}$$

## 2. Computation of $\widehat{\psi}$

2.1. **Change of measure.** Under $\widehat{\mathbb{P}}$, $\widehat{V}$ is a standard Brownian motion. So, using a standard representation of a Brownian bridge, we can rewrite $\widehat{\psi}$ as

$$\widehat{\psi}(y_{0:\tau},x,z)=\mathbb{E}\left(\exp\left(\int_0^\tau h\left(x\left(1-\frac{s}{\tau}\right)+z\frac{s}{\tau}+\sigma\left(B_s-\frac{s}{\tau}B_\tau\right)\right)dy_s\right.\right.$$
$$\left.\left.-\frac{h^2}{2}\int_0^\tau\left(x\left(1-\frac{s}{\tau}\right)+z\frac{s}{\tau}+\sigma\left(B_s-\frac{s}{\tau}B_\tau\right)\right)^2ds\right)\right),$$

where $B$ is a standard Brownian motion (under $\mathbb{P}$). As we want to compute the above integral, where $B$ is the only random variable involved, we can suppose that $B$ is adapted to the filtration $\mathcal{F}$. We have (using the change of variable $s'=s/\tau$ and the scaling property of the Brownian motion)

$$\begin{aligned}\widehat{\psi}(y_{0:\tau},x,z)&=\mathbb{E}\left(\exp\left(\int_0^1 h\left(x(1-s')+zs'+B_{\tau s'}-s'B_\tau\right)dy_{\tau s'}\right.\right.\\&\qquad\left.\left.-\frac{h^2\tau}{2}\int_0^1\left(x(1-s')+zs'+B_{\tau s'}-s'B_\tau\right)^2ds'\right)\right)\\&=\mathbb{E}\left(\exp\left(\int_0^1 h(x(1-s')+zs'+\sqrt{\tau}(B_{s'}-s'B_1))dy_{\tau s'}\right.\right.\\&\qquad\left.\left.-\frac{h^2\tau}{2}\int_0^1(x(1-s')+zs'+\sqrt{\tau}(B_{s'}-s'B_1))^2ds\right)\right)\end{aligned}$$

In the spirit of [MY08] (Section 2.1), we define a new probability $\mathbb{Q}$ by (for all $t$)

$$\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_t}=\exp\left(-\frac{h^2\tau^2}{2}\int_0^1 B_s^2ds-h\tau\int_0^1 B_sdB_s\right).$$

By Girsanov's theorem, under the probability $\mathbb{Q}$, the process

$$\beta_t=B_t+\int_0^t h\tau B_sds\,,\ \forall t\geq 0 \tag{2.1}$$

is a Brownian motion. We get

$$\begin{aligned}\widehat{\psi}(y_{0:\tau},x,z)&=\exp\left(\int_0^1 h(x(1-s)+zs)dy_{\tau s}-\frac{h^2\tau}{2}\int_0^1(x(1-s)+zs)^2ds\right)\\&\times\mathbb{E}^{\mathbb{Q}}\left(\exp\left(\int_0^1 h\sqrt{\tau}(B_s-sB_1)dy_{\tau s}-h^2\tau^{3/2}\int_0^1(x(1-s)+zs)(B_s-sB_1)ds\right.\right.\\&\qquad\left.\left.-\frac{h^2\tau^2}{2}\int_0^1 s^2B_1^2-2sB_sB_1ds+h\tau\int_0^1 B_sdB_s\right)\right).\end{aligned}\tag{2.2}$$

Using the integration by parts formula, we can rewrite the last expectation as

(2.3)

$$\begin{aligned}&\mathbb{E}^{\mathbb{Q}}\left(\exp\left(-h\sqrt{\tau}\int_0^1(y_{\tau s}-\int_0^1 y_{\tau u}du)dB_s+h^2\tau^{3/2}\int_0^1\left(-x\frac{(1-s)^2}{2}+z\frac{s^2}{2}+\frac{x}{6}-\frac{z}{6}\right)dB_s\right.\right.\\&\qquad\left.\left.+h^2\tau^2B_1\left(\frac{B_1}{3}-\int_0^1\frac{s^2}{2}dB_s\right)+h\tau\left(\frac{B_1^2}{2}-\frac{1}{2}\right)\right)\right)=\\&\mathbb{E}^{\mathbb{Q}}\left(\exp\left(-h\sqrt{\tau}\int_0^1(y_{\tau s}-\int_0^1 y_{\tau u}du)dB_s+h^2\tau^{3/2}x\int_0^1 sdB_s+\frac{h^2\tau^{3/2}}{2}(z-x)\int_0^1 s^2dB_s\right.\right.\\&\qquad\left.\left.-h^2\tau^{3/2}\left(\frac{x}{3}+\frac{z}{6}\right)B_1+\left(\frac{h^2\tau^2}{3}+\frac{h\tau}{2}\right)B_1^2-\frac{h^2\tau^2}{2}B_1\int_0^1 s^2dB_s-\frac{h\tau}{2}\right)\right).\end{aligned}$$

2.2. **Covariances computation.** The last expectation contains an exponential of a polynomial of degree 2 of 4 Gaussians:

$$G_1 = B_1\,,\; G_2 = \int_0^1 s dB_s\,,\; G_3 = \int_0^1 s^2 dB_s\,,\; G_4 = \int_0^1 \left( y_{\tau s} - \int_0^1 y_{\tau u} du \right) dB_s\,.$$

So this expectation can be expressed as a function of the covariance matrix of these Gaussians. We compute here the covariances which do not depend on $y_{0:\tau}$. We need the following Lemma.

**Lemma 2.1.** *For any $t > 0$, for any function $g : \mathbb{R} \mapsto \mathbb{R}$ which is measurable with respect to the Lebesgue measure and such that $\int_0^t g(s)^2 ds < \infty$, we have :*

$$\int_0^t g(s) dB_s = \int_0^t \left( g(s) - \theta e^{\theta s} \int_s^t e^{-\theta u} g(u) du \right) d\beta_s\,,$$

*with*

$$\theta = h\tau\,. \tag{2.4}$$

*Proof.* Under $\mathbb{Q}$, $B$ is an Ornstein-Uhlenbeck process (see Equation (2.1)). We can write $B$ as the strong solution of (2.1):

$$B_t = e^{-\theta t} \int_0^t e^{\theta s} d\beta_s\,,\; \forall t \geq 0\,. \tag{2.5}$$

We use Ito's formula to compute:

$$\begin{aligned}
&\int_0^t \left( g(s) - \theta e^{\theta s} \int_s^t e^{-\theta u} g(u) du \right) d\beta_s \\
&\quad = \int_0^t \left( g(s) - \theta e^{\theta s} \int_0^t e^{-\theta u} g(u) du \right) d\beta_s + \int_0^t \left( \theta e^{\theta s} \int_0^s e^{-\theta u} g(u) du \right) d\beta_s \\
&= \int_0^t g(s) d\beta_s - \left( \int_0^t e^{-\theta u} g(u) du \right) \left( \int_0^t \theta e^{\theta s} d\beta_s \right) + \int_0^t \left( \theta e^{\theta s} \int_0^s e^{-\theta u} g(u) du \right) d\beta_s \\
&= \int_0^t g(s) d\beta_s - \int_0^t e^{-\theta u} g(u) \left( \int_0^u \theta e^{\theta s} d\beta_s \right) du - \int_0^t \left( \int_0^s e^{-\theta u} g(u) du \right) \theta e^{\theta s} d\beta_s \\
&\qquad + \int_0^t \left( \theta e^{\theta s} \int_0^s e^{-\theta u} g(u) du \right) d\beta_s \\
&\qquad\qquad = \int_0^t g(s) d\beta_s - \int_0^t \theta g(u) B_u du = \int_0^t g(s) dB_s\,.
\end{aligned}$$

□

**Lemma 2.2.** *We have, for all $s, t \geq 0$,*

$$g(s) - \theta e^{\theta s} \int_s^t e^{-\theta u} g(u) du = \begin{cases} e^{\theta(s-t)} & \text{if } g(u) = 1,\ \forall u, \\ \left(t + \frac{1}{\theta}\right) e^{\theta(s-t)} - \frac{1}{\theta} & \text{if } g(u) = u,\ \forall u, \\ \left(t^2 + \frac{2t}{\theta} + \frac{2}{\theta^2}\right) e^{\theta(s-t)} - \left(\frac{2s}{\theta} + \frac{2}{\theta^2}\right) & \text{if } g(u) = u^2,\ \forall u\,. \end{cases}$$

*Proof.* The proof in the case $g(u) = 1$ is straightforward. We compute, for all $s, t \geq 0$:

$$\begin{aligned}
s - \theta e^{\theta s} \int_s^t u e^{-\theta u} du &= s - \theta e^{\theta s} \left[ \left( -\frac{u}{\theta} - \frac{1}{\theta^2} \right) e^{-\theta u} \right]_s^t \\
&= s - \theta \left( -\frac{t}{\theta} - \frac{1}{\theta^2} \right) e^{\theta(s-t)} + \theta \left( -\frac{s}{\theta} - \frac{1}{\theta^2} \right) \\
&= \left( t + \frac{1}{\theta} \right) e^{\theta(s-t)} - \frac{1}{\theta}\,,
\end{aligned}$$

$$\begin{aligned}
s^2-\theta e^{\theta s}\int_s^t u^2 e^{-\theta u}du &= s^2-\theta e^{\theta s}\left[\left(-\frac{u^2}{\theta}-\frac{2u}{\theta^2}-\frac{2}{\theta^3}\right)e^{-\theta u}\right]_s^t\\
&= s^2-\theta e^{\theta s}\left(-\frac{t^2}{\theta}-\frac{2t}{\theta^2}-\frac{2}{\theta^3}\right)e^{-\theta t}+\theta e^{\theta s}\left(-\frac{s^2}{\theta}-\frac{2s}{\theta^2}-\frac{2}{\theta^3}\right)e^{-\theta s}\\
&= \left(t^2+\frac{2t}{\theta}+\frac{2}{\theta^2}\right)e^{\theta(s-t)}-\left(\frac{2s}{\theta}+\frac{2}{\theta^2}\right).
\end{aligned}$$

□

**Lemma 2.3.** *We have:*

$$\mathrm{Var}^{\mathbb{Q}}(G_1)=\frac{1-e^{-2\theta}}{2\theta},$$

$$\mathrm{Var}^{\mathbb{Q}}(G_2)=\left(1+\frac{1}{\theta}\right)^2\frac{(1-e^{-2\theta})}{2\theta}+\frac{1}{\theta^2}-\left(\frac{2}{\theta^2}+\frac{2}{\theta^3}\right)(1-e^{-\theta}),$$

$$\mathrm{Var}^{\mathbb{Q}}(G_3) = \left(1+\frac{2}{\theta}+\frac{2}{\theta^2}\right)^2\frac{(1-e^{-2\theta})}{2\theta} + \left(\frac{2}{\theta}+\frac{2}{\theta^2}\right)^3\frac{\theta}{6} - \frac{8}{6\theta^5} - \frac{4}{\theta^2}\left(1+\frac{2}{\theta}+\frac{2}{\theta^2}\right)$$

$$\mathrm{Cov}^{\mathbb{Q}}(G_1,G_2)=\left(\frac{1}{2\theta}+\frac{1}{2\theta^2}\right)(1-e^{-2\theta})+\frac{e^{-\theta}-1}{\theta^2},$$

$$\mathrm{Cov}^{\mathbb{Q}}(G_1,G_3)=\left(\frac{1}{2\theta}+\frac{1}{\theta^2}+\frac{1}{\theta^3}\right)(1-e^{-2\theta})-\frac{2}{\theta^2},$$

$$\mathrm{Cov}^{\mathbb{Q}}(G_2,G_3)=\left(1+\frac{1}{\theta}\right)\left(\frac{1}{2\theta}+\frac{1}{\theta^2}+\frac{1}{\theta^3}\right)(1-e^{-2\theta})-\left(\frac{1}{\theta^2}+\frac{2}{\theta^3}+\frac{2}{\theta^4}\right)(1-e^{-\theta})-\frac{1}{\theta^2}.$$

*Proof.* Lemma 2.1 tells us that the variables $G_1$, $G_2$, $G_3$, $G_4$ are centered Gaussians under $\mathbb{Q}$. Using Lemma 2.2, we compute the following expectations:

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}(G_1^2) &= \int_0^1 e^{2\theta(s-1)}ds\\
&= \frac{1-e^{-2\theta}}{2\theta},
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}(G_2^2) &= \int_0^1\left(\left(1+\frac{1}{\theta}\right)e^{\theta(s-1)}-\frac{1}{\theta}\right)^2 ds\\
&= \int_0^1\left(1+\frac{1}{\theta}\right)^2 e^{2\theta(s-1)}+\frac{1}{\theta^2}-\frac{2}{\theta}\left(1+\frac{1}{\theta}\right)e^{\theta(s-1)}ds\\
&= \left[\left(1+\frac{1}{\theta}\right)^2\frac{e^{2\theta(s-1)}}{2\theta}+\frac{s}{\theta^2}-\left(1+\frac{1}{\theta}\right)\frac{2e^{\theta(s-1)}}{\theta^2}\right]_0^1\\
&= \left(1+\frac{1}{\theta}\right)^2\frac{(1-e^{-2\theta})}{2\theta}+\frac{1}{\theta^2}-\left(\frac{2}{\theta^2}+\frac{2}{\theta^3}\right)(1-e^{-\theta}),
\end{aligned}$$

$$\mathbb{E}^{\mathbb{Q}}(G_3^2)=\int_0^1\left(\left(1+\frac{2}{\theta}+\frac{2}{\theta^2}\right)e^{\theta(s-1)}-\left(\frac{2s}{\theta}+\frac{2}{\theta^2}\right)\right)^2ds=$$

$$\int_0^1\left(1+\frac{2}{\theta}+\frac{2}{\theta^2}\right)^2e^{2\theta(s-1)}+\left(\frac{2s}{\theta}+\frac{2}{\theta^2}\right)^2-2\left(1+\frac{2}{\theta}+\frac{2}{\theta^2}\right)e^{\theta(s-1)}\left(\frac{2s}{\theta}+\frac{2}{\theta^2}\right)ds=$$

$$\left[\left(1+\frac{2}{\theta}+\frac{2}{\theta^2}\right)^2\frac{e^{2\theta(s-1)}}{2\theta}+\left(\frac{2s}{\theta}+\frac{2}{\theta^2}\right)^3\frac{\theta}{6}-2\left(1+\frac{2}{\theta}+\frac{2}{\theta^2}\right)\left(\frac{2s}{\theta^2}\right)e^{\theta(s-1)}\right]_0^1=$$

$$\left(1+\frac{2}{\theta}+\frac{2}{\theta^2}\right)^2\frac{(1-e^{-2\theta})}{2\theta}+\left(\frac{2}{\theta}+\frac{2}{\theta^2}\right)^3\frac{\theta}{6}-\frac{8}{6\theta^5}-\frac{4}{\theta^2}\left(1+\frac{2}{\theta}+\frac{2}{\theta^2}\right),$$

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}(G_1G_2) &= \int_0^1 e^{\theta(s-1)} \times \left( \left(1+\frac{1}{\theta}\right) e^{\theta(s-1)} - \frac{1}{\theta} \right) ds \\
&= \left[ \left(1+\frac{1}{\theta}\right) \frac{e^{2\theta(s-1)}}{2\theta} - \frac{e^{\theta(s-1)}}{\theta^2} \right]_0^1 = \left( \frac{1}{2\theta} + \frac{1}{2\theta^2} \right)(1-e^{-2\theta}) + \frac{e^{-\theta}-1}{\theta^2} \,,
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}(G_1G_3) &= \int_0^1 e^{\theta(s-1)} \left( \left(1+\frac{2}{\theta}+\frac{2}{\theta^2}\right) e^{\theta(s-1)} - \left( \frac{2s}{\theta} + \frac{2}{\theta^2} \right) \right) ds \\
&= \left[ \left(1+\frac{2}{\theta}+\frac{2}{\theta^2}\right) \frac{e^{2\theta(s-1)}}{2\theta} - \frac{2s}{\theta^2} e^{\theta(s-1)} \right]_0^1 = \left( \frac{1}{2\theta} + \frac{1}{\theta^2} + \frac{1}{\theta^3} \right)(1-e^{-2\theta}) - \frac{2}{\theta^2} \,,
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}(G_2G_3) &= \int_0^1 \left( \left(1+\frac{1}{\theta}\right) e^{\theta(s-1)} - \frac{1}{\theta} \right) \times \left( \left(1+\frac{2}{\theta}+\frac{2}{\theta^2}\right) e^{\theta(s-1)} - \left( \frac{2s}{\theta} + \frac{2}{\theta^2} \right) \right) ds \\
&= \int_0^1 \left(1+\frac{1}{\theta}\right)\left(1+\frac{2}{\theta}+\frac{2}{\theta^2}\right) e^{2\theta(s-1)} - \frac{1}{\theta}\left(1+\frac{2}{\theta}+\frac{2}{\theta^2}\right) e^{\theta(s-1)} \\
&\qquad - \left(1+\frac{1}{\theta}\right) e^{\theta(s-1)} \left( \frac{2s}{\theta} + \frac{2}{\theta^2} \right) + \frac{1}{\theta}\left( \frac{2s}{\theta} + \frac{2}{\theta^2} \right) ds \\
&= \left[ \left(1+\frac{1}{\theta}\right)\left(1+\frac{2}{\theta}+\frac{2}{\theta^2}\right) \frac{e^{2\theta(s-1)}}{2\theta} - \left( \frac{1}{\theta} + \frac{2}{\theta^2} + \frac{2}{\theta^3} \right) \frac{e^{\theta(s-1)}}{\theta} \right. \\
&\qquad \left. - \left(1+\frac{1}{\theta}\right) \frac{2s}{\theta^2} e^{\theta(s-1)} + \left( \frac{s^2}{\theta^2} + \frac{2s}{\theta^3} \right) \right]_0^1
\end{aligned}$$

□

Let $U_1, U_2, U_3, U_4$ be i.i.d. of law $\mathcal{N}(0,1)$. We can find $\alpha, \beta, \gamma, a, b, c, \lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathbb{R}$ such that (under $\mathbb{Q}$)

$$\begin{pmatrix} G_1 \\ G_3 \\ G_2 \\ G_4 \end{pmatrix} \overset{\text{law}}{=} \begin{pmatrix} \alpha U_1 & & & \\ \beta U_1 & +\gamma U_2 & & \\ a U_1 & +b U_2 & +c U_3 & \\ \lambda_1 U_1 & +\lambda_2 U_2 & +\lambda_3 U_3 & +\lambda_4 U_4 \end{pmatrix} .$$

(There is no mistake here, we do intend to look at the vector $(G_1, G_3, G_2, G_4)$.) Indeed, we take

$$\alpha = \sqrt{\mathrm{Var}^{\mathbb{Q}}(G_1)} \,,\ \beta = \frac{\mathrm{Cov}^{\mathbb{Q}}(G_1, G_3)}{\alpha} \,,\ \gamma = \sqrt{\mathrm{Var}^{\mathbb{Q}}(G_3) - \beta^2} \,, \tag{2.6}$$

$$a = \frac{\mathrm{Cov}^{\mathbb{Q}}(G_1, G_2)}{\alpha} \,,\ b = \frac{\mathrm{Cov}^{\mathbb{Q}}(G_2, G_3) - a\beta}{\gamma} \,,\ c = \sqrt{\mathrm{Var}^{\mathbb{Q}}(G_2) - a^2 - b^2} \,. \tag{2.7}$$

And we find $\lambda_1, \dots, \lambda_4$ by solving

$$\left\{ \begin{array}{llllcl} \alpha\lambda_1 & & & & = & \mathrm{Cov}^{\mathbb{Q}}(G_1, G_4) \\ \beta\lambda_1 & +\gamma\lambda_2 & & & = & \mathrm{Cov}^{\mathbb{Q}}(G_3, G_4) \\ a\lambda_1 & +b\lambda_2 & +c\lambda_3 & & = & \mathrm{Cov}^{\mathbb{Q}}(G_2, G_4) \\ \lambda_1^2 & +\lambda_2^2 & +\lambda_3^2 & +\lambda_4^2 & = & \mathrm{Var}^{\mathbb{Q}}(G_4) \,. \end{array} \right. \tag{2.8}$$

We observe that $\alpha, \beta, \gamma, a, b, c$ can be written explicitly in terms of the parameters of the problem.

2.3. **Integral computation.** The last part of (2.3) is equal to

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}} \left( \exp\left( \left( \frac{\theta^2}{3} + \frac{\theta}{2} \right) G_1^2 - \frac{\theta^2}{2} G_1 G_3 - h^2 \tau^{3/2} \left( \frac{x}{3} + \frac{z}{6} \right) G_1 \right.\right. \\
&\qquad \left.\left. + h^2 \tau^{3/2} x G_2 + \frac{h^2 \tau^{3/2}}{2} (z-x) G_3 - h\sqrt{\tau} G_4 - \frac{\theta}{2} \right) \right) = \\
&\int_{u_1,\dots,u_4 \in \mathbb{R}} \exp\left( \left( \frac{\theta^2}{3} + \frac{\theta}{2} \right) \alpha^2 u_1^2 - \frac{\theta^2}{2} \alpha u_1 (\beta u_1 + \gamma u_2) - h^2 \tau^{3/2} \left( \frac{x}{3} + \frac{z}{6} \right) \alpha u_1 \right.
\end{aligned} \tag{2.9}$$

$$+h^2\tau^{3/2}x(au_1+bu_2+cu_3)+\frac{h^2\tau^{3/2}}{2}(z-x)(\beta u_1+\gamma u_2)-h\sqrt{\tau}(\lambda_1 u_1+\cdots+\lambda_4 u_4)-\frac{\theta}{2}\Big)$$

$$\frac{\exp\left(-\frac{(u_1^2+\cdots+u_4^2)}{2}\right)}{(2\pi)^2}du_1\dots du_4=$$

$$\int_{u_1,\dots,u_4\in\mathbb{R}}\exp\left\{-\frac{1}{2\sigma_1^2}\left[u_1-\sigma_1^2\left(-\frac{\theta^2\alpha\gamma}{2}u_2-h^2\tau^{3/2}\left(\frac{x}{3}+\frac{z}{6}\right)\alpha+h^2\tau^{3/2}xa\right.\right.\right.$$
$$\left.\left.+\frac{h^2\tau^{3/2}}{2}(z-x)\beta-h\sqrt{\tau}\lambda_1\right)\right]^2$$
$$+\frac{\sigma_1^2}{2}\left[-\frac{\theta^2\alpha\gamma}{2}u_2-h^2\tau^{3/2}\left(\frac{x}{3}+\frac{z}{6}\right)\alpha+h^2\tau^{3/2}xa+\frac{h^2\tau^{3/2}}{2}(z-x)\beta-h\sqrt{\tau}\lambda_1\right]^2$$
$$+h^2\tau^{3/2}x(bu_2+cu_3)+\frac{h^2\tau^{3/2}}{2}(z-x)\gamma u_2-h\sqrt{\tau}(\lambda_2u_2+\cdots+\lambda_4u_4)-\frac{\theta}{2}$$
$$\left.-\frac{(u_2^2+\cdots+u_4^2)}{2}\right\}\frac{1}{(2\pi)^2}du_1\dots du_4\,,$$

where

$$\sigma_1^2=\left(2\left(-\left(\frac{\theta^2}{3}+\frac{\theta}{2}\right)\alpha^2+\frac{\theta^2\alpha\beta}{2}+\frac{1}{2}\right)\right)^{-1}. \tag{2.10}$$

As the above expectation is finite then $\sigma_1^2$ is well defined. We set

$$m_1=\sigma_1^2\left(-h^2\tau^{3/2}\left(\frac{x}{3}+\frac{z}{6}\right)\alpha+h^2\tau^{3/2}xa+\frac{h^2\tau^{3/2}}{2}(z-x)\beta-h\sqrt{\tau}\lambda_1\right). \tag{2.11}$$

The above expectation (2.9) is equal to:

$$\int_{u_1,\dots,u_4\in\mathbb{R}}\exp\left(-\frac{1}{2\sigma_1^2}\left[u_1+\sigma_1^2\frac{\theta^2\alpha\gamma}{2}u_2-m_1\right]^2\right. \tag{2.12}$$
$$+\left(\frac{\sigma_1^2\theta^2\alpha\gamma}{2}\right)^2u_2^2\frac{1}{2\sigma_1^2}+\frac{m_1^2}{2\sigma_1^2}-\frac{1}{2\sigma_1^2}\times 2\left(\frac{\sigma_1^2\theta^2\alpha\gamma}{2}\right)m_1u_2$$
$$+h^2\tau^{3/2}x(bu_2+cu_3)+\frac{h^2\tau^{3/2}}{2}(z-x)\gamma u_2$$
$$\left.-h\sqrt{\tau}(\lambda_2u_2+\lambda_3u_3+\lambda_4u_4)-\frac{(u_2^2+u_3^2+u_4^2)}{2}-\frac{\theta}{2}\right)\frac{1}{(2\pi)^2}du_1\dots du_4=$$
$$\int_{u_1,\dots,u_4\in\mathbb{R}}\exp\left(-\frac{1}{2\sigma_1^2}\left[u_1+\sigma_1^2\frac{\theta^2\alpha\gamma}{2}u_2-m_1\right]^2\right.$$
$$-\frac{1}{2\sigma_2^2}\left[u_2-\sigma_2^2\left(h^2\tau^{3/2}xb+\frac{h^2\tau^{3/2}}{2}(z-x)\gamma-h\sqrt{\tau}\lambda_2-\frac{\theta^2\alpha\gamma m_1}{2}\right)\right]^2$$
$$\left.+\frac{m_2^2}{2\sigma_2^2}+\frac{m_1^2}{2\sigma_1^2}+h^2\tau^{3/2}xcu_3-h\sqrt{\tau}(\lambda_3u_3+\lambda_4u_4)-\frac{(u_3^2+u_4^2)}{2}-\frac{\theta}{2}\right)\frac{1}{(2\pi)^2}\,,$$

where

$$\sigma_2^2=\left(2\left(-\frac{\sigma_1^2\theta^4\alpha^2\gamma^2}{8}+\frac{1}{2}\right)\right)^{-1}, \tag{2.13}$$

and

$$m_2=\sigma_2^2\left(h^2\tau^{3/2}xb+\frac{h^2\tau^{3/2}}{2}(z-x)\gamma-h\sqrt{\tau}\lambda_2-\frac{\theta^2\alpha\gamma}{2}m_1\right). \tag{2.14}$$

Then (2.12) is equal to:

$$\int_{u_1,\dots,u_4\in\mathbb{R}} \exp\left(-\frac{1}{2\sigma_1^2}\left[u_1+\sigma_1^2\frac{\theta^2\alpha\gamma}{2}u_2-m_1\right]^2-\frac{1}{2\sigma_2^2}\left[u_2-m_2\right]^2+\frac{m_2^2}{2\sigma_2^2}+\frac{m_1^2}{2\sigma_1^2}\right.$$
$$-\frac{1}{2}\left[u_3-h^2\tau^{3/2}cx+h\sqrt{\tau}\lambda_3\right]^2-\frac{1}{2}\left[u_4+h\sqrt{\tau}\lambda_4\right]^2$$
$$\left.+\frac{1}{2}(-h^2\tau^{3/2}cx+h\sqrt{\tau}\lambda_3)^2+\frac{1}{2}(-h\sqrt{\tau}\lambda_4)^2-\frac{\theta}{2}\right)\frac{1}{(2\pi)^2}du_1\dots du_4=$$
$$\sigma_1\sigma_2\exp\left(\frac{m_1^2}{2\sigma_1^2}+\frac{m_2^2}{2\sigma_2^2}+\frac{1}{2}(-h^2\tau^{3/2}cx+h\sqrt{\tau}\lambda_3)^2+\frac{1}{2}(h\sqrt{\tau}\lambda_4)^2-\frac{\theta}{2}\right) \tag{2.15}$$

2.4. **Asymptotic $\tau\to+\infty$.** From (2.2), (2.3), (2.9), (2.12), (2.15), we see that $\widehat{\psi}(y_{0:\tau},x,z)\propto\exp(P(x,z))$ with $P$ a polynomial of degree 2 in $x$, $z$ ("$\propto$" stands for "proportional to"). Let us write $-A_2(\theta)$ for the coefficient of $x^2$ in $P$, $-B_2(\theta)$ for the coefficient of $z^2$ in $P$, $C_1(\theta)$ for the coefficient of $xz$ in $P$ , $A_1(\theta)$ for the coefficient of $x$ in $P$, $B_1(\theta)$ for the coefficient of $z$ in $P$ and $C_0(\theta)$ for the "constant" coefficient. We will write $A_1^{y_{0:\tau}}(\theta)=A_1(y_{0:\tau},\theta)$ (or simply $A_1^{y_{0:\tau}}$), etc, when in want of stressing the dependency in $y$. When there will be no ambiguity, we will drop the $y_.$ superscript. The coefficients $A_2^{y_{0:\tau}}$, $B_2^{y_{0:\tau}}$, $C_1^{y_{0:\tau}}$ do not depend on $y$ as it will be seen below. We have

$$\widehat{\psi}(y_{0:\tau},x,z)=\sigma_1\sigma_2\exp\left(-A_2x^2-B_2z^2+A_1^{y_{0:\tau}}x+B_1^{y_{0:\tau}}z+C_1^{y_{0:\tau}}xz+C_0^{y_{0:\tau}}\right)\,. \tag{2.16}$$

We are interested in the limit $\tau\to+\infty$, with $h$ being fixed (or equivalently $\theta\to+\infty$ with $h$ being fixed).

**Lemma 2.4.** *We have*

$$A_2(\theta)\underset{\theta\to+\infty}{\longrightarrow}\frac{h}{2}\,,\ B_2(\theta)\underset{\theta\to+\infty}{\longrightarrow}\frac{h}{2}\,,\ C_1(\theta)=-\frac{3h}{2\theta}+o\left(\frac{1}{\theta}\right)\,.$$

*Proof.* The coefficient of $x^2$ in $P$ is

$$-A_2(\theta)=-h\frac{\theta}{6}+\frac{\sigma_1^2}{2}h\theta^3\left(-\frac{\alpha}{3}-\frac{\beta}{2}+a\right)^2$$
$$+\frac{\sigma_2^2}{2}h\theta^3\left(b-\frac{\gamma}{2}-\frac{\theta^2\alpha\gamma}{2}\sigma_1^2\left(-\frac{\alpha}{3}-\frac{\beta}{2}+a\right)\right)^2+\frac{1}{2}h\theta^3c^2\,.$$

We compute (using [WR15] software):

$$\alpha=\frac{1}{\sqrt{2\theta}}+o\left(\frac{1}{\theta^n}\right)\,,\ \forall n\geq 1\,, \tag{2.17}$$

$$\mathrm{Cov}^{\mathbb{Q}}(G_1,G_3)=\frac{1}{2\theta}-\frac{1}{\theta^2}+\frac{1}{\theta^3}+o\left(\frac{1}{\theta^3}\right)\,, \tag{2.18}$$

$$\beta=\frac{1}{\sqrt{2\theta}}-\frac{\sqrt{2}}{\theta^{3/2}}+\frac{\sqrt{2}}{\theta^{5/2}}+o\left(\frac{1}{\theta^{5/2}}\right)\,, \tag{2.19}$$

$$\beta^2=\frac{1}{2\theta}-\frac{2}{\theta^2}+\frac{4}{\theta^3}-\frac{4}{\theta^4}+o\left(\frac{1}{\theta^4}\right)\,, \tag{2.20}$$

$$\mathrm{Var}^{\mathbb{Q}}(G_3)=\frac{1}{2\theta}-\frac{2}{3\theta^2}+o\left(\frac{1}{\theta^4}\right)\,, \tag{2.21}$$

$$\gamma=\frac{2}{\theta\sqrt{3}}-\frac{\sqrt{3}}{\theta^2}+\frac{\sqrt{3}}{4\theta^3}+\frac{3\sqrt{3}}{8\theta^4}+o\left(\frac{1}{\theta^4}\right)\,, \tag{2.22}$$

$$\mathrm{Cov}^{\mathbb{Q}}(G_1,G_2)=\frac{1}{2\theta}-\frac{1}{2\theta^2}+o\left(\frac{1}{\theta^3}\right)\,, \tag{2.23}$$

$$\sigma_1^2 = \frac{6}{\theta} + \frac{18}{\theta^2} + \frac{18}{\theta^3} - \frac{54}{\theta^4} + o\left(\frac{1}{\theta^4}\right) , \tag{2.24}$$

$$a = \frac{1}{\sqrt{2\theta}} - \frac{1}{\theta^{3/2}\sqrt{2}} + o\left(\frac{1}{\theta^{5/2}}\right) , \tag{2.25}$$

$$\mathrm{Var}^{\mathbb{Q}}(G_2) = \frac{1}{2\theta} - \frac{3}{2\theta^3} + o\left(\frac{1}{\theta^3}\right) , \tag{2.26}$$

$$b = \frac{\sqrt{3}}{2\theta} - \frac{\sqrt{3}}{4\theta^2} - \frac{9\sqrt{3}}{16\theta^3} + o\left(\frac{1}{\theta^3}\right) , \tag{2.27}$$

$$c^2 = \frac{1}{4\theta^2} - \frac{5}{4\theta^3} + o\left(\frac{1}{\theta^3}\right) , \tag{2.28}$$

$$\sigma_2^2 = \frac{1}{3\theta^2} - \frac{1}{\theta} + 2 + o\left(\frac{1}{\theta^3}\right) . \tag{2.29}$$

From which we deduce

$$b - \frac{\gamma}{2} - \frac{\theta^2\alpha\gamma}{2}\sigma_1^2\left(-\frac{\alpha}{3} - \frac{\beta}{2} + a\right) = \frac{\sqrt{3}}{2\theta^3} + o\left(\frac{1}{\theta^3}\right) ,$$

$$\frac{\sigma_2^2}{2}h\theta^3\left(b - \frac{\gamma}{2} - \frac{\theta^2\alpha\gamma}{2}\sigma_1^2\left(-\frac{\alpha}{3} - \frac{\beta}{2} + a\right)\right)^2 \underset{\theta\to+\infty}{\longrightarrow} 0 ,$$

$$\frac{1}{2}h\theta^3c^2 = h\left(\frac{\theta}{8} - \frac{5}{8} + o(1)\right) ,$$

$$\frac{\sigma_1^2}{2}h\theta^3\left(-\frac{\alpha}{3} - \frac{\beta}{2} + a\right)^2 = h\left(\frac{\theta}{24} + \frac{1}{8} + o(1)\right) ,$$

$$A_2(\theta) \underset{\theta\to+\infty}{\longrightarrow} \frac{h}{2} . \tag{2.30}$$

The coefficient of $z^2$ in $P$ is

$$-B_2(\theta) = -h\frac{\theta}{6} + \frac{\sigma_1^2}{2}h\theta^3\left(-\frac{\alpha}{6} + \frac{\beta}{2}\right)^2 + \frac{\sigma_2^2}{2}h\theta^3\left(\frac{\gamma}{2} - \frac{\theta^2\alpha\gamma}{2}\sigma_1^2\left(-\frac{\alpha}{6} + \frac{\beta}{2}\right)\right)^2 . \tag{2.31}$$

We have:

$$\frac{\sigma_2^2}{2}h\theta^3\left(\frac{\gamma}{2} - \frac{\theta^2\alpha\gamma}{2}\sigma_1^2\left(-\frac{\alpha}{6} + \frac{\beta}{2}\right)\right)^2 \underset{\theta\to+\infty}{\longrightarrow} 0 ,$$

$$\frac{\sigma_1^2}{2}h\theta^3\left(-\frac{\alpha}{6} + \frac{\beta}{2}\right)^2 = h\left(\frac{\theta}{6} - \frac{1}{2} + o(1)\right) ,$$

$$B_2(\theta) \underset{\theta\to+\infty}{\longrightarrow} \frac{h}{2} . \tag{2.32}$$

The coefficient of $xz$ in $P$ is

$$\begin{aligned} C_1(\theta) = -h\frac{\theta}{6} &+ \sigma_1^2h\theta^3\left(-\frac{\alpha}{3} - \frac{\beta}{2} + a\right)\left(-\frac{\alpha}{6} + \frac{\beta}{2}\right) \\ &+ \sigma_2^2h\theta^3\left(b - \frac{\gamma}{2} - \frac{\theta^2\alpha\gamma}{2}\sigma_1^2\left(-\frac{\alpha}{3} - \frac{\beta}{2} + a\right)\right)\left(\frac{\gamma}{2} - \frac{\theta^2\alpha\gamma}{2}\sigma_1^2\left(-\frac{\alpha}{6} + \frac{\beta}{2}\right)\right) \end{aligned} \tag{2.33}$$

(it does not depend on $y_{0:\tau}$). We have:

$$\sigma_2^2h\theta^3\left(b - \frac{\gamma}{2} - \frac{\theta^2\alpha\gamma}{2}\sigma_1^2\left(-\frac{\alpha}{3} - \frac{\beta}{2} + a\right)\right)\left(\frac{\gamma}{2} - \frac{\theta^2\alpha\gamma}{2}\sigma_1^2\left(-\frac{\alpha}{6} + \frac{\beta}{2}\right)\right) \underset{\theta\to+\infty}{\longrightarrow} 0 ,$$

$$\sigma_1^2h\theta^3\left(-\frac{\alpha}{3} - \frac{\beta}{2} + a\right)\left(-\frac{\alpha}{6} + \frac{\beta}{2}\right) = h\left(\frac{\theta}{6} - \frac{3}{2\theta} + o\left(\frac{1}{\theta}\right)\right) ,$$

$$C_1(\theta) = \frac{3h}{2\theta} + o\left(\frac{1}{\theta}\right) . \tag{2.34}$$

□

Let us set, for all $s \leq t$,

$$\mathcal{W}_{s,t} = \sup_{(s_1,s_2)\in[s,t]} |W_{s_1} - W_{s_2}| \,,\; \mathcal{V}_{s,t} = \sup_{(s_1,s_2)\in[s,t]} |V_{s_1} - V_{s_2}| \,.$$

**Definition 2.5.** Suppose we have functions $f_1$, $f_2$ going from some set $F$ to $\mathbb{R}$. We write

$$f_1 \preceq f_2$$

if there exists a constant $B$ in $\mathbb{R}_+$, which does not depend on the parameters of our problem, such that $f_1(z) \leq B f_2(z)$, for all $z$ in $F$.
In the particular case when we are dealing with functions of a parameter $\Delta \in \mathbb{R}$ , we write

$$f_1 \underset{\Delta}{\preceq} f_2 \text{ or } f_1(\Delta) \underset{\Delta}{\preceq} f_2(\Delta)$$

if there exists a constant $B_1$ in $\mathbb{R}_+$, which does not depend on the parameters of our problem, and a constant $\Delta_0$, which may depend on the parameters of our problem, such that

$$\Delta \geq \Delta_0 \Rightarrow f_1(\Delta) \leq B_1 f_2(\Delta) \,.$$

If, in addition, $\Delta_0$ depends continuously on the parameter $\tau$, we write

$$f_1 \underset{\Delta,c}{\preceq} f_2 \,.$$

The notation $\underset{\Delta}{\preceq}$ is akin to the notation $O(\dots)$. It has the advantage that one can single out which asymptotic we are studying.

If we have $f_1 \underset{\Delta,c}{\preceq} f_2$, say for $\tau \geq \tau_0$ ($\tau_0 > 0$), then there exists a constant $B_1$ and a continuous function $\Delta_0$ such that, for all $\tau \geq \tau_0$ and $\Delta \geq \Delta_0(\tau)$, $f_1(\Delta) \leq B_1 f_2(\Delta)$. In particular, for any $\tau_1 > \tau_0$, if $\tau \in [\tau_0, \tau_1]$ and $\Delta \geq \sup_{t\in[\tau_0,\tau_1]} \Delta_0(t)$ then $f_1(\Delta) \leq B_1 f_2(\Delta)$. We then say that

$$f_1 \underset{\Delta}{\preceq} f_2, \text{ uniformly for } \tau \in [\tau_0, \tau_1] \,.$$

We state here (without proof) useful properties concerning the above Definition.

**Lemma 2.6.** *Suppose we have functions $f$, $f_1$, $f_2$, $h_1$, $h_2$.*

- *If $f \leq f_1 + f_2$ and $f_1 \preceq f_2$ then $f \preceq f_2$.*
- *If $f \leq f_1 + f_2$ and $\log(f_1) \underset{\Delta}{\preceq} h_1$ and $\log(f_2) \underset{\Delta}{\preceq} h_2$, with $h_1(\Delta)$, $h_2(\Delta) \underset{\Delta\to\infty}{\longrightarrow} -\infty$, then $\log(f) \underset{\Delta}{\preceq} \sup(h_1, h_2)$.*

**Lemma 2.7.** *For all $k \in \mathbb{N}$,*

$$\left|B_1(Y_{k\tau:(k+1)\tau}, \theta) - B_1(Y_{(k+1)\tau:(k+2)\tau}, \theta)\right| \preceq Mh\tau^2 + h\mathcal{V}_{k\tau,(k+2)\tau} + h\mathcal{W}_{k\tau,(k+2)\tau} \,,$$

*and*

$$|B_1(Y_{0:\tau}, \theta)| \preceq Mh\tau^2 + h\mathcal{V}_{0,2\tau} + h\mathcal{W}_{0,2\tau} \,.$$

We need the following result to prove the above lemma.

**Lemma 2.8.** *For all $k \in \mathbb{N}$, $s \in [k, k+1]$*

$$\left|Y_{\tau s} - \int_k^{k+1} Y_{\tau u} du - Y_{\tau(s+1)} + \int_k^{k+1} Y_{\tau(u+1)} du\right| \preceq h\tau^2 M + h\tau \mathcal{V}_{k\tau,(k+1)\tau} + \mathcal{W}_{k\tau,(k+1)\tau} \,,$$

$$\left|\int_0^1 \frac{e^{-\theta}\sinh(\theta s)}{\theta} dY_{\tau k+\tau s} - \int_0^1 \frac{e^{-\theta}\sinh(\theta s)}{\theta} dY_{\tau(k+1)+\tau s}\right| \preceq \frac{M\tau + \mathcal{V}_{k\tau,(k+1)\tau} + \mathcal{W}_{k\tau,(k+1)\tau}}{\theta} \,.$$

*And, for all $s \in [0,1]$,*

$$\left|Y_{\tau s} - \int_0^1 Y_{\tau u} du\right| \preceq h\tau^2 M + h\tau \mathcal{V}_{0,\tau} + h\tau \mathcal{W}_{0,\tau} \,,$$

$$\left|\int_0^1 \frac{e^{-\theta}\sinh(\theta s)}{\theta} dY_{\tau s}\right| \preceq \frac{M\tau + \mathcal{V}_{0,\tau} + \mathcal{W}_{0,\tau}}{\theta}\,.$$

*Proof.* We write the proof only for the first two formulas. We have, for all $k \in \mathbb{N}$, $s \in [k, k+1]$,

$$\begin{aligned}
&\left|Y_{\tau s} - \int_k^{k+1} Y_{\tau u}du - Y_{\tau(s+1)} + \int_k^{k+1} Y_{\tau(u+1)}du\right| \\
&= \left|-\int_{\tau s}^{\tau(s+1)} hX_u du + W_{\tau s} - W_{\tau(s+1)} - \int_k^{k+1}\left(-h\int_{\tau u}^{\tau(u+1)} X_v dv + W_{\tau u} - W_{\tau(u+1)}\right)du\right| \\
&\preceq \left|\int_k^{k+1} h\left(\int_{\tau s}^{\tau(s+1)} X_v dv - \int_{\tau u}^{\tau(u+1)} X_v dv\right)du\right| + \mathcal{W}_{k\tau,(k+2)\tau} \\
&= \left|h\int_k^{k+1}\left(\int_{\tau s}^{\tau(s+1)} X_v - X_{v+\tau(u-s)}dv\right)du\right| + \mathcal{W}_{k\tau,(k+2)\tau} \\
&= \left|h\int_k^{k+1}\left(\int_{\tau s}^{\tau(s+1)}\int_{v+\tau(u-s)}^{v} f(X_t)dt + V_v - V_{v+\tau(u-s)}dvdu\right)\right| + \mathcal{W}_{k\tau,(k+2)\tau} \\
&\leq h\tau^2 M + h\tau\mathcal{V}_{k\tau,(k+2)\tau} + \mathcal{W}_{k\tau,(k+2)\tau}\,,
\end{aligned}$$

and (using integration by parts)

$$\begin{aligned}
&\left|\int_0^1 \frac{e^{-\theta}\sinh(\theta s)}{\theta}dY_{\tau k+\tau s} - \int_0^1 \frac{e^{-\theta}\sinh(\theta s)}{\theta}dY_{\tau(k+1)+\tau s}\right| \\
&= \left|\int_0^\tau \frac{e^{-\theta}\sinh(hs)}{\theta}(h(X_{\tau k+s} - X_{\tau(k+1)s})ds + dW_{\tau k+s} - dW_{\tau(k+1)+s})\right| \\
&\leq \left|\int_0^\tau \frac{e^{-\theta}\sinh(hs)}{\theta}h(X_{\tau k+s} - X_{\tau(k+1)+s})ds\right| \\
&\quad + \left|\frac{e^{-\theta}\sinh(\theta)}{\theta}(W_{\tau(k+1)} - W_{\tau(k+2)})\right| \\
&\quad + \left|\int_0^\tau (W_{\tau k+s} - W_{\tau(k+1)+s})\frac{e^{-\theta}\cosh(hs)}{\tau}ds\right| \\
&\preceq \int_0^\tau \frac{e^{-\theta}\sinh(hs)}{\theta}h(\tau M + \mathcal{V}_{k\tau,(k+2)\tau})ds + \frac{e^{-\theta}\sinh(\theta)}{\theta}\mathcal{W}_{k\tau,(k+2)\tau} + \int_0^\tau \mathcal{W}_{k\tau,(k+2)\tau}\frac{e^{-\theta}\cosh(hs)}{\tau}ds \\
&\preceq \frac{h(M\tau + \mathcal{V}_{k\tau,(k+2)\tau})}{h\theta} + \frac{\mathcal{W}_{k\tau,(k+2)\tau}}{\theta}\,.
\end{aligned}$$

□

*Proof of Lemma 2.7.* We write the proof in the case $k = 0$. From (2.2), (2.15), we deduce

$$\begin{aligned}
B_1(Y_{0:\tau},\theta) &= -\sigma_1^2 h\theta^2\left(-\frac{\alpha}{6} + \frac{\beta}{2}\right)\lambda_1(Y_{0:\tau}) + h\int_0^1 sdY_{\tau s} \\
&\quad + \sigma_2^2 h\theta^2\left(\frac{\gamma}{2} - \frac{\theta^2\alpha\gamma\sigma_1^2}{2}\left(-\frac{\alpha}{6} + \frac{\beta}{2}\right)\right)\left(-\lambda_2(Y_{0:\tau}) + \frac{\theta^2\alpha\gamma\sigma_1^2}{2}\lambda_1(Y_{0:\tau})\right)\,.
\end{aligned} \tag{2.35}$$

For further use, we also write the formula for $A_1(Y_{0:\tau},\theta)$:

$$\begin{aligned}
A_1(Y_{0:\tau},\theta) &= h\int_0^1(1-s)dY_{s\tau} + \sigma_1^2 h\theta^2\left(\frac{\alpha}{3} + \frac{\beta}{2} - a\right)\lambda_1(Y_{0:\tau}) \\
&+ \sigma_2^2 h\theta^2\left(b - \frac{\gamma}{2} - \frac{\theta^2\alpha\gamma\sigma_1^2}{2}\left(-\frac{\alpha}{3} - \frac{\beta}{2} + a\right)\right)\left(-\lambda_2(Y_{0:\tau}) + \frac{\theta^2\alpha\gamma\sigma_1^2}{2}\lambda_1(Y_{0:\tau})\right) - h\theta^2\lambda_3(Y_{0:\tau})c.
\end{aligned} \tag{2.36}$$

We have to remember here that $\lambda_1$, $\lambda_2$, $\lambda_3$ are functions of $y_{0:\tau}$. So we might write $\lambda_1(y_{0:\tau})$, ... to stress this dependency (and the same goes for other quantities). From Lemmas 2.1, 2.2, we get ($g_1$, $g_2$ defined below)

$$\begin{aligned}\mathrm{Cov}^{\mathbb{Q}}(G_1,G_4)(Y_{0:\tau}) &= \int_0^1 e^{\theta(s-1)}\times\left(g_1(s)-\theta e^{\theta s}\int_s^1 e^{-\theta u}g_1(u)du\right)ds\\ &= \int_0^1 g_1(s)e^{\theta(s-1)}ds-\int_0^1 e^{-\theta u}g_1(u)\int_0^u \theta e^{2\theta s-\theta}dsdu\\ &= \int_0^1 g_1(s)e^{\theta(s-1)}ds-\int_0^1 e^{-\theta u}g_1(u)\left(\frac{e^{2\theta u}-1}{2}\right)e^{-\theta}du\\ &= \int_0^1 g_1(s)e^{-\theta}\cosh(\theta s)ds\,, \qquad (2.37)\end{aligned}$$

$$\mathrm{Cov}^{\mathbb{Q}}(G_1,G_4)(Y_{\tau:2\tau})=\int_0^1 g_2(s)e^{-\theta}\cosh(\theta s)ds\,, \qquad (2.38)$$

with

$$g_1(s)=Y_{\tau s}-\int_0^1 Y_{\tau u}du\,,\; g_2(s)=Y_{\tau(s+1)}-\int_0^1 Y_{\tau(u+1)}du\,,$$

and

$$\begin{aligned}\mathrm{Cov}(G_3,G_4)(Y_{0:\tau}) &= \int_0^1\left(\left(1+\frac{2}{\theta}+\frac{2}{\theta^2}\right)e^{\theta(s-1)}-\left(\frac{2s}{\theta}+\frac{2}{\theta^2}\right)\right)\\ &\quad\times\left(g_1(s)-\theta e^{\theta s}\int_s^1 e^{-\theta u}g_1(u)du\right)ds\\ &=\left(1+\frac{2}{\theta}+\frac{2}{\theta^2}\right)\mathrm{Cov}(G_1,G_4)(Y_{0:\tau})-\int_0^1\frac{2s}{\theta}g_1(s)ds\\ &\quad+\int_0^1 e^{-\theta u}g_1(u)\int_0^u\left(2s+\frac{2}{\theta}\right)e^{\theta s}dsdu\\ &=\left(1+\frac{2}{\theta}+\frac{2}{\theta^2}\right)\mathrm{Cov}(G_1,G_4)(Y_{0:\tau})-\int_0^1\frac{2s}{\theta}g_1(s)ds\\ &\quad+\int_0^1 e^{-\theta u}g_1(u)\frac{2u}{\theta}e^{\theta u}du\\ &=\left(1+\frac{2}{\theta}+\frac{2}{\theta^2}\right)\mathrm{Cov}(G_1,G_4)(Y_{0:\tau})\,,\end{aligned} \qquad (2.39)$$

$$\mathrm{Cov}(G_3,G_4)(Y_{\tau:2\tau})=\left(1+\frac{2}{\theta}+\frac{2}{\theta^2}\right)\mathrm{Cov}(G_1,G_4)(Y_{\tau:2\tau})-\frac{2}{\theta^2}\int_0^1 g_2(u)e^{-\theta u}du\,. \qquad (2.40)$$

From (2.8), (2.17)-(2.29), we deduce (using again [WR15])

$$-\sigma_1^2h\theta^2\left(-\frac{\alpha}{6}+\frac{\beta}{2}\right)\lambda_1(Y_{0:\tau})=-h(2\theta+O(1))\mathrm{Cov}^{\mathbb{Q}}(G_1,G_4)(Y_{0:\tau})\,,$$

$$\sigma_2^2h\theta^2\left(\frac{\gamma}{2}-\frac{\theta^2\alpha\gamma\sigma_1^2}{2}\left(-\frac{\alpha}{6}+\frac{\beta}{2}\right)\right)=hO\left(\theta\right)\,,$$

$$\begin{aligned}-\lambda_2(Y_{0:\tau})+\frac{\theta^2\alpha\gamma\sigma_1^2}{2}\lambda_1(Y_{0:\tau}) &= \mathrm{Cov}^{\mathbb{Q}}(G_1,G_4)(Y_{0:\tau})\left(-\frac{1}{\gamma}\left(1+\frac{2}{\theta}+\frac{2}{\theta^2}-\frac{\beta}{\alpha}\right)+\frac{\beta}{\alpha\gamma}+\frac{\theta^2\gamma\sigma_1^2}{2}\right)\\ &= \mathrm{Cov}^{\mathbb{Q}}(G_1,G_4)(Y_{0:\tau})\times O\left(\frac{1}{\theta}\right)+\int_0^1 g_1(u)e^{-\theta u}du\times O\left(\frac{1}{\theta}\right)\,. \qquad (2.41)\end{aligned}$$

So we get

(2.42)
$$-\sigma_1^2 h\theta^2\left(-\frac{\alpha}{6}+\frac{\beta}{2}\right)\lambda_1(Y_{0:\tau})+\sigma_2^2 h\theta^2\left(\frac{\gamma}{2}-\frac{\theta^2\alpha\gamma\sigma_1^2}{2}\left(-\frac{\alpha}{6}+\frac{\beta}{2}\right)\right)\left(-\lambda_2(Y_{0:\tau})+\frac{\theta^2\alpha\gamma\sigma_1^2}{2}\lambda_1(Y_{0:\tau})\right)$$
$$=-2h(\theta+O(1))\mathrm{Cov}^{\mathbb{Q}}(G_1,G_4)(Y_{0:\tau})\,.$$

We have

$$\begin{aligned}
\mathrm{Cov}^{\mathbb{Q}}(G_1,G_4)(Y_{0:\tau}) &= \int_0^1\left(Y_{\tau s}-\int_0^1 Y_{\tau u}du\right)e^{-\theta}\cosh(\theta s)ds\\
&= \left(Y_\tau-\int_0^1 Y_{\tau u}du\right)\frac{e^{-\theta}\sinh(\theta)}{\theta}-\int_0^1\frac{e^{-\theta}\sinh(\theta s)}{\theta}dY_{\tau s}\\
&= \int_0^1 sdY_{\tau s}\times\frac{e^{-\theta}\sinh(\theta)}{\theta}-\int_0^1\frac{e^{-\theta}\sinh(\theta s)}{\theta}dY_{\tau s}\,, \qquad (2.43)
\end{aligned}$$

and so

(2.44)
$$\begin{aligned}
-2h(\theta+O(1))\mathrm{Cov}^{\mathbb{Q}}(G_1,G_4)(Y_{0:\tau})+h\int_0^1 sdY_{\tau s} &= -2h(\theta+O(1))\left(\int_0^1 sdY_{\tau s}\right)\left(\frac{1}{2\theta}-\frac{e^{-2\theta}}{2\theta}\right)\\
&\quad +h\int_0^1 sdY_{\tau s}+h(\theta+O(1))\times 2\int_0^1\frac{e^{-\theta}\sinh(\theta s)}{\theta}dY_{\tau s}\\
&= h\left(\int_0^1 sdY_{\tau s}\right)\times O\left(\frac{1}{\theta}\right)-h(\theta+O(1))\times 2\int_0^1\frac{e^{-\theta}\sinh(\theta s)}{\theta}dY_{\tau s}
\end{aligned}$$

And so, using Lemma 2.8, Equations (2.42), (2.43), (2.44) (as similar formulas of the ones above are valid if we replace $Y_{0:\tau}$ by $Y_{0:2\tau}$), we get

$$\begin{aligned}
|B_1(Y_{0:\tau},\theta)-B_1(Y_{\tau:2\tau},\theta)| &\preceq \frac{1}{\tau}(h\tau^2M+h\tau\mathcal{V}_{0,2\tau}+\mathcal{W}_{0,2\tau})+h\theta\left(\frac{M\tau^2+\mathcal{V}_{0,2\tau}+\mathcal{W}_{0,2\tau}}{\theta}\right)\\
&\preceq Mh\tau^2+h\mathcal{V}_{0,2\tau}+h\mathcal{W}_{0,2\tau}\,.
\end{aligned}$$

□

## 3. Definitions and useful notions

We follow here the ideas of [OR05].

### 3.1. Notations.

We state here notations and definitions that will be useful throughout the paper.

- The set $\mathbb{R}$, $\mathbb{R}^2$ are endowed, respectively, with $\mathcal{B}(\mathbb{R})$, $\mathcal{B}(\mathbb{R}^2)$, their Borel tribes.
- The set of probability distributions on a measurable space $(E,\mathcal{F})$ and the set of nonnegative measures on $(E,\mathcal{F})$ are denoted by $\mathcal{P}(E)$ and $\mathcal{M}^+(E)$ respectively. We write $\mathcal{C}(E)$ for the set of continuous function on a topological space $E$ and $\mathcal{C}_b^+(E)$ for the set of bounded, continuous, nonnegative functions on $E$.
- When applied to measures, $\|\dots\|$ stands for the total variation norm (for $\mu$, $\nu$ probabilities on a measurable space $(F,\mathcal{F})$, $\|\mu-\nu\|=\sup_{A\in\mathcal{F}}|\mu(A)-\nu(A)|$).
- For any nonnegative kernel $K$ on a measurable space $E$ and any $\mu\in\mathcal{M}^+(E)$, we set
$$K\mu(dv')=\int_E\mu(dv)K(v,dv')\,.$$
- If we have a sequence of nonnegative kernels $K_1$, $K_2$, … on some measured spaces $E_1$, $E_2$, … (meaning that for all $i\geq 1$, $x\in E_{i-1}$, $K_i(x,.)$ is a nonnegative measure on $E_i$, then for all $i<j$, we define the kernel

$$K_{i+1:j}(x_i,dx_j)=\int_{x_{i+1}\in E_{i+1}}\dots\int_{x_{j-1}\in E_{j-1}}K_{i+1}(x_i,dx_{i+1})K_{i+2}(x_{i+1},dx_{i+2})\dots K_j(x_{j-1},dx_j)\,.$$

– For any measurable space $E$ and any nonzero $\mu \in \mathcal{M}^+(E)$, we define the normalized nonnegative measure,
$$\overline{\mu} = \frac{1}{\mu(E)}\mu\,.$$
– For any measurable space $E$ and any nonnegative kernel $K$ defined on $E$, we define the normalized nonnegative nonlinear operator $\overline{K}$ on $\mathcal{M}^+(E)$, taking values in $\mathcal{P}(E)$, and defined by
$$\overline{K}(\mu) = \frac{K\,\mu}{(K\,\mu)(E)} = \frac{K\,\overline{\mu}}{(K\,\overline{\mu})(E)} = \overline{K}(\overline{\mu})\,,$$
for any $\mu \in \mathcal{M}^+(E)$ such that $K\,\mu(E) \neq 0$, and defined by $\overline{K}(\mu) = 0$ otherwise.
– A kernel $K$ from a measurable space $E_1$ into another measurable space $E_2$ is said to be $\epsilon$-mixing ($\epsilon \in (0,1)$) if there exists $\lambda$ in $\mathcal{M}^+(E_2)$ and $\epsilon_1, \epsilon_2 > 0$ such that, for all $x_1$ in $E_1$,
$$\epsilon_1\lambda(.) \leq K(x_1,.) \leq \frac{1}{\epsilon_2}\lambda(.)\,\text{with}\;\epsilon_1\epsilon_2 = \epsilon^2\,.$$
This property implies that, for all $A$, $\mu$, $\overline{K}(\mu)(A) \geq \epsilon^2\overline{\lambda}(A)$. If $\overline{K}$ is Markov, this last inequality implies that $\overline{K}$ is $(1-\epsilon^2)$-contracting in total variation (see [DG01] p. 161-162 for more details):
$$\forall \mu,\nu \in \mathcal{P}(E),\; \|\overline{K}(\mu) - \overline{K}(\nu)\| \leq (1-\epsilon^2)\|\mu-\nu\|\,.$$
– For any measurable space $E$ and any $\psi : E \to \mathbb{R}^+$ (measurable) and $\mu \in \mathcal{M}^+(E)$, we set
$$\langle \mu, \psi\rangle = \int_E \psi(x)\mu(dx)\,.$$
If in addition, $\langle\mu,\psi\rangle > 0$, we set
$$\psi \bullet \mu(dv) = \frac{1}{\langle \mu,\psi\rangle} \times \psi(v)\mu(dv)\,.$$
– For $\mu$ and $\mu'$ in $\mathcal{M}^+(E)$ ($(E,\mathcal{F})$ being a measurable space), we say that $\mu$ and $\mu'$ are comparable if there exist positive constants $a$ and $b$ such that, for all $A \in \mathcal{F}$,
$$a\mu'(A) \leq \mu(A) \leq b\mu'(A)\,.$$
We then define the Hilbert metric between $\mu$ and $\mu'$ by
$$h(\mu,\mu') = \log\left(\frac{\sup_{A\in\mathcal{F}:\mu'(A)>0}\frac{\mu(A)}{\mu'(A)}}{\inf_{A\in\mathcal{F}:\mu'(A)>0}\frac{\mu(A)}{\mu'(A)}}\right)\,.$$
It is easily seen (see for instance [Oud00], Chapter 2) that, for any nonnegative kernal $K$ and any $A$ in $\mathcal{F}$,
$$h(K\mu, K\mu') \leq h(\mu,\mu')\,, \tag{3.1}$$
$$h(\overline{\mu},\overline{\mu'}) \leq h(\mu,\mu')\,, \tag{3.2}$$
$$\exp(-h(\mu,\mu')) \leq \frac{\mu(A)}{\mu'(A)} \leq \exp(h(\mu,\mu')),\text{ if } \mu'(A) > 0\,. \tag{3.3}$$
In addition, we have the following relation with the total variation norm:
$$\|\overline{\mu} - \overline{\mu'}\| \leq \frac{2}{\log(3)}h(\mu,\mu')\,. \tag{3.4}$$
– We set $\widetilde{Q}$ to be the transition of the chain $(X_{k\tau}, X_{(k+1)\tau})_{k\geq 0}$.
– We write $\propto$ between two quantities if they are equal up to a multiplicative constant.
– For $\psi : \mathbb{R}^2 \to \mathbb{R}$, we write $\psi(0,.)$ for the function such that, for all $x$ in $\mathbb{R}$, $\psi(0,.)(x) = \psi(0,x)$.

We suppose here that the observation $(Y_t)_{t\geq 0}$ is fixed. For $k \in \mathbb{N}^*$ and $x, z \in \mathbb{R}$, we define

$$\psi_k(x,z) = \psi(Y_{(k-1)\tau:k\tau}, x, z) \tag{3.5}$$

(the density $\psi$ is defined in Lemma 1.3). For $x_1 \in \mathbb{R}$, $x_2 \in \mathbb{R}$ and $n \in \mathbb{N}^*$, we introduce the nonnegative kernel

$$R_n(x_1, dx_2) = \psi_n(x_1, x_2) Q(x_1, dx_2) \,.$$

Using the above notations, we now have, for all $n \in \mathbb{N}^*$, and for all probability law $\pi'_0$ (with $(\pi'_t)_{t\geq 0}$ defined in Equation (1.3))

$$\pi_{n\tau} = \overline{R}_n(\pi_{(n-1)\tau}) \,,\, \pi'_{n\tau} = \overline{R}_n(\pi'_{(n-1)\tau})$$

and for $0 < m < n$,

$$\pi_{n\tau} = \overline{R}_n\overline{R}_{n-1}\dots\overline{R}_m(\pi_{(m-1)\tau}) \,,\, \pi'_{n\tau} = \overline{R}_n\overline{R}_{n-1}\dots\overline{R}_m(\pi'_{(m-1)\tau}).$$

3.2. **Representation of the optimal filter as the law of a Markov chain.** Regardless of the notations of the other sections, we suppose we have a Markov chain $(\mathfrak{X}_n)_{n\geq 0}$ taking values in measured spaces $E_0$, $E_1$, $\dots$, with nonnegative kernels $\mathfrak{Q}_1$, $\mathfrak{Q}_2$, $\dots$ (it might be a non-homogeneous Markov chain). Suppose we have potentials $\Psi_1 : E_1 \rightarrow \mathbb{R}_+$, $\Psi_2 : E_2 \rightarrow \mathbb{R}_+$, $\dots$ (measurable functions with values in $\mathbb{R}_+$) and a law $\eta_0$ on $E_0$. We are interested in the sequence of probability measures $(\eta_k)_{k\geq 1}$ , respectively on $E_1$, $E_2$, $\dots$, defined by

$$\forall k \geq 1 \,,\, \forall \varphi \in \mathcal{C}_b^+(E_k) \,,\, \eta_k(f) = \frac{\mathbb{E}_{\eta_0}(\varphi(\mathfrak{X}_k)\prod_{1\leq i\leq k}\Psi_i(\mathfrak{X}_i))}{\mathbb{E}_{\eta_0}(\prod_{1\leq i\leq k}\Psi_i(\mathfrak{X}_i))} \,, \tag{3.6}$$

where $\eta_0 \in \mathcal{P}(E_0)$ and the index $\eta_0$ means we start with $\mathfrak{X}_0$ of law $\eta_0$ . We will say that $(\eta_k)_{k\geq 0}$ is a Feynman-Kac sequence on $(E_k)_{k\geq 0}$ based on the transitions $(\mathfrak{Q}_k)_{k\geq 1}$, the potentials $(\Psi_k)_{k\geq 1}$ and the initial law $\eta_0$. Suppose we have another law $\eta'_0$, we then set

$$\forall k \geq 1 \,,\, \forall \varphi \in \mathcal{C}_b^+(E_k) \,,\, \eta'_k(f) = \frac{\mathbb{E}_{\eta'_0}(\varphi(\mathfrak{X}_k)\prod_{1\leq i\leq k}\Psi_i(\mathfrak{X}_i))}{\mathbb{E}_{\eta'_0}(\prod_{1\leq i\leq k}\Psi_i(\mathfrak{X}_i))} \,.$$

If the functions $\Psi_k$'s are likelihood associated to observations of a Markov chain with transitions $\mathfrak{Q}_1$, $\mathfrak{Q}_2$, $\dots$ and initial law $\eta_0$, then the measures $\eta_k$'s are optimal filters. We fix $n \geq 1$. We would like to express $\eta_n$ as the marginal law of some Markov process. We will do so using ideas from [DG01]. We set, for all $k \in \{1, \dots, n\}$,

$$\mathfrak{R}_k(x, dx') = \Psi_k(x')\mathfrak{Q}_k(x, dx') \,.$$

We suppose that, for all $k$, $\mathfrak{R}_k$ is $\epsilon_k$-mixing.By a simple recursion, we have, for all $n$,

$$\overline{\mathfrak{R}_{1:n}} = \overline{\mathfrak{R}}_n\overline{\mathfrak{R}}_{n-1}\dots\overline{\mathfrak{R}}_1 \,. \tag{3.7}$$

We set, for all $k \in \{0, 1, \dots, n-1\}$,

$$\Psi_{n|k}(x) = \int_{x_{k+1}\in E_{k+1}} \dots \int_{x_n \in E_n} \mathfrak{R}_{k+1}(x, dx_{k+1}) \prod_{k+2\leq i\leq n} \mathfrak{R}_i(x_{i-1}, dx_i) \,.$$

If $k = n$, we set $\Psi_{n|n}$ to be constant equal to 1. For $k \in \{1, 2, \dots, n\}$, we set

$$\mathfrak{S}_{n|k}(x, dx') = \frac{\Psi_{n|k+1}(x')}{\Psi_{n|k}(x)}\mathfrak{R}_{k+1}(x, dx') \,.$$

From [DG01], we get the following result (a simple proof can also be found in [OR05], Proposition 3.1).

**Proposition 3.1.** *The operators $(\mathfrak{S}_{n|k})_{0\leq k\leq n-1}$ are Markov kernels. For all $k \in \{0, \dots, n-1\}$, $\mathfrak{S}_{n|k}$ is $\epsilon_{k+1}$-mixing. We have*

$$\eta_n = \mathfrak{S}_{n|n-1}\mathfrak{S}_{n|n-1}\dots\mathfrak{S}_{n|0}(\Psi_{n|0} \bullet \eta_0) \,,$$
$$\eta'_n = \mathfrak{S}_{n|n-1}\mathfrak{S}_{n|n-1}\dots\mathfrak{S}_{n|0}(\Psi_{n|0} \bullet \eta'_0) \,,$$

*and*

$$\|\eta_n - \eta'_n\| \leq \prod_{1\leq k\leq n}(1-\epsilon_k^2) \times \|\Psi_{n|0} \bullet \eta_0 - \Psi_{n|0} \bullet \eta'_0\| \,.$$

Following the computations of [OR05], p. 434, we have, for all measurable $\Psi : \mathbb{R}^2 \to \mathbb{R}^+$,

$$\|\Psi \bullet \eta_0 - \Psi \bullet \eta_0'\| \leq 2 \inf \left(1, \frac{\|\Psi\|}{\langle \eta_0, \Psi \rangle} \|\eta_0 - \eta_0'\|\right) . \tag{3.8}$$

For all $x$ in $E_0$, as $\mathfrak{R}_1$ is $\epsilon_1$-mixing,

$$\begin{aligned} \frac{\Psi_{n|0}(x)}{\langle \eta_0, \Psi_{n|0} \rangle} &= \frac{\int_{z\in E_1} \mathfrak{R}_{2:n}(z, E_n)\mathfrak{R}_1(x, dz)}{\int_{y\in E_0}\int_{z\in E_1} \mathfrak{R}_{2:n}(z, E_n)\mathfrak{R}_1(y, dz)\eta_0(dy)} \\ \text{(for some } \epsilon_1', \epsilon_1'', \lambda_1 \text{with } \epsilon_1'\epsilon_1'' = \epsilon_1^2) &\leq \frac{\int_{z\in E_1} \mathfrak{R}_{2:n}(z, E_n)\frac{1}{\epsilon_1'}\lambda_1(dz)}{\int_{z\in E_1} \mathfrak{R}_{2:n}(z, E_n)\epsilon_1''\lambda_1(dz)} = \frac{1}{\epsilon_1^2} . \end{aligned} \tag{3.9}$$

## 4. Truncated filter

We introduce in this section a filter built with truncated likelihoods. We will call it truncated filter or robust filter, the use of the adjective "robust" refers to the fact that it has stability properties (it appears in the proof of Proposition 6.3 below).

4.1. **Integrals of the potential .** We are look here at $\widehat{\psi}(y_{0:\tau}, x, z)$ for some $x$, $z$ in $\mathbb{R}$ and a fixed observation $y_{0:\tau}$ between 0 and $\tau$. All what will be said is also true for observations between $k\tau$ and $(k+1)\tau$ (for any $k$ in $\mathbb{N}$). From Equations (2.33), (2.35), (2.36), we see that $A_1^{y_{0:\tau}}$, $B_1^{y_{0:\tau}}$ are polynomials of degree 1 in $\lambda_1, \dots, \lambda_3$ and that $C_1$ does not depend on $y_{0:\tau}$. We fix $x$ and $z$ in $\mathbb{R}$. Recall that, by Equation (2.8), Lemmas 2.1 and 2.2, $\lambda_1$, $\lambda_2$, $\lambda_3$ are functions of $y_{0:\tau}$ and that they can be expressed as polynomials of degree 1 of integrals of deterministic functions against $dy_{\tau s}$ (this requires some integrations by parts). Under the law $\widetilde{\mathbb{P}}$, conditioned to $X_0 = x$, $X_\tau = z$, we can write

$$X_t = \left(1 - \frac{s}{\tau}\right) x + \frac{s}{\tau} z + \widetilde{B}_s - \frac{s}{\tau}\widetilde{B}_\tau ,$$

where $(\widetilde{B}_s)_{s\geq 0}$ is a Brownian motion, independent of $W$. And we can write

$$A_1^{Y_{0:\tau}} = \alpha_1 + \int_0^\tau (f_1(s)dW_s + f_2(s)dX_s) , \tag{4.1}$$

for some constant $\alpha_1$, and some deterministic functions $f_1$, $f_2$ (and the same goes for $B_1^{Y_{0:\tau}}$).

We set

$$p_{1,1} = \left(1 - \frac{C_1^2}{4A_2B_2}\right)_+^2 , \; p_{2,1} = -\frac{C_1}{2B_2} , \; p_{2,2} = 1 .$$

From now on, we suppose the following.

**Hypothesis 2.** *We fix a parameter $\iota \in (1/2, 1)$. The parameters $\tau$, $h$, $\Delta$ are such that*

$$A_2 \geq \frac{h}{4} , \; B_2 \geq \frac{h}{4} , C_1 \leq \frac{h}{8} , \; \frac{1}{1 + p_{2,1}} - 6B_2p_{2,1}\theta^{1-\iota} > 0 , \; p_{1,1} > \frac{1}{2} , \; |p_{2,1}| \leq \frac{1}{2} . \tag{4.2}$$

This is possible because of Lemma 2.4 and because this Lemma implies: $p_{2,1} = O(\theta^{-1})$, $p_{1,1} \underset{\theta\to+\infty}{\longrightarrow} 1$ by

Let us set

$$\kappa = \begin{bmatrix} A_2 & -\frac{C_1}{2} \\ -\frac{C_1}{2} & B_2 \end{bmatrix} . \tag{4.3}$$

If we take

$$P = \begin{bmatrix} p_{1,1} & 0 \\ p_{2,1} & p_{2,2} \end{bmatrix} = \begin{bmatrix} \left(1 - \frac{C_1^2}{4A_2B_2}\right)^{1/2} & 0 \\ -\frac{C_1}{2B_2} & 1 \end{bmatrix} , \tag{4.4}$$

then

$$\kappa = P^T \begin{bmatrix} A_2 & 0 \\ 0 & B_2 \end{bmatrix} P . \tag{4.5}$$

We have

$$P^{-1}=\left[\begin{array}{cc}\frac{1}{p_{1,1}} & 0\\ -\frac{p_{2,1}}{p_{1,1}} & 1\end{array}\right]\,.$$

First, we have to rule out the case where $A_1^{y_{0:\tau}}$ and $B_1^{y_{0:\tau}}$ are colinear.

**Lemma 4.1.** *The quantities $A_1^{y_{0:\tau}}$ and $B_1^{y_{0:\tau}}$ are not colinear.*

*Proof.* Suppose there exists $\lambda\in\mathbb{R}$ such that $B_1^{y_{0:\tau}}=\lambda A_1^{y_{0:\tau}}$ for $\lambda_W$-almost all $y_{0:\tau}$. We have, for all $\varphi$ in $\mathcal{C}_b^+(\mathbb{R})$, using Lemma 1.3 (remember Equations (2.2), (2.3), (2.9), (2.12), (2.15))

$$\begin{aligned}(4.6)\quad \int_{\mathcal{C}([0;\tau])}&\varphi(A_1^{y_{0:\tau}})\psi(y_{0:\tau},x,z)\lambda_W(dy_{0:\tau})\\ &\leq e^{2M|x-z|+\tau\left(M+\frac{M^2}{2}\right)}\int_{\mathcal{C}([0;\tau])}\varphi(A_1^{y_{0:\tau}})\widehat{\psi}(y_{0:\tau},x,z)\lambda_W(dy_{0:\tau})\\ &=\sigma_1\sigma_2 e^{2M|x-z|+\tau\left(M+\frac{M^2}{2}\right)}\int_{\mathbb{R}}\varphi(t)\exp(-A_2x^2-B_2z^2+C_1xz+tx+\lambda tz)\Psi'(t)dt\,,\end{aligned}$$

where

$$\Psi'(t)=\mathbb{E}^{\mathbb{P}}(\exp(C_0^{W_{0:\tau}})|A_1^{W_{0:\tau}}=t)\,.$$

We know the integral over the whole domain is finite (because $\int_{\mathcal{C}([0,1])}\psi(y_{0:\tau},x,z)=1$ and because of Lemma 1.3). We introduce $\Psi_1'$ such that

$$\Psi'(t)=\exp(-\frac{1}{4}(t,\lambda t)\kappa^{-1}(t,\lambda t)^T)\Psi_1'(t)\,,$$

and

$$\forall(t_1,t_2)\in\mathbb{R}^2,\ \mathcal{Q}(t_1,t_2)=\exp\left(-\frac{1}{4}(t_1,t_2)\kappa^{-1}(t_1,t_2)^T\right)\,.$$

We have, for all $t$,

$$e^{M|X_0-X_t|-\frac{Mt}{2}-\frac{M^2t}{2}}\leq\left.\frac{d\mathbb{P}}{d\widetilde{\mathbb{P}}}\right|_{\mathcal{F}_t}\leq e^{M|X_t-X_0|+\frac{Mt}{2}}$$

(this can be deduced from the computations above Lemma 1.2). Thus, we have, for all $\varphi$ (the first equality being a consequence of Lemma 1.3)

$$\begin{aligned}\int_{\mathcal{C}([0;1])}\varphi(A_1^{y_{0:\tau}})\psi(y_{0:\tau},x,z)\lambda_W(dy_{0:\tau}) &= \mathbb{E}^{\mathbb{P}}(\varphi(A^{Y_{0:\tau}})|X_0=x,X_\tau=z)\\ &=\frac{\mathbb{E}^{\widetilde{\mathbb{P}}}\left(\varphi(A^{Y_{0:\tau}})\left.\frac{d\mathbb{P}}{d\widetilde{\mathbb{P}}}\right|_{\mathcal{F}_\tau}\middle|X_0=x,X_\tau=z\right)}{\mathbb{E}^{\widetilde{\mathbb{P}}}\left(\left.\frac{d\mathbb{P}}{d\widetilde{\mathbb{P}}}\right|_{\mathcal{F}_\tau}\middle|X_0=x,X_\tau=z\right)}\\ (4.7)\qquad(\text{ by Equation (4.1)}) &\geq e^{-2M|x-z|-\tau\left(M+\frac{M^2}{2}\right)}\int_{\mathbb{R}}\varphi(t)\mathcal{Q}''_{x,z}(t)dt\end{aligned}$$

for some Gaussian density $\mathcal{Q}''_{x,z}$.

From Equations (4.6), (4.7), we deduce that, for $(x,z)$ fixed, we have for almost all $t$,

$$(4.8)\qquad e^{-2M|x-z|-\tau\left(M+\frac{M^2}{2}\right)}\mathcal{Q}''_{x,z}(t)\leq e^{2M|x-z|+\tau\left(M+\frac{M^2}{2}\right)}\sigma_1\sigma_2\mathcal{Q}((t,\lambda t)-2(x,z)\kappa)\Psi_1'(t)\,.$$

In the same way, for $(x,z)$ fixed, we have for almost all $t$,

$$(4.9)\qquad e^{2M|x-z|+\tau\left(M+\frac{M^2}{2}\right)}\mathcal{Q}''_{x,z}(t)\geq e^{-2M|x-z|-\tau\left(M+\frac{M^2}{2}\right)}\sigma_1\sigma_2\mathcal{Q}((t,\lambda t)-2(x,z)\kappa)\Psi_1'(t)\,.$$

The density $\mathcal{Q}''_{x,z}(t)$ is of the form

$$\mathcal{Q}''_{x,z}(t)=\frac{1}{\sqrt{2\pi\sigma_0^2}}\exp\left(-\frac{1}{2\sigma_0^2}(t-(a_0x+b_0z))^2\right)\,,$$

with $\sigma_0$, $a_0$, $b_0$ independent of $(x,z)$. So, looking at the above inequalities in $(x,z)=(0,0)$, we see there exists $\epsilon>0$ and a constant $C_\epsilon$, such that, for almost all $t$ in $(-\epsilon,\epsilon)$,

$$(C_\epsilon\sigma_1\sigma_2)^{-1}e^{-\tau(2M+M^2)}\leq\Psi_1'(t)\leq C_\epsilon(\sigma_1\sigma_2)^{-1}e^{\tau(2M+M^2)}\,. \tag{4.10}$$

For any $t$, the quantities $\log(\mathcal{Q}''_{x,z}(t))$, $\log(\mathcal{Q}((t,\lambda t)-2(x,z)\kappa)$ are polynomials in $x$, $z$, of degree less than 2. Using the above remarks and studying adequate sequences $(x_n,z_n)_{n\geq 0}$ (for example, with $x_n \underset{n\to+\infty}{\longrightarrow} +\infty$, $z_n$ remaining in a neighborhood of 0), one can show that the coefficients in $x^2$, $z^2$ and $xz$ in these two polynomials the same. We then have

$$\frac{a_0^2}{2\sigma_0^2}=A_2\,,\ \frac{b_0^2}{2\sigma_0^2}=B_2\,,\ \frac{a_0b_0}{\sigma_0^2}=C_1\,.$$

By Hypothesis 2, we have

$$\frac{a_0b_0}{\sigma_0^2}=2\sqrt{A_2B_2}\geq\frac{h}{2}>\frac{h}{8}$$

and $C_1\leq h/8$, which is not possible, hence the result. □

We can now write for any test function $\varphi$ in $\mathcal{C}_b^+([0,\tau])$ (remember Equation (2.16))

$$\int_{\mathcal{C}([0,\tau])}\varphi(A_1^{y_{0:\tau}},B_1^{y_{0:\tau}})\widehat{\psi}(y_{0:\tau},x,z)\lambda_W(dy_{0:\tau})= \tag{4.11}$$
$$\sigma_1\sigma_2\int_{\mathbb{R}^k}\varphi(t_1,t_2)\exp\left[-A_2x^2-B_2z^2+C_1xz+t_1x\ +t_2z\right]\times\Psi(t_1,t_2)dt_1dt_2\,,$$

where

$$\Psi(t_1,t_1)=\mathbb{E}^{\mathbb{P}}(\exp(C_0^{W_{0:\tau}})|A_1^{W_{0:\tau}}=t_1,B_1^{W_{0:\tau}}=t_2)\,.$$

We know the integral over the whole domain is finite (because $\int_{\mathcal{C}([0,1])}\psi(y_{0:\tau},x,z)\lambda_W(dy_{0:\tau})=1$ and because of Lemma 1.3). Let us define $\Psi_1$ by the formula

$$\Psi(t_1,t_2)=\exp\left(-\frac{1}{4}(t_1,t_2)\kappa^{-1}(t_1,t_2)^T\right)\Psi_1(t_1,t_2)\,.$$

The next result tells us that, somehow, $\log(\Psi_1(t_1,t_2))$ is negligible before $t_1^2+t_2^2$ (when $(t_1,t_2)\to+\infty$).

**Lemma 4.2.** *There exists a constant $C_1'(h,\tau)$ (continuous in $(h,\tau)$) and $\epsilon>0$ such that for all $(x,z)$ and for almost all $(t_1,t_2)$ in $B(2(x,z)\kappa,\epsilon)$ (the ball of center $2(x,z)\kappa$ and radius $\epsilon$),*

$$\frac{1}{C_1'(h,\tau)}\exp\left(-4M|x-z|-\tau\left(2M+M^2\right)\right)\leq\Psi_1(t_1,t_2) \tag{4.12}$$
$$\leq C_1'(h,\tau)\exp\left(4M|x-z|+\tau\left(2M+M^2\right)\right)\,.$$

*Proof.* We fix $(x,z)$ in $\mathbb{R}^2$. Similarly as (4.6), we get, for all $\varphi\in\mathcal{C}_b^+(\mathbb{R}^2)$,

$$\begin{aligned}\int_{\mathcal{C}([0,\tau])}&\varphi(A_1^{y_{0:\tau}},B_1^{y_{0:\tau}})\psi(y_{0:\tau},x,z)\lambda_W(dy_{0:\tau})\\
&\leq\int_{\mathcal{C}([0,\tau])}\varphi(A_1^{y_{0:\tau}},B_1^{y_{0:\tau}})e^{2M|x-z|+\tau\left(M+\frac{M^2}{2}\right)}\widehat{\psi}(y_{0:\tau},x,z)\lambda_W(dy_{0:\tau})\\
&=\sigma_1\sigma_2e^{2M|x-z|+\tau\left(M+\frac{M^2}{2}\right)}\int_{\mathbb{R}^2}\varphi(t_1,t_2)e^{-A_2x^2-B_2z^2+C_1xz+t_1x+t_2z}\Psi(t_1,t_2)dt_1dt_2\\
&=\sigma_1\sigma_2e^{2M|x-z|+\tau\left(M+\frac{M^2}{2}\right)}\int_{\mathbb{R}^2}\varphi(t_1,t_2)\exp\left(-\frac{1}{4}((t_1,t_2)-2(x,z)\kappa)\kappa^{-1}((t_1,t_2)^T-2\kappa(x,z)^T)\right)\\
&\qquad\times\Psi_1(t_1,t_2)dt_1dt_2\,.\end{aligned}$$

Similarly as (4.7), we get, for all $\varphi$,

$$\int_{\mathcal{C}([0,\tau])}\varphi(A_1^{y_{0:\tau}},B_1^{y_{0:\tau}})\psi(y_{0:\tau},x,z)\lambda_W(dy_{0:\tau})\leq e^{2M|z-x|+\tau\left(M+\frac{M^2}{2}\right)}\int_{\mathbb{R}^2}\varphi(t_1,t_2)\mathcal{Q}'_{x,z}(t_1,t_2)dt_1dt_2\,,$$

$$e^{-2M|z-x|-\tau\left(M+\frac{M^2}{2}\right)}\int_{\mathbb{R}^2}\varphi(t_1,t_2)\mathcal{Q}'_{x,z}(t_1,t_2)dt_1dt_2 \leq \int_{\mathcal{C}([0,\tau])}\varphi(A_1^{y_{0:\tau}},B_1^{y_{0:\tau}})\psi(y_{0:\tau},x,z)\lambda_W(dy_{0:\tau})\,,$$

for some Gaussian density $\mathcal{Q}'_{x,z}$ with covariance matrix which does not depend on $x$, $z$ (see Equation (4.1)). We then have, a.s. in $(t_1,t_2)$ (for the Lebesgue measure),

$$\mathcal{Q}'_{x,z}(t_1,t_2) \leq \sigma_1\sigma_2 e^{4M|x-z|+\tau\left(2M+M^2\right)}\mathcal{Q}((t_1,t_2)-2(x,z)\kappa)\Psi_1(t_1,t_2)\,. \tag{4.13}$$

Using the lower bound in the inequality in Lemma 1.3, we get in the same way, a.s. in $(t_1,t_2)$,

$$\mathcal{Q}'_{x,z}(t_1,t_2) \geq \sigma_1\sigma_2 e^{-4M|x-z|-\tau\left(2M+M^2\right)}\mathcal{Q}((t_1,t_2)-2(x,z)\kappa)\Psi_1(t_1,t_2)\,. \tag{4.14}$$

So, we deduce from Equation (4.14), that there exists $\epsilon_1>0$ such that, for all $(x,z)$ and for almost all $(t_1,t_2)$ in $B(2(x,z)\kappa,\epsilon_1)$

$$\Psi_1(t_1,t_2) \leq C'_1(\tau,h)e^{4M|x-z|+\tau\left(2M+M^2\right)}\,,$$

for some function $C'_1(\tau,h)$ of the parameters $\tau$, $h$ (continuous in $(h,\tau)$). One can also see that $\mathcal{Q}'_{x,z}$ reaches it maximum at $(x,z)\kappa'$, where $\kappa'$ is fixed in $\mathcal{M}_{2,2}(\mathbb{R})$ (the set of $2\times 2$ matrices with coefficients in $\mathbb{R}$). From Equation (4.13), we get that there exists $\epsilon_2>0$ such that, for all $x$, $z$ and for almost all $(t_1,t_2)$ in $B((x,z)\kappa',\epsilon'_2)$,

$$\begin{aligned}\mathcal{Q}((t_1,t_2)-2(x,z)\kappa) \geq \frac{1}{2}\mathcal{Q}'_{x,z}((x,z)\kappa')(\sigma_1\sigma_2)^{-1}e^{-4M|x-z|-\tau\left(4M+2M^2\right)}(C'_1(h,\tau))^{-1}\\ \times\exp\left(-4M\left|\frac{1}{2}(x,z)\kappa'\kappa^{-1}(1,-1)^T\right|\right)\,,\end{aligned}$$

and so, by continuity,

$$\begin{aligned}\mathcal{Q}((x,z)\kappa'-2(x,z)\kappa) \geq \frac{1}{2}\mathcal{Q}'_{x,z}((x,z)\kappa')(\sigma_1\sigma_2)^{-1}e^{-4M|x-z|-\tau\left(4M+2M^2\right)}(C'_1(h,\tau))^{-1}\\ \times\exp\left(-4M\left|\frac{1}{2}(x,z)\kappa'\kappa^{-1}(1,-1)^T\right|\right)\,,\end{aligned}$$

If $\kappa'\neq 2\kappa$, we can find a sequence $(x_n,z_n)$ such that $x_n^2+z_n^2 \underset{n\to+\infty}{\longrightarrow} +\infty$ and

$$\log(\mathcal{Q}((x_n,z_n)\kappa'-2(x_n,z_n)\kappa)) \preceq -(x_n^2+z_n^2)\,,$$

whereas

$$\begin{aligned}\log\left(\mathcal{Q}'_{x,z}((x_n,z_n)\kappa')(\sigma_1\sigma_2)^{-1}e^{-4M|x_n-z_n|-\tau\left(4M+2M^2\right)}\exp\left(-4M\left|\frac{1}{2}(x,z)\kappa'\kappa^{-1}(1,-1)^T\right|\right)\right)\\ \succeq -|x_n-z_n|\,,\end{aligned}$$

which is not possible. So $\kappa'=2\kappa$.

So, we get from Equation (4.13) that there exists $\epsilon_3>0$ such that for all $x$, $z$, and for almost all $(t_1,t_2)$ in $B(2(x,z)\kappa,\epsilon_3)$,

$$\Psi_1(t_1,t_2) \geq \frac{e^{-4M|x-z|-\tau(2M+M^2)}}{C'_1(h,\tau)}$$

(with, possibly, a new $C'_1(h,\tau)$). □

**Lemma 4.3.** *If we have a set $\mathcal{A}=\{y_{0:\tau}\in\mathcal{C}([0;\tau])\,:\,(A_1^{y_{0:\tau}},B_1^{y_{0:\tau}})\in\mathcal{B}\}$ for some subset $\mathcal{B}$ of $\mathbb{R}$, then*

$$\begin{aligned}\int_{\mathcal{A}}\widehat{\psi}(y_{0:\tau},x,z)\lambda_W(dy_{0:\tau}) &\leq \sigma_1\sigma_2C'_1(h,\tau)\\ &\times\int_{(t_2,t_2)\in\mathcal{B}}\exp\left\{-\frac{A_2^{-1}}{4}\left(\left|\frac{t_1}{p_{1,1}}-\frac{p_{2,1}t_2}{p_{1,1}}-2A_2p_{1,1}x\right|-4M\left|\frac{1}{p_{1,1}}+\frac{p_{2,1}}{p_{1,1}}\right|\right)_+^2\right.\\ &\qquad\left.-\frac{B_2^{-1}}{4}\left(|t_2-2B_2(p_{2,1}x+z)|-4M\right)_+^2\right\}\end{aligned}$$

$$\times \exp\left(\left|16M^2(1,-1)\kappa^{-1}(1,-1)^T\right| + \tau\left(2M+M^2\right) + 8M|x-z|\right) dt_1 dt_2\,,$$

*and*

$$\int_{\mathcal{A}} \widehat{\psi}(y_{0:\tau},x,z)\lambda_W(dy_{0:\tau}) \geq \sigma_1\sigma_2 C_1'(h,\tau)$$
$$\times \int_{(t_2,t_2)\in\mathcal{B}} \exp\left\{-\frac{A_2^{-1}}{4}\left(\left|\frac{t_1}{p_{1,1}} - \frac{p_{2,1}t_2}{p_{1,1}} - 2A_2 p_{1,1}x\right| + 4M\left|\frac{1}{p_{1,1}} + \frac{p_{2,1}}{p_{1,1}}\right|\right)^2\right.$$
$$\left. - \frac{B_2^{-1}}{4}\left(|t_2 - 2B_2(p_{2,1}x+z)| + 4M\right)^2\right\}$$
$$\times \exp\left(-\left|16M^2(1,-1)\kappa^{-1}(1,-1)^T\right| - \tau\left(2M+M^2\right) - 8M|x-z|\right) dt_1 dt_2\,.$$

*Proof.* We have

$$\int_{\mathcal{A}} \widehat{\psi}(y_{0:\tau},x,z)\lambda_W(dy_{0:\tau})$$
$$= \sigma_1\sigma_2 \int_{\mathcal{B}} \exp\left(-(x,z)\kappa(x,z)^T + t_1x + t_2z\right)\exp\left(-\frac{1}{4}(t_1,t_2)\kappa^{-1}(t_1,t_2)^T\right)\Psi_1(t_1,t_2)dt_1dt_2$$
$$= \sigma_1\sigma_2 \int_{\mathcal{B}} \exp\left(-\frac{1}{4}[(t_1,t_2)^T - 2\kappa(x,z)^T]^T\kappa^{-1}[(t_1,t_2)^T - 2\kappa(x,z)^T]\right)$$
$$\times \Psi_1\left(2\kappa\frac{1}{2}\kappa^{-1}(t_1,t_2)^T\right)dt_1dt_2$$

(by Lemma 4.2)

$$\leq \sigma_1\sigma_2 \int_{\mathcal{B}} \exp\left(-\frac{1}{4}[(t_1,t_2)^T - 2\kappa(x,z)^T]^T\kappa^{-1}[(t_1,t_2)^T - 2\kappa(x,z)^T]\right)$$
$$\times C_1'(h,\tau)\exp\left(4M\times|(1,-1)\kappa^{-1}(t_1,t_2)^T| + \tau\left(2M+M^2\right)\right)dt_1dt_2\,,$$

and we can bound by below by

$$\sigma_1\sigma_2 \int_{\mathcal{B}} \exp\left(-\frac{1}{4}[(t_1,t_2) - 2\kappa(x,z)]^T\kappa^{-1}[(t_1,t_2) - 2\kappa(x,z)]\right)$$
$$\times \frac{1}{C_1'(h,\tau)}\exp\left(-4M\times|(1,-1)\kappa^{-1}(t_1,t_2)^T| - \tau\left(2M+M^2\right)\right)dt_1dt_2\,,$$

For $(t_1,t_2)\in\mathbb{R}^2$, we have, for any $\delta\in\{-1,1\}$

$$\exp\left(-\frac{1}{4}[(t_1,t_2)^T - 2\kappa(x,z)^T]^T\kappa^{-1}[(t_1,t_2)^T - 2\kappa(x,z)^T] + 4\delta M\times(1,-1)\kappa^{-1}(t_1,t_2)^T\right)$$
$$= \exp\left(-\frac{1}{4}[(t_1,t_2)^T - 2\kappa(x,z)^T - 8\delta M(1,-1)^T]^T\kappa^{-1}[(t_1,t_2)^T - 2\kappa(x,z)^T - 8\delta M(1,-1)^T]\right.$$
$$\left.+16M^2(1,-1)\kappa^{-1}(1,-1)^T + 8\delta M(x,z)(1,-1)^T\right)$$
$$= \exp\left(-\frac{1}{4}[(P^{-1})^T(t_1,t_2)^T - 2\begin{bmatrix} A_2 & 0 \\ 0 & B_2\end{bmatrix}P(x,z)^T - 8\delta M(P^{-1})^T(1,-1)^T]^T\right.$$
$$\left.\times\begin{bmatrix} A_2^{-1} & 0 \\ 0 & B_2^{-1}\end{bmatrix}[(P^{-1})^T(t_1,t_2)^T - 2\begin{bmatrix} A_2 & 0 \\ 0 & B_2\end{bmatrix}P(x,z)^T - 8\delta M(P^{-1})^T(1,-1)^T]\right)$$
$$\times\exp(16M^2(1,-1)\kappa^{-1}(1,-1)^T + 8\delta M(x,z)(1,-1)^T)\,.$$

From there, we get the result. □

4.2. **Truncation.** In the following, the parameter $\Delta > 0$ is to be understood as a truncation level. For $k\geq 0$ and $\Delta>0$, we set (for all $b$)

$$C_{k+1}(\Delta,b) = \{z\,:\,|2B_2(1+p_{2,1})z - b|\leq\Delta\} \tag{4.15}$$

(which indeed does not depend on $k$),

$$C_{k+1}(\Delta) = C_{k+1}(\Delta, B_1^{y_{k\tau:(k+1)\tau}})$$

and

$$m_{k+1}(b) = \frac{b}{2B_2(1+p_{2,1})}\,, \tag{4.16}$$

(which indeed does not depend on $k$) and

$$m_{k+1} = m_{k+1}(B_1^{Y_{k\tau:(k+1)\tau}})\,.$$

We suppose that $m_0$ is a point in the support of $\pi_0$ (the law of $X_0$) and we set

$$C_0(\Delta) = \left[m_0 - \frac{\Delta}{2B_2(1+p_{2,1})}\,,\, m_0 + \frac{\Delta}{2B_2(1+p_{2,1})}\right].$$

From Hypothesis 2 and Lemma 2.7, we see that there exists a universal constant $C$ such that

$$|m_k - m_{k-1}| \leq \begin{cases} C(M\tau^2 + \mathcal{V}_{(k-2)\tau,k\tau} + \mathcal{W}_{(k-2)\tau,k\tau}) & \text{if } k \geq 2\,, \\ |m_0| + C(M\tau^2 + \mathcal{V}_{0,2\tau} + \mathcal{W}_{0,2\tau}) & \text{if } k = 1\,. \end{cases} \tag{4.17}$$

For the simplicity of the proofs, we add here an assumption. We set

$$d(\Delta) = \frac{\Delta}{1+p_{2,1}} - 6B_2p_{2,1}\theta^{1-\iota}\Delta - 4M\,, \tag{4.18}$$

and

$$\begin{aligned} T(\Delta) = {} & \frac{C\sqrt{\tau}\exp\left(-\frac{1}{2}\left(\frac{\theta^{1-\iota}\Delta}{6C\sqrt{2\tau}}\right)^2\right)}{\theta^{1-\iota}\Delta} \\ & + \left(M\frac{(1+p_{2,1})}{p_{1,1}} + \sqrt{A_2}\right) C_1'(h,\tau)\sigma_1\sigma_2 p_{1,1}\frac{B_2 e^{63M\tau + \frac{9\tau M}{2} + 640M^2}}{d(\Delta)} \exp\left(-\frac{1}{4B_2}d(\Delta)^2\right)\,. \end{aligned} \tag{4.19}$$

**Hypothesis 3.** *We suppose that $m_0$ is chosen such that*

$$\pi_0(C_0(\Delta)^{\complement}) \preceq T(\Delta)\,,$$

*for $\Delta$ bigger than some $\Delta_0 > 0$. We assume $\Delta \geq \Delta_0$.*

We define, for all $k \geq 1$, $x$ and $x'$ in $\mathbb{R}$ (recall that $\psi_k$ is defined in Equation 3.5)

$$\psi_k^\Delta(x,x') = \psi_k(x,x')1_{C_k(\Delta)}(x')\,, \tag{4.20}$$

$$D_k = |m_k - m_{k-1}|\,, \tag{4.21}$$

and for $D \geq 0$,

$$\xi_1(D,\Delta) = \frac{1}{\sqrt{2\pi\tau}}\exp\left(-\frac{\left(D + \frac{\Delta}{B_2(1+p_{2,1})}\right)^2}{2\tau}\right)\exp\left(-M\left(D + \frac{\Delta}{B_2(1+p_{2,1})}\right) - \left(\frac{\tau}{2} + \frac{\tau^2}{2}\right)M\right)\,, \tag{4.22}$$

$$\xi_2(D,\Delta) = \frac{1}{\sqrt{2\pi\tau}}\exp\left(-\frac{\left(\left(D - \frac{\Delta}{B_2(1+p_{2,1})}\right)_+\right)^2}{2\tau}\right)\exp\left(M\left(D + \frac{\Delta}{B_2(1+p_{2,1})}\right) + \frac{\tau}{2}M\right)\,, \tag{4.23}$$

and,

$$R_k^\Delta(x,dx') = \begin{cases} \psi_k^\Delta(x,x')Q(x,dx') & \text{if } x \in C_{k-1}(\Delta)\,, \\ \psi_k^\Delta(x,x')\xi_1(D_k,\Delta)dx' & \text{if } x \notin C_{k-1}(\Delta)\,. \end{cases} \tag{4.24}$$

We define $(\pi_n^\Delta)_{n\geq 0}$ by the following

$$\begin{cases} \pi_0^\Delta = \pi_0 \\ \pi_{k\tau}^\Delta = \overline{R}_k^\Delta \overline{R}_{k-1}^\Delta \dots \overline{R}_1^\Delta(\pi_0) \quad \text{for all } k \geq 1\,. \end{cases}$$

The next lemma tells us that the measures $\pi_k$ are concentrated on the compacts $C_k(\Delta)$.

**Lemma 4.4.** *If*

$$\theta^{1-\iota}\Delta > 3|m_0| + 3CM\tau^2\,,\ d(\Delta) > 0\,, \tag{4.25}$$

*then have, for all $k \geq 0$,*

$$\mathbb{E}(\pi_{k\tau}(C_k(\Delta)^\complement) \preceq T(\Delta)\,.$$

*Proof.* We suppose $k \geq 1$. For a measure $\mu$ in $\mathcal{M}^+(\mathbb{R})$, we define

$$\widetilde{Q}\mu(dx, dx') = \mu(dx)Q(x, dx')\,,\ \forall x, x' \in \mathbb{R},$$

(recall $\widetilde{Q}$ has been defined as a Markov kernel on $\mathbb{R}^2$, so the above is an extension of the definition of $\widetilde{Q}$). We have

$$\pi_k(C_k(\Delta)^\complement) \times 1_{|m_k - m_{k-1}| \leq \Delta\theta^{1-\iota}} =$$
$$1_{|m_k - m_{k-1}| \leq \Delta\theta^{1-\iota}} \int_{(x,x')\in\mathbb{R}^2} \frac{\psi_k(x,x')}{\langle \widetilde{Q}\pi_{k-1}, \psi_k\rangle} 1_{C_k(\Delta)^\complement}(x)(1_{C_{k-1}(2\Delta)}(x) + 1_{C_{k-1}(2\Delta)^\complement}(x))\widetilde{Q}\pi_{k-1}(dx, dx')\,.$$

and (using the same computations as in [LO03], proof of Proposition 5.3, [Oud00], p. 66)

(4.26)

$$\mathbb{E}\left( 1_{|m_k - m_{k-1}| \leq \Delta\theta^{1-\iota}} \int_{\mathbb{R}^2} \frac{\psi_k(x,x')}{\langle \widetilde{Q}\pi_{k-1}, \psi_k\rangle} 1_{C_k(\Delta)^\complement}(x') 1_{C_{k-1}(2\Delta)}(x) \widetilde{Q}\pi_{k-1}(dx, dx') \middle| \, Y_{0:(k-1)\tau} \right)$$
$$= \int_{y\in\mathcal{C}([0,\tau])} 1_{|m_k(B_1^y) - m_{k-1}| \leq \Delta\theta^{1-\iota}} \left( \int_{\mathbb{R}^2} \frac{\psi(y,x,x')}{\langle \widetilde{Q}\pi_{k-1}, \psi_k\rangle} 1_{C_k(\Delta, B_1^y)^\complement}(x') 1_{C_{k-1}(2\Delta)}(x) \widetilde{Q}\pi_{k-1}(dx, dx') \right)$$
$$\times \left( \int_{(u,u')\in\mathbb{R}^2} \widetilde{Q}\pi_{k-1}(u, du')\psi(y,u,u') \right) \lambda_W(dy)$$
$$\text{(by Fubini's theorem)}$$
$$= \int_{y\in\mathcal{C}([0,1])} 1_{|m_k(B_1^y) - m_{k-1}| \leq \Delta\theta^{1-\iota}} \int_{\mathbb{R}^2} \psi(y,x,x') 1_{C_k(\Delta, B_1^y)^\complement}(x') 1_{C_{k-1}(2\Delta)}(x) \widetilde{Q}\pi_{k-1}(dx, dx') \lambda_W(dy)$$
$$\text{(using Lemma 1.3 and Lemma 4.3)}$$
$$\leq \sigma_1\sigma_2 C_1'(h,\tau) \int_{(x,x')\in\mathbb{R}^2} \int_{(t_1,t_2)\in\mathbb{R}^2\,:\,|m_k(t_2) - m_{k-1}| \leq \Delta\theta^{1-\iota}} e^{10M|x-x'| + \tau(4M+2M^2) + 16M^2 \left|(1,-1)\kappa^{-1}(1,-1)^T\right|}$$
$$\times e^{-\frac{1}{4A_2}\left(\left|\frac{t_1}{p_{1,1}} - \frac{p_{2,1}t_2}{p_{1,1}} - 2A_2 p_{1,1}x\right| - 4M\left|\frac{1+p_{2,1}}{p_{1,1}}\right|\right)_+^2} \times e^{-\frac{1}{4B_2}\left(\left|t_2 - 2B_2(p_{2,1}x + x')\right| - 4M\right)_+^2}$$
$$\times 1_{C_k(\Delta, t_2)^\complement}(x') 1_{C_{k-1}(2\Delta)}(x) dt_1 dt_2 \widetilde{Q}\pi_{k-1}(dx, dx')\,.$$

By a similar computation, we get

$$\mathbb{E}\left( \int_{\mathbb{R}^2} \frac{\psi_k(x,x')}{\langle \widetilde{Q}\pi_{k-1}, \psi_k\rangle} 1_{C_k(\Delta)^\complement}(x) 1_{C_{k-1}(2\Delta)^\complement}(x) \widetilde{Q}\pi_{k-1}(dx, dx') \middle| \, Y_{0:(k-1)\tau} \right)$$
$$= \int_{y\in\mathcal{C}([0,1])} \int_{\mathbb{R}^2} \psi(y,x,x') 1_{C_k(\Delta, B_1^y)^\complement}(x') 1_{C_{k-1}(2\Delta)^\complement}(x) \widetilde{Q}\pi_{k-1}(dx, dx') \lambda_W(dy)$$
$$\leq \int_{\mathbb{R}^2} 1_{C_{k-1}(2\Delta)^\complement}(x) \widetilde{Q}\pi_{k-1}(dx, dx') \leq \pi_{k-1}(C_{k-1}(2\Delta)^\complement)\,.$$

For $x \in C_{k-1}(2\Delta)$, $x' \in C_k(\Delta, t_2)^\complement$,

$$
\begin{aligned}
(4.27)\quad & |t_2 - 2B_2(p_{2,1}x + x')| \\
&= \left| -2B_2 x' + \frac{t_2}{p_{2,1}+1} + \frac{p_{2,1}}{p_{2,1}+1} t_2 - 2B_2 p_{2,1} x \right| \\
&\geq \frac{\Delta}{1+p_{2,1}} - 2B_2 p_{2,1} |m_k(t_2) - m_{k-1}| - 2B_2 p_{2,1} |m_{k-1} - x| \\
&\geq \frac{\Delta}{1+p_{2,1}} - 6B_2 p_{2,1} \theta^{1-\iota}\Delta \,,\ \text{ if } |m_k(t_2) - m_{k-1}| \leq \theta^{1-\iota}\Delta \,.
\end{aligned}
$$

So

$$
\begin{aligned}
\mathbb{E}(\pi_k(C_k(\Delta)^{\complement})) &\leq \mathbb{P}(|m_k - m_{k-1}| \geq \theta^{1-\iota}\Delta) + \mathbb{E}(\pi_{k-1}(C_{k-1}(2\Delta)^{\complement})) \\
&+ \sigma_1\sigma_2 C_1'(h,\tau) \int_{(x,x')\in\mathbb{R}^2} \int_{(t_1,t_2)\in\mathbb{R}^2 \,:\, |t_2 - 2B_2(p_{2,1}x+x')| > \frac{\Delta}{1+p_{2,1}} - 6\Delta\theta^{1-\iota}B_2 p_{2,1}} \\
&\quad e^{10M|x-x'| + \tau(4M+2M^2) + 16M^2\left|(1,-1)\kappa^{-1}(1,-1)^T\right|} \\
&\quad \times e^{-\frac{1}{4A_2}\left(\left|\frac{t_1}{p_{1,1}} - \frac{p_{2,1}t_2}{p_{1,1}} - 2A_2 p_{1,1} x\right| - 4M\left|\frac{1+p_{2,1}}{p_{1,1}}\right|\right)_+^2} \\
&\quad \times e^{-\frac{1}{4B_2}\left(|t_2 - 2B_2(p_{2,1}x+x')| - 4M\right)_+^2} dt_1 dt_2 \widetilde{Q}\pi_{k-1}(dx,dx') \,.
\end{aligned}
$$

We have, for all $x \geq 0$,

$$
\begin{aligned}
\mathbb{P}(\mathcal{V}_{0,2\tau} \geq x) &= \mathbb{P}(\sup_{s\in[0,2\tau]} V_s - \inf_{s\in[0,2\tau]} V_s \geq x) \\
&\leq \mathbb{P}(\sup_{s\in[0,2\tau]} V_s \geq x/2) + \mathbb{P}(-\inf_{s\in[0,2\tau]} V_s \geq x/2) \\
&= 4\mathbb{P}(V_{2\tau} \geq x/2) \\
&= 2\mathbb{P}(2|V_{2\tau}| \geq x) \,.
\end{aligned}
$$

And so, we can bound (for all $x$)

$$
\mathbb{P}(\mathcal{V}_{(k-2)_+\tau,k\tau} \geq x) \leq 2\mathbb{P}(2|W_{2\tau}| \geq x) \,, \tag{4.28}
$$

$$
\mathbb{P}(\mathcal{W}_{(k-2)_+\tau,k\tau} \geq x) \leq 2\mathbb{P}(2|W_{2\tau}| \geq x) \,. \tag{4.29}
$$

So (with the constant $C$ defined in Equation (4.17)), using the inequality

$$
\forall z > 0 \,,\ \int_z^{+\infty} \frac{\exp\left(-\frac{t^2}{2\sigma^2}\right)}{\sqrt{2\pi\sigma^2}} dt \leq \frac{\sigma \exp\left(-\frac{z^2}{2\sigma^2}\right)}{z\sqrt{2\pi}} \,, \tag{4.30}
$$

and using Equation (4.17), as $\theta^{1-\iota}\Delta \geq 3|m_0| + 3CM\tau^2$, we get

$$
\begin{aligned}
\mathbb{P}(|m_k - m_{k-1}| \geq \theta^{1-\iota}\Delta) &\leq 4\mathbb{P}\left(2 \times |W_{2\tau}| \geq \frac{\theta^{1-\iota}\Delta}{3C}\right) \\
&= 4\mathbb{P}\left(|W_1| \geq \frac{\theta^{1-\iota}\Delta}{6C\sqrt{2\tau}}\right) \\
&\leq \frac{48C\sqrt{\tau}}{\theta^{1-\iota}\Delta\sqrt{\pi}} \exp\left(-\frac{1}{2}\left(\frac{\theta^{1-\iota}\Delta}{6C\sqrt{2\tau}}\right)^2\right) \,. \qquad (4.31)
\end{aligned}
$$

For all $x$, $x'$, we have

$$
\begin{aligned}
(4.32)\quad & \int_{(t_1,t_2)\in\mathbb{R}^2 \,:\, |t_2 - 2B_2(p_{2,1}x+x')| > \frac{\Delta}{1+p_{2,1}} - 6\Delta\theta^{1-\iota}B_2 p_{2,1}} \\
&\quad e^{-\frac{1}{4A_2}\left(\left|\frac{t_1}{p_{1,1}} - \frac{p_{2,1}t_2}{p_{1,1}} - 2A_2 p_{1,1} x\right| - 4M\left|\frac{1+p_{2,1}}{p_{1,1}}\right|\right)_+^2} \\
&\quad \times e^{-\frac{1}{4B_2}\left(|t_2 - 2B_2(p_{2,1}x+x')| - 4M\right)_+^2} dt_1 dt_2
\end{aligned}
$$

$$\text{(change of variables : } \begin{pmatrix} t_1' \\ t_2' \end{pmatrix} = (P^{-1})^T \begin{pmatrix} t_1 \\ t_2 \end{pmatrix})$$

$$= \int_{(t_1',t_2')\in\mathbb{R}^2 \,:\, |t_2'-2B_2(p_{2,1}x+x')|>\frac{\Delta}{1+p_{2,1}}-6\Delta\theta^{1-\iota}B_2p_{2,1}} e^{-\frac{1}{4A_2}\left(\left|t_1'-2A_2p_{1,1}x\right|-4M\left|\frac{1+p_{2,1}}{p_{1,1}}\right|\right)_+^2}$$

$$\times e^{-\frac{1}{4B_2}\left(\left|t_2'-2B_2(p_{2,1}x+x')\right|-4M\right)_+^2} p_{1,1}dt_1'dt_2'$$

$$\text{(by (4.30))} \leq \left(8M\frac{|1+p_{2,1}|}{p_{1,1}} + 2\sqrt{\pi A_2}\right)$$

$$\times \frac{4B_2}{d(\Delta)} \exp\left(-\frac{1}{4B_2}d(\Delta)^2\right) p_{1,1}\,.$$

We have, by Lemma 1.2 and Hypothesis 2,

$$\text{(4.33)} \quad \int_{(x,x')\in\mathbb{R}^2} e^{10M|x-x'|+\tau(4M+2M^2)+16M^2(1,-1)\kappa^{-1}(1,-1)^T}\widetilde{Q}\pi_{k-1}(dx,dx')$$

$$\leq \int_{(x,x')\in\mathbb{R}^2} \frac{1}{\sqrt{2\pi\tau}} \exp\left(-\frac{(x-x')^2}{2\tau} + 11M|x-x'| + \tau\left(\frac{9}{2}M+2M^2\right) + 640M^2\right) dx'\pi_{k-1}(dx)$$

$$\leq 2\exp\left(\frac{121}{2}M^2\tau + \tau\left(\frac{9}{2}M+2M^2\right) + 640M^2\right).$$

So we have

$$\mathbb{E}(\pi_k(C_k(\Delta)^{\complement})) \leq \mathbb{E}(\pi_{k-1}(C_{k-1}(2\Delta)^{\complement})) + \frac{48C\sqrt{\tau}}{\theta^{1-\iota}\Delta\sqrt{\pi}}\exp\left(-\frac{1}{2}\left(\frac{\theta^{1-\iota}\Delta}{6C\sqrt{2\tau}}\right)^2\right)$$

$$+\left(8M\frac{|1+p_{2,1}|}{p_{1,1}} + 2\sqrt{\pi A_2}\right)C_1'(h,\tau)\sigma_1\sigma_2p_{1,1}\frac{8B_2}{d(\Delta)}$$

$$\times\exp\left(-\frac{1}{4B_2}d(\Delta)^2 + 63M^2\tau + \frac{9}{2}\tau M + 640M^2\right).$$

Then, by recurrence,

$$\mathbb{E}(\pi_k(C_k(\Delta)^{\complement})) \leq \mathbb{E}(\pi_0(C_0(2^k\Delta)^{\complement})) + \sum_{i=0}^{k-1}\left[\frac{48C\sqrt{\tau}}{2^i\theta^{1-\iota}\Delta\sqrt{\pi}}\exp\left(-\frac{1}{2}\left(\frac{2^i\theta^{1-\iota}\Delta}{6C\sqrt{2\tau}}\right)^2\right)\right.$$

$$\left.+\left(8M\frac{|1+p_{2,1}|}{p_{1,1}} + 2\sqrt{\pi A_2}\right)C_1'(h,\tau)\sigma_1\sigma_2p_{1,1}\frac{8B_2e^{68M^2\tau+\frac{9}{2}\tau M+640M^2}}{d(2^i\Delta)}\exp\left(-\frac{1}{4B_2}d(2^i\Delta)^2\right)\right].$$

We have

$$\begin{aligned}\frac{1}{d(\Delta)} + \frac{1}{d(2\Delta)} + \frac{1}{d(4\Delta)} + \dots \quad &\leq \quad \frac{1}{d(\Delta)} + \frac{1}{d(\Delta)} + \frac{1}{d(\Delta)+2\Delta\left(\frac{1}{1+p_{2,1}} - 4B_2p_{2,1}\theta^{1-\iota}\right)} \\ &\quad + \frac{1}{d(\Delta)+4\Delta\left(\frac{1}{1+p_{2,1}} - 4B_2p_{2,1}\theta^{1-\iota}\right)} + \dots \\ &\leq \quad \frac{2}{d(\Delta)} + \frac{1}{\Delta\left(\frac{1}{1+p_{2,1}} - 4B_2p_{2,1}\theta^{1-\iota}\right)} \leq \frac{3}{d(\Delta)}\,.\end{aligned}$$

So

$$\mathbb{E}(\pi_k(C_k(\Delta)^{\complement})) \leq \mathbb{E}(\pi_0(C_0(2^k\Delta)^{\complement})) + \frac{96C\sqrt{\tau}}{\tau^{1-\iota}\Delta\sqrt{\pi}}\exp\left(-\frac{1}{2}\left(\frac{\tau^{1-\iota}\Delta}{6C\sqrt{2\tau}}\right)^2\right)$$

$$+\left(8M\frac{|1+p_{2,1}|}{p_{1,1}} + 2\sqrt{\pi A_2}\right)C_1'(h,\tau)\sigma_1\sigma_2p_{1,1}8B_2e^{63M^2\tau+\frac{9}{2}\tau M+640M^2}$$

$$\times \frac{3}{d(\Delta)} \exp\left(-\frac{1}{4B_2} d(\Delta)^2\right)$$

□

**Corollary 4.5.** *We suppose that $\pi'_0 \in \mathcal{P}(\mathbb{R})$ is such that $\pi_0$ and $\pi'_0$ are comparable. We suppose that $(\pi'_t)_{t\geq 0}$ is defined by Equation (1.3). Under the assumption of the previous Lemma, we have, for all $k \geq 0$,*

$$\mathbb{E}(\pi'_{k\tau}(C_k(\Delta)^{\complement}) \preceq T(\Delta) e^{2h(\pi_0,\pi'_0)} .$$

*Proof.* By Equations (3.1), (3.2), we have, for all $k$,

$$h(\pi'_{k\tau}, \pi_{k\tau}) \leq h(\pi'_0, \pi_0) . \tag{4.34}$$

So, by Equation (3.3),

$$\begin{aligned} \mathbb{E}(\pi'_{k\tau}(C_k(\Delta)^{\complement}) &\leq \mathbb{E}(e^{h(\pi'_0,\pi_0)} \pi_{k\tau}(C_k(\Delta)^{\complement})) \\ &\preceq T(\Delta) e^{h(\pi_0,\pi'_0)} . \end{aligned}$$

□

The next result tells us that $\overline{R}_k$ and $\overline{R}_k^{\Delta}$ ($k \geq 1$) are close in some sense (recall that $\pi_{k\tau} = \overline{R}_k(\pi_{(k-1)\tau})$)

**Proposition 4.6.** *We suppose that $\Delta$ satisfies the assumption of the previous Lemma (Equation (4.25)). We suppose that $(\pi'_{n\tau})_{n\geq 0}$ satisfies the assumptions of the above Corollary. For all $k \geq 1$, we have*

$$\mathbb{E}(\|\pi'_{k\tau} - \overline{R}_k^{\Delta}(\pi'_{(k-1)\tau})\|) \preceq T(\Delta) e^{h(\pi_0,\pi'_0)} .$$

*Proof.* We define measures on $\mathbb{R}^2$:

$$\mu = \widetilde{Q}\pi'_{(k-1)\tau} ,$$

$$\begin{aligned} \mu'(dx, dx') = 1_{C_k(\Delta)}(x') \widetilde{Q}(1_{C_{k-1}(\Delta)} \pi'_{(k-1)\tau})(dx, dx') \\ + \pi'_{(k-1)\tau}(C_{k-1}(\Delta)^{\complement}) \xi_1(D_k, \Delta) 1_{C_k(\Delta)}(x') dx dx' , \end{aligned}$$

where (by a slight abuse of notation)

$$\widetilde{Q}(1_{C_{k-1}(\Delta)} \pi'_{(k-1)\tau})(dx, dx') = 1_{C_{k-1}(\Delta)}(x) \pi'_{(k-1)\tau}(dx) Q(x, dx') .$$

By the definition of $R^{\Delta}$ (Equation (4.24)) and computing as in [OR05], p.433 (or as in [Oud00], p.66), we get

$$\begin{aligned} \|\pi'_{k\tau} - \overline{R}_k^{\Delta}(\pi'_{(k-1)\tau})\| &= \|\psi_k \bullet \mu - \psi_k \bullet \mu'\| \\ \text{(using Equation (3.8))} \quad &\leq 2 \int_{\mathbb{R}^2} \frac{\psi_k(x,x')}{\langle \widetilde{Q}\pi'_{k-1}, \psi_k \rangle} \times [1_{C_k(\Delta)^{\complement}}(x') \widetilde{Q}(1_{C_{k-1}(\Delta)} \pi'_{(k-1)\tau})(dx, dx') \\ &\quad + \widetilde{Q}(1_{C_{k-1}(\Delta)^c} \pi'_{(k-1)\tau})(dx, dx') \\ &\quad + \pi'_{(k-1)\tau}(C_{k-1}(\Delta)^{\complement}) \xi_1(D_k, \Delta) 1_{C_k(\Delta)}(x') dx dx'] . \end{aligned}$$

We have, by Lemma 1.3,

$$\begin{aligned} \mathbb{E}\left( 1_{[0,\theta^{1-\iota}\Delta]}(|m_k - m_{k-1}|) \int_{\mathbb{R}^2} \frac{\psi_k(x,x')}{\langle \widetilde{Q}\pi'_{k-1}, \psi_k \rangle} \times 1_{C_k(\Delta)^{\complement}}(x') \widetilde{Q}(1_{C_{k-1}(\Delta)} \pi'_{(k-1)\tau})(dx, dx') \,\middle|\, Y_{0:(k-1)\tau} \right) \\ = \int_{y_{0:\tau} \in \mathcal{C}([0,\tau])} 1_{[0,\theta^{1-\iota}\Delta]}(|m_k(B_1^{y_{0:\tau}}) - m_{k-1}(B_1^{Y_{(k-2)\tau:(k-1)\tau}})|) \\ \times \int_{\mathbb{R}^2} \frac{\psi(y_{0:\tau}, x, x')}{\langle \widetilde{Q}\pi'_{k-1}, \psi(y_{0:\tau}, ., .) \rangle} 1_{C_k(\Delta, B_1^{y_{0:\tau}})^{\complement}}(x') \widetilde{Q}(1_{C_{k-1}(\Delta)} \pi'_{(k-1)\tau})(dx, dx') \\ \times \left( \int_{\mathbb{R}^2} \widetilde{Q}\pi_{(k-1)\tau}(du, du') \psi(y_{0:\tau}, u, u') \right) \lambda_W(dy_{0:\tau}) \end{aligned}$$

$$
\begin{aligned}
&\text{(using Equations (3.3), (4.34))}\\
&\leq e^{2h(\pi_0,\pi_0')}\int_{y_{0:\tau}\in\mathcal{C}([0,\tau])}1_{[0,\theta^{1-\iota}\Delta]}(|m_k(B_1^{y_{0:\tau}})-m_{k-1}(B_1^{Y_{(k-2)\tau:(k-1)\tau}})|)\\
&\times\int_{\mathbb{R}^2}\psi(y_{0:\tau},x,x')1_{C_k(\Delta,B_1^y)^{\complement}}(x')1_{C_{k-1}(\Delta)}(x)\widetilde{Q}\pi_{k-1}(dx,dx')\lambda_W(dy_{0:\tau})\\
&\text{(using (4.26), (4.27), (4.32), (4.33) and the fact that } C_{k-1}(\Delta)\subset C_{k-1}(2\Delta))\\
&\preceq T(\Delta)e^{2h(\pi_0,\pi_0')}\,.
\end{aligned}
$$

We have

$$
\mathbb{E}\left(\int_{\mathbb{R}^2}\frac{\psi_k(x,x')}{\langle\widetilde{Q}\pi_{k-1}',\psi_k\rangle}\times\widetilde{Q}(1_{C_{k-1}(\Delta)^{\complement}}\pi_{(k-1)\tau}')(dx,dx')\middle|\,Y_{0:(k-1)\tau}\right)=
$$

$$
\int_{y_{0:\tau}\in\mathcal{C}([0,\tau])}\int_{\mathbb{R}^2}\psi(y_{0:\tau},x,x')1_{C_{k-1}(\Delta)^c}(x)\widetilde{Q}\pi_{(k-1)\tau}'(dx,dx')\lambda_W(dy_{0:\tau})=\pi_{(k-1)\tau}'(C_{k-1}(\Delta)^{\complement})
$$

and

$$
\begin{aligned}
&\mathbb{E}\left(\int_{\mathbb{R}^2}\frac{\psi_k(x,x')}{\langle\widetilde{Q}\pi_{k-1}',\psi_k\rangle}\times\pi_{(k-1)\tau}'(C_{k-1}(\Delta)^{\complement})\xi_1(D_k,\Delta)1_{C_k(\Delta,B_1^y)}(x')dxdx'\middle|\,Y_{0:(k-1)\tau}\right)\\
&=\int_{y_{0:\tau}\in\mathcal{C}([0,\tau])}\int_{\mathbb{R}^2}\psi(y_{0:\tau},x,x')\pi_{(k-1)\tau}'(C_{k-1}(\Delta)^{\complement})\xi_1(D_k,\Delta)1_{C_k(\Delta,B_1^y)}(x')dxdx'\lambda_W(dy_{0:\tau})\\
&\leq\pi_{(k-1)\tau}'(C_{k-1}(\Delta)^{\complement})\,.
\end{aligned}
$$

So, using (4.31) and Corollary 4.5, we get the result. □

## 5. New formula for the optimal filter

We have reduced the problem to a discrete-time problem. For all $n$, $\pi_{n\tau}$ is the marginal of a Feynman-Kac sequence based on the transition $\widetilde{Q}$ and the potentials $(\psi_k)_{k\geq1}$. We wish to apply the same method as [OR05]. We restrict the state space to the compacts $(C_k(\Delta))_{k\geq0}$. But, even when restricted to compacts, $\widetilde{Q}$ cannot be mixing, so we cannot apply [OR05] directly. The purpose of this Section is to find another representation of the sequence $(\pi_{n\tau})_{n\geq0}$ as a Feynman-Kac sequence, in such a way that the underlying Markov operators would be mixing, when restricted to compacts. Looking at Equation (3.6), we see that a Feynman-Kac sequence is a result of the deformation of a measure on trajectories (we weight the trajectories with potentials $(\psi_k)_{k\geq1}$). The main idea of the following is to incorporate the deformations delicately (in two steps), in order to retain something of the mixing property of the operator $Q$ (which is mixing when restricted to compacts).

In this Section, we work with a fixed observation $(Y_s)_{s\geq0}=(y_s)_{s\geq0}$.

5.1. **Filter based on partial information.** We define, for all $x=(x_1,x_2)$, $x'=(x_1',x_2')$ in $\mathbb{R}^2$, $k$ in $\mathbb{N}^*$, $n$ in $\mathbb{N}^*$, $n\geq k$,

$$
\text{for } k\geq2\,,\ \widetilde{R}_k^{\Delta}(x,dx')=\begin{cases}1_{C_{k-1}(\Delta)}(x_1')\psi_k^{\Delta}(x')\widetilde{Q}^2(x,dx') & \text{if } x_2\in C_{k-2}(\Delta)\,,\\ 1_{C_{k-1}(\Delta)}(x_1')\psi_k^{\Delta}(x')\xi_1(D_{k-1},\Delta)dx' & \text{otherwise}\,,\end{cases}\tag{5.1}
$$

$$
\psi_{2n|2k}^{\Delta}(x)=\begin{cases}\widetilde{R}_{2n}^{\Delta}\widetilde{R}_{2n-2}^{\Delta}\dots\widetilde{R}_{2k+2}^{\Delta}(x,\mathbb{R}^2) & \text{if } k\leq n-1\,,\\ 1 & \text{if } k=n\,,\end{cases}\tag{5.2}
$$

(so $\psi_{2n|2k}^{\Delta}(x)$ does not depend on $x^{(1)}$),

$$
S_{2n|2k}^{\Delta}(x,dx')=\begin{cases}\frac{\psi_{2n|2k+2}^{\Delta}(x')}{\psi_{2n|2k}^{\Delta}(x)}\widetilde{R}_{2k+2}^{\Delta}(x,dx') & \text{if } k\leq n-1\,,\\ dx' & \text{if } k=n-1\,.\end{cases}
$$

These notations come from [DG01]. As $Q$ has a density with respect to the Lebesgue measure on $\mathbb{R}$, so has $S_{2n|2k}^{\Delta}$ (with respect to the Lebesgue measure on $\mathbb{R}^2$). We write $(x,x')\in E^2\mapsto S_{2n|2k}^{\Delta}(x,x')$

for this density. We fix $n$ in $\mathbb{N}^*$ in the rest of this subsection. We define $S^{\Delta,(p)}_{2n|2k}$, $\psi^{\Delta,(p)}_{2n|2k}$, for $0 \le k \le n$, with the same formulas used above to define $S^{\Delta}_{2n|2k}$, $\psi^{\Delta}_{2n|2k}$, except we replace $\psi^{\Delta}_{2n}$ by 1. For all $D > 0$, we set (recall the definitions of $\xi_1$, $\xi_2$ in (4.22), (4.23))

$$\epsilon(D,\Delta) = \frac{\xi_1(D,\Delta)}{\xi_2(D,\Delta)}\,, \tag{5.3}$$

and, for all $k$,

$$\epsilon_k = \epsilon(D_k,\Delta)\,.$$

**Lemma 5.1.** *For $k \le n-1$, $S^{\Delta}_{2n|2k}$ is a Markov operator and $S^{\Delta}_{2n|2k}$ is $(1-\epsilon^2_{2k+1})$-contracting for the total variation norm, $S^{\Delta,(p)}_{2n|2k}$ is a Markov operator and $S^{\Delta,(p)}_{2n|2k}$ is $(1-\epsilon^2_{2k+1})$-contracting for the total variation norm*

*Proof.* We write the proof only for the kernels $S^{\Delta}_{\dots}$, it would be very similar for the kernels $S^{\Delta,(p)}_{\dots}$. By Proposition 3.1, $S^{\Delta}_{2n|2k}$ is a Markov operator. We set, for all $k \ge 1$, $x_1$, $x_2$ in $\mathbb{R}$,

$$\lambda_k(dx_1,dx_2) = 1_{C_{k-1}(\Delta)}(x_1)1_{C_k(\Delta)}(x_2)\psi_k(x_1,x_2)dx_1dx_2\,.$$

By Lemma 1.2, we have, for all $x_1$, $x_2$, $z_1$, $z_2$ in $\mathbb{R}$, $k \ge 2$ (we use here the second line of Equation (5.1))

$$\xi_1(D_{k-1},\Delta)\lambda_k(dz_1,dz_2) \le \widetilde{R}^{\Delta}_k(x_1,x_2,dz_1,dz_2) \le \xi_2(D_{k-1},\Delta)\lambda_k(dz_1,dz_2)\,. \tag{5.4}$$

So $\widetilde{R}^{\Delta}_k$ is $\sqrt{\epsilon_{k-1}}$-mixing. So, for all $x$ in $\mathbb{R}^2$, all $k$ such that $0 \le k \le n-1$ (the convention being that, if $k = n-1$, $(\widetilde{R}^{\Delta}_{2n}\dots\widetilde{R}^{\Delta}_{2k+4})(y,dz) = \delta_y(dz)$)

$$\begin{aligned}\psi^{\Delta}_{2n|2k}(x) &= \int_{\mathbb{R}^2\times\mathbb{R}^2}(\widetilde{R}^{\Delta}_{2n}\dots\widetilde{R}^{\Delta}_{2k+4})(y,dz)\widetilde{R}^{\Delta}_{2k+2}(x,dy)\\ &\le \int_{\mathbb{R}^2\times\mathbb{R}^2}(\widetilde{R}^{\Delta}_{2n}\dots\widetilde{R}^{\Delta}_{2k+4})(y,dz)\xi_2(D_{2k+1},\Delta)\lambda_{2k+2}(dy)\,,\end{aligned}$$

and, for $x'$ in $\mathbb{R}^2$,

$$\widetilde{R}^{\Delta}_{2k+2}(x,dx') \ge \xi_1(D_{2k+1},\Delta)\lambda_{2k+2}(dx')\,,$$

so

$$S^{\Delta}_{2n|2k}(x,dx') \ge \frac{\xi_1(D_{2k+1},\Delta)}{\xi_2(D_{2k+1},\Delta)} \times \frac{\widetilde{R}^{\Delta}_{2k+4:2n}(x',\mathbb{R}^2)\lambda_{2k+2}(dx')}{\int_{\mathbb{R}^2}\widetilde{R}^{\Delta}_{2k+4:2n}(y,\mathbb{R})\lambda_{2k+2}(dy)}\,. \tag{5.5}$$

In the same way as above, we can also obtain

$$S^{\Delta}_{2n|2k}(x,dx') \le \frac{\xi_2(D_{2k+1},\Delta)}{\xi_1(D_{2k+1},\Delta)} \times \frac{\widetilde{R}_{2k+4:2n}(x',\mathbb{R}^2)\lambda_{2k+2}(dx')}{\int_{\mathbb{R}^2}\widetilde{R}_{2k+4:2n}(y,\mathbb{R}^2)\lambda_{2k+2}(dy)}\,. \tag{5.6}$$

This implies that $S^{\Delta}_{2n|2k}$ is $(1-\epsilon^2_{2k+1})$-contracting for the total variation norm (see Subsection 3.1). One can also use Proposition 3.1 to prove this result. We did it this way because we will re-use Equations (5.5), (5.6). □

We set $Z_0$ to be of the form $Z_0 = (0, Z_0^{(2)})$, with $Z_0^{(2)}$ a random variable. We set $(Z_{2k})_{0\le k\le n}$ to be a non-homogeneous Markov chain with kernels $S^{\Delta}_{2n|0}$, $S^{\Delta}_{2n|2}$, …, $S^{\Delta}_{2n|2n-2}$ (for $k$ in $\{1,2,\dots,n\}$, the law of $Z_{2k}$ knowing $Z_{2k-2}$ is $S^{\Delta}_{2n|2k-2}(Z_{2k-2},.)$). For $Z_{2k}$ being a element of this chain, we denote by $Z^{(1)}_{2k}$ and $Z^{(2)}_{2k}$ its first and second component respectively. Recalling Proposition 3.1 (or Proposition 3.1, p. 428 in [OR05], or similar results in [DG01]), if the law of $Z_0$ is chosen properly, then $Z^{(2)}_{2n}$ has the same law as $X_{2n}$ knowing $Y_{\tau:2\tau}$, …, $Y_{(2n-1)\tau:2n\tau}$, henceforth the title of this Subsection.

*Remark* 5.2. We have that, for all $k \ge 1$, $Z^{(2)}_{2k}$ takes values in $C_{2k}(\Delta)$ and $Z^{(1)}_{2k}$ takes values in $C_{2k-1}(\Delta)$.

We set $(Z^{(p)}_{2k})_{0\le k\le n}$ to be a non-homogeneous Markov chain with $Z^{(p)}_0 = Z_0$ and with kernel $S^{\Delta,(p)}_{2n|0}$, $S^{\Delta,(p)}_{2n|2}$, …, $S^{\Delta,(p)}_{2n|2n-2}$.

5.2. **New Markov chains.** We set $U_{2k+1}=(Z^{(2)}_{2k},Z^{(1)}_{2k+2})$ for $k\in\{0,1,\dots,n-1\}$ and $U^{(1)}_{2n+1}=Z^{(2)}_{2n}$. We set $U^{(p)}_{2k+1}=(Z^{(p)(2)}_{2k},Z^{(p)(1)}_{2k+2})$ for $k\in\{0,1,2,\dots,n-1\}$.

**Lemma 5.3.** *The sequence $(U_1,U_3,\dots,U_{2n-1},U^{(1)}_{2n+1})$ is a non-homogeneous Markov chain. The sequence $(U^{(p)}_1,U^{(p)}_3,\dots,U^{(p)}_{2n-3},U^{(p)(2)}_{2n-1})$ is a non-homogeneous Markov chain.*

If $Z^{(2)}_0$ is of law $\mu$, then the law of $U_1$ is given by, for all $(z,z')$ in $\mathbb{R}^2$,

$$\mathbb{P}(U_1\in(dz,dz'))=\int_{x\in\mathbb{R}}S^{\Delta}_{2n|0}((0,z),(dz',dx))\mu(dz)\,. \tag{5.7}$$

We write $S^U_{2n|2k+1}$ for the transition kernel between $U_{2k-1}$ and $U_{2k+1}$ (for $k=1,2,\dots,n-1$) and $S^U_{2n|2n+1}$ for the transition between $U_{2n-1}$ and $U^{(1)}_{2n+1}$. We write $S^{(p)U}_{2n|2k+1}$ for the transition kernel between $U^{(p)}_{2n|2k-1}$ and $U^{(p)}_{2n|2k+1}$ (for $k=1,2,\dots,n-1$)

*Proof.* We write the proof only for $(U_1,U_3,\dots,U_{2n-1},U^{(1)}_{2n+1})$, it would be very similar for the sequence $(U^{(p)}_1,U^{(p)}_3,\dots,U^{(p)}_{2n-1})$. Let $\varphi$ be a test function (in $\mathcal{C}^+_b(\mathbb{R}^2)$). For $k\in\{1,\dots,n-1\}$, we have (for $z^{(2)}_0$, $z^{(1)}_2$, … ,$z^{(1)}_{2k}$ in $\mathbb{R}$)

$$\begin{aligned}&\mathbb{E}(\varphi(U_{2k+1})|U_1=(z^{(2)}_0,z^{(1)}_2),\dots,U_{2k-1}=(z^{(2)}_{2k-2},z^{(1)}_{2k}))=\\&\qquad\mathbb{E}(\varphi(Z^{(2)}_{2k},Z^{(1)}_{2k+2})|Z^{(2)}_0=z^{(2)}_0,\dots,Z^{(2)}_{2k-2}=z^{(2)}_{2k-2},Z^{(1)}_{2k}=z^{(1)}_{2k})=\\&\qquad\qquad\mathbb{E}(\varphi(Z^{(2)}_{2k},Z^{(1)}_{2k+2})|Z^{(2)}_{2k-2}=z^{(2)}_{2k-2},Z^{(1)}_{2k}=z^{(1)}_{2k})\,,\end{aligned}$$

as $S^{\Delta}_{2n|2k-2}(z^{(1)}_{2k-2},z^{(2)}_{2k-2},.,.)$ does not depend on $z^{(1)}_{2k-2}$. So the quantity above is equal to, for any $z\in C_{2k-1}(\Delta)$,

$$\begin{aligned}\int_{\mathbb{R}^2}\varphi(z^{(2)}_{2k},z^{(1)}_{2k+2})&\frac{S^{\Delta}_{2n|2k-2}((z,z^{(2)}_{2k-2}),(z^{(1)}_{2k},z^{(2)}_{2k}))}{\int_{\mathbb{R}}S^{\Delta}_{2n|2k-2}((z,z^{(2)}_{2k-2}),(z^{(1)}_{2k},z'))dz'}\\&\times\left(\int_{\mathbb{R}}S^{\Delta}_{2n|2k}((z^{(1)}_{2k},z^{(2)}_{2k}),(z^{(1)}_{2k+2},z^{(2)}_{2k+2})dz^{(2)}_{2k+2}\right)dz^{(2)}_{2k}dz^{(1)}_{2k+2}\,.\end{aligned}$$

A similar computation can be made for $\mathbb{E}(\varphi(U^{(1)}_{2n+1})|U_1,\dots U_{2n-1})$. ☐

We set, for all $k$,

$$\begin{aligned}\epsilon'(D,\Delta)=\exp\Bigg[&-\frac{1}{2}B_2p^2_{2,1}\left(\frac{\Delta}{B_2(1+p_{2,1})}+D\right)^2-B_2\Delta p_{2,1}\left(\frac{\Delta}{B_2(1+p_{2,1})}+D\right)\\&-2M\left(\frac{\Delta}{B_2(1+p_{2,1})}+D\right)-\tau\left(M+\frac{M^2}{2}\right)\Bigg]\,,\end{aligned}\tag{5.8}$$

$$\epsilon'_{2k}=\epsilon'(|m_{2k}-m_{2k-1}|,\Delta)\,.$$

**Proposition 5.4.** *For any $k=1,2,\dots,n$, the Markov kernel $S^U_{2n|2k+1}$ is $(\epsilon^2_{2k-1}(\epsilon'_{2k})^2)$-contracting. For any $k=1,2,\dots,n-1$, the Markov kernel $S^{(p)U}_{2n|2k+1}$ is $(\epsilon^2_{2k-1}(\epsilon'_{2k})^2)$-contracting.*

Before going into the proof of the above Proposition, we need the following technical results. We are interested in the bounds appearing in Lemma 4.3. We suppose that $t_1$, $t_2$, $x$, $z$ in $\mathbb{R}$ are fixed. To simplify the computations, we introduce the following notations:

$$\begin{pmatrix}t'_1\\t'_2\end{pmatrix}=(P^{-1})^T\begin{pmatrix}t_1\\t_2\end{pmatrix}\,,$$

$$M_1=\frac{2M(|1+p_{2,1}|)}{p_{1,1}}\,.$$

**Lemma 5.5.** *Suppose that, for some $D \geq 0$,*

$$|x-z| \leq D\,,$$

$$|2B_2(p_{1,2}+1)z - t'_2| \leq \Delta$$

*then*

$$\begin{aligned}\exp\left(-\frac{1}{4B_2}(t'_2 - 2B_2(p_{2,1}+1)z)^2 - B_2p_{2,1}^2D^2 - p_{2,1}D\Delta\right)\\ \leq \exp\left(-\frac{1}{4B_2}(t'_2 - 2B_2(p_{2,1}x+z))^2\right)\\ \leq \exp\left(-\frac{1}{4B_2}(t'_2 - 2B_2(p_{2,1}+1)z)^2 + p_{2,1}D\Delta\right).\end{aligned}$$

*Proof.* We have

$$\begin{aligned}&\exp\left(-\frac{1}{4B_2}(t'_2 - 2B_2(p_{2,1}x+z))^2\right)\\ &\qquad = \exp\left(-\frac{1}{4B_2}(t'_2 - 2B_2(p_{2,1}+1)z) + 2B_2p_{2,1}(z-x))^2\right)\\ &\leq \exp\left(-\frac{1}{4B_2}(t'_2 - 2B_2(p_{2,1}+1)z)^2 - B_2p_{2,1}^2(z-x)^2 + |p_{2,1}(z-x)| \times |t'_2 - 2B_2(p_{2,1}+1)z|\right)\\ &\qquad \leq \exp\left(-\frac{1}{4B_2}(t'_2 - 2B_2(p_{2,1}+1)z)^2 + p_{2,1}D\Delta\right)\end{aligned}$$

and

$$\begin{aligned}&\exp\left(-\frac{1}{4B_2}(t'_2 - 2B_2(p_{2,1}x+z))^2\right)\\ &\geq \exp\left(-\frac{1}{4B_2}(t'_2 - 2B_2(p_{2,1}+1)z)^2 - B_2p_{2,1}^2(z-x)^2 - |p_{2,1}(z-x)| \times |t'_2 - 2B_2(p_{2,1}+1)z|\right)\\ &\qquad \geq \exp\left(-\frac{1}{4B_2}(t'_2 - 2B_2(p_{2,1}+1)z)^2 - B_2p_{2,1}^2D^2 - p_{2,1}D\Delta\right)\end{aligned}$$

□

*Proof of Proposition 5.4.* We write the proof in the case $k \in \{1,2,\dots,n-1\}$ and for $S^U_{2n|2k+1}$ (the other cases being very similar). Let $\varphi$ be a test function (in $\mathcal{C}_b^+(\mathbb{R})$). By Remark 5.2, we have that $U^{(2)}_{2k-1}$ takes its values in $C_{2k-1}(\Delta)$. We write, for any $z^{(2)}_{2k-2} \in \mathbb{R}$, $z^{(1)}_{2k} \in C_{2k-1}(\Delta)$, $z \in \mathbb{R}$ (like in the proof of Lemma 5.3)

$$\begin{aligned}(5.9)\quad &\mathbb{E}(\varphi(U_{2k+1})|U_{2k-1} = (z^{(2)}_{2k-2}, z^{(1)}_{2k})) =\\ &\int_{\mathbb{R}^2} \varphi(z^{(2)}_{2k}, z^{(1)}_{2k+2}) \frac{S^\Delta_{2n|2k-2}((z, z^{(2)}_{2k-2}), (z^{(1)}_{2k}, z^{(2)}_{2k}))}{\int_{\mathbb{R}} S^\Delta_{2n|2k-2}((z, z^{(2)}_{2k-2}), (z^{(1)}_{2k}, z'))dz'}\\ &\times \left(\int_{\mathbb{R}} S^\Delta_{2n|2k}((z^{(1)}_{2k}, z^{(2)}_{2k}), (z^{(1)}_{2k+2}, z^{(2)}_{2k+2})dz^{(2)}_{2k+2}\right) dz^{(2)}_{2k} dz^{(1)}_{2k+2} \geq\\ &\text{(by Equations (5.5), (5.6))}\\ &\int_{\mathbb{R}^2} \varphi(z^{(2)}_{2k}, z^{(1)}_{2k+2}) \epsilon^2_{2k-1} \frac{\widetilde{R}^\Delta_{2n:2k+2}((z^{(1)}_{2k}, z^{(2)}_{2k}), \mathbb{R}^2) 1_{C_{2k}(\Delta)}(z^{(2)}_{2k}) \psi_{2k}(z^{(1)}_{2k}, z^{(2)}_{2k})}{\int_{\mathbb{R}} \widetilde{R}^\Delta_{2n:2k+2}((z^{(1)}_{2k}, z'), \mathbb{R}^2) 1_{C_{2k}(\Delta)}(z') \psi_{2k}(z^{(1)}_{2k}, z')dz'}\\ &\times \left(\int_{\mathbb{R}} S^\Delta_{2n|2k}((z^{(1)}_{2k}, z^{(2)}_{2k}), (z^{(1)}_{2k+2}, z^{(2)}_{2k+2})dz^{(2)}_{2k+2}\right) dz^{(2)}_{2k} dz^{(1)}_{2k+2}\end{aligned}$$

From Lemma 1.3, we get, for all $z_{2k}$ such that $z_{2k}^{(2)} \in C_{2k}(\Delta)$, using the same kind of computation as in the proof of Lemma 4.3,

$$
\begin{aligned}
&\psi_{2k}(z_{2k}^{(1)}, z_{2k}^{(2)}) \geq \sigma_1\sigma_2 e^{-2M|z_{2k}^{(1)}-z_{2k}^{(2)}|-\tau\left(M+\frac{M^2}{2}\right)} \\
&\times\exp\left(-A_2(z_{2k}^{(1)})^2 - B_2(z_{2k}^{(2)})^2 + C_1 z_{2k}^{(1)} z_{2k}^{(2)} + A_1^{y(2k-1)\tau:(2k)\tau} z_{2k}^{(1)} + B_1^{y(2k-1)\tau:(2k)\tau} z_{2k}^{(2)} + C_0^{y(2k-1)\tau:(2k)\tau}\right) \\
&= \sigma_1\sigma_2 e^{-2M|z_{2k}^{(1)}-z_{2k}^{(2)}|-\tau\left(M+\frac{M^2}{2}\right)} \times\exp\left[-\frac{1}{4A_2}\left(\frac{A_1^{y(2k-1)\tau:(2k)\tau}}{p_{1,1}} - \frac{p_{2,1}B_1^{y(2k-1)\tau:(2k)\tau}}{p_{1,1}} - 2A_2 p_{1,1} z_{2k}^{(1)}\right)^2\right. \\
&\qquad - \frac{1}{4B_2}\left(B_1^{y(2k-1)\tau:(2k)\tau} - 2B_2(p_{2,1}z_{2k}^{(1)} + z_{2k}^{(2)})\right)^2 + C_0^{y(2k-1)\tau:(2k)\tau} \\
&\qquad \left.+\frac{1}{4}(A_1^{y(2k-1)\tau:(2k)\tau}, B_1^{y(2k-1)\tau:(2k)\tau})\kappa^{-1}(A_1^{y(2k-1)\tau:(2k)\tau}, B_1^{y(2k-1)\tau:(2k)\tau})^T\right] \\
&\qquad \geq \text{(by Lemma 5.5) } \sigma_1\sigma_2 e^{-2M|z_{2k}^{(1)}-z_{2k}^{(2)}|+C_0^{y(2k-1)\tau:(2k)\tau}} \times e^{-\tau\left(M+\frac{M^2}{2}\right)} \\
&\qquad \times\exp\left[-\frac{1}{4A_2}\left(\frac{A_1^{y(2k-1)\tau:(2k)\tau}}{p_{1,1}} - \frac{p_{2,1}B_1^{y(2k-1)\tau:(2k)\tau}}{p_{1,1}} - 2A_2 p_{1,1} z_{2k}^{(1)}\right)^2\right. \\
&\quad - \frac{1}{4B_2}\left(B_1^{y(2k-1)\tau:(2k)\tau} - 2B_2(p_{2,1}+1)z_{2k}^{(2)}\right)^2 - B_2 p_{2,1}^2(z_{2k}^{(1)} - z_{2k}^{(2)})^2 - \Delta p_{2,1}|z_{2k}^{(1)} - z_{2k}^{(2)}| \\
&\qquad\qquad \left.+\frac{1}{4}(A_1^{y(2k-1)\tau:(2k)\tau}, B_1^{y(2k-1)\tau:(2k)\tau})\kappa^{-1}(A_1^{y(2k-1)\tau:(2k)\tau}, B_1^{y(2k-1)\tau:(2k)\tau})^T\right].
\end{aligned}
$$

We set

$$
\begin{aligned}
\psi_{2k}^{(1)}(x) &= \exp\left(-\frac{1}{4A_2}\left(\frac{A_1^{y(2k-1)\tau:(2k)\tau}}{p_{1,1}} - \frac{p_{2,1}B_1^{y(2k-1)\tau:(2k2)\tau}}{p_{1,1}} - 2A_2 p_{1,1} x\right)^2\right), \\
\psi_{2k}^{(2)}(x) &= \exp\left(-\frac{1}{4B_2}\left(B_1^{y(2k-1)\tau:(2k)\tau} - 2B_2(p_{2,1}+1)x\right)^2\right)
\end{aligned}
$$

In the same way as above:

$$
\begin{aligned}
&\psi_{2k}(z_{2k}^{(1)}, z_{2k}^{(2)}) \leq \sigma_1\sigma_2 e^{2M|z_{2k}^{(1)}-z_{2k}^{(2)}|+\tau\left(M+\frac{M^2}{2}\right)+C_0^{y(2k-1)\tau:(2k)\tau}} \psi_{2k}^{(1)}(z_{2k}^{(1)})\psi_{2k}^{(2)}(z_{2k}^{(2)}) \\
&\times\exp\left(\Delta p_{2,1}|z_{2k}^{(1)} - z_{2k}^{(2)}| + \frac{1}{4}(A_1^{y(2k-1)\tau:(2k)\tau}, B_1^{y(2k-1)\tau:(2k)\tau})\kappa^{-1}(A_1^{y(2k-1)\tau:(2k)\tau}, B_1^{y(2k-1)\tau:(2k)\tau})^T\right).
\end{aligned}
$$

Looking back at (5.9), we get

$$
\begin{aligned}
&\mathbb{E}(\varphi(U_{2k+1})|U_{2k-1} = (z_{2k-2}^{(2)}, z_{2k}^{(1)})) \geq \\
&\int_{\mathbb{R}^2} \varphi(z_{2k}^{(2)}, z_{2k+2}^{(1)})\epsilon_{2k-1}^2(\epsilon'_{2k})^2 \frac{\widetilde{R}^{\Delta}_{2n:2k+2}((z_{2k}^{(1)}, z_{2k}^{(2)}), \mathbb{R}^2)1_{C_{2k}(\Delta)}(z_{2k}^{(2)})\psi_{2k}^{(2)}(z_{2k}^{(2)})}{\int_{\mathbb{R}} \widetilde{R}^{\Delta}_{2n:2k+2}((z_{2k}^{(1)}, z'), \mathbb{R}^2)1_{C_{2k}(\Delta)}(z_{2k}^{(2)})\psi_{2k}^{(2)}(z')dz'} \\
&\qquad\qquad \times\left(\int_{\mathbb{R}} S^{\Delta}_{2n|2k}((z_{2k}^{(1)}, z_{2k}^{(2)}), (z_{2k+2}^{(1)}, z_{2k+2}^{(2)})dz_{2k+2}^{(2)}\right) dz_{2k}^{(2)} dz_{2k+2}^{(1)}.
\end{aligned}
$$

As $R_{2n:2k+2}((z_{2k}^{(1)}, z'), .)$ and $S^{\Delta}_{2n|2k}((z_{2k}^{(1)}, z'), .)$ do not depend on $z_{2k}^{(1)}$ for any $z'$, we get that $S^U_{2n|2k+1}$ is $(1-\epsilon_{2k-1}^2(\epsilon'_{2k})^2)$-contracting (remember Section 3.1).

□

5.3. **New representation.**

**Proposition 5.6.** *Let $n \geq 1$. If we suppose that $Z_0^{(2)}$ is of law $\psi^{\Delta}_{2n|0}(0, .) \bullet \mu$, then, for all test function $\varphi$ (in $\mathcal{C}_b^+(\mathbb{R})$),*

$$
\frac{\mathbb{E}(\varphi(U_{2n+1}^{(1)})\prod_{1\leq i\leq n}\psi^{\Delta}_{2i-1}(U_{2i-1}))}{\mathbb{E}(\prod_{1\leq i\leq n}\psi^{\Delta}_{2i-1}(U_{2i-1}))} = \left(\overline{R}^{\Delta}_{2n}\overline{R}^{\Delta}_{2n-1}\dots\overline{R}^{\Delta}_1(\mu)\right)(\varphi). \tag{5.10}
$$

*If we suppose that $Z_0^{(2)}$ is of law $\psi_{2n|0}^{\Delta,(p)}(0,.)\bullet\mu$, then, for all test function $\varphi$ (in $\mathcal{C}_b^+(\mathbb{R})$),*

$$\frac{\mathbb{E}(\varphi(U_{2n-1}^{(p)(2)})\prod_{1\leq i\leq n}\psi_{2i-1}^{\Delta}(U_{2i-1}^{(p)}))}{\mathbb{E}(\prod_{1\leq i\leq n}\psi_{2i-1}^{\Delta}(U_{2i-1}^{(p)}))}=\left(\overline{R}_{2n-1}^{\Delta}\overline{R}_{2n-2}^{\Delta}\dots\overline{R}_{1}^{\Delta}(\mu)\right)(\varphi)\,. \tag{5.11}$$

*Remark* 5.7. Recall that we are working with a fixed observation $(Y_s)_{s\geq0}=(y_s)_{s\geq0}$. The above Proposition tells that, for all $n$, $\overline{R}_n^{\Delta}\overline{R}_{n-1}^{\Delta}\dots\overline{R}_1^{\Delta}(\mu)$ can be written as the $n$-th term of a Feynman-Kac sequence based on mixing kernels (by Proposition 5.4). We can apply Proposition 3.1 to this Feynman-Kac sequence. This representation and this result are also true for a measure $\overline{R}_n^{\Delta}\overline{R}_{n-1}^{\Delta}\dots\overline{R}_k^{\Delta}(\eta)$ for any $k\leq n$, $\eta$ probability measure on $\mathbb{R}$.

*Proof.* We write the proof only for Equation (5.10). The computation leading to Equation (5.11) would be very similar. It would simplify nicely because we replace $\psi_{2n}^{\Delta}$ by 1 in the definition of the $S_{2n|\dots}^{\Delta(p)}$, $\psi_{2n|\dots}^{\Delta(p)}$.

We have, for any test function $\varphi$ (in $\mathcal{C}_b^+(\mathbb{R})$),

$$\begin{aligned}
&\mathbb{E}(\varphi(U_{2n+1}^{(1)})\prod_{1\leq i\leq n}\psi_{2i-1}^{\Delta}(U_{2i-1}))=\\
&\int_{\mathbb{R}\times(\mathbb{R}^2)^n}\varphi(z_{2n}^{(2)})\prod_{0\leq k\leq n-1}\left[S_{2n|2k}^{\Delta}(z_{2k},z_{2k+2})\psi_{2k+1}^{\Delta}(z_{2k}^{(2)},z_{2k+2}^{(1)})\right](\psi_{2n|0}^{\Delta}(0,.)\bullet\mu)(dz_0)dz_2\dots dz_{2n}=\\
&\qquad\int_{\mathbb{R}\times(\mathbb{R}^2)^n}\varphi(z_{2n}^{(2)})\prod_{0\leq k\leq n-1}\left[\frac{\psi_{2n|2k+2}^{\Delta}(z_{2k+2})}{\psi_{2n|2k}^{\Delta}(z_{2k})}\widetilde{R}_{2k+2}^{\Delta}(z_{2k},dz_{2k+2})\psi_{2k+1}^{\Delta}(z_{2k}^{(2)},z_{2k+2}^{(1)})\right]\\
&\qquad\qquad\times(\psi_{2n|0}^{\Delta}(0,.)\bullet\mu)(dz_0)dz_2\dots dz_{2n}=\\
&\int_{\mathbb{R}\times(\mathbb{R}^2)^n}\varphi(z_{2n}^{(2)})\prod_{0\leq k\leq n-1}\left[\widetilde{R}_{2k+2}^{\Delta}(z_{2k},dz_{2k+2})\psi_{2k+1}^{\Delta}(z_{2k}^{(2)},z_{2k+2}^{(1)})\right]\frac{1}{\mu(\psi_{2n|0}^{\Delta}(0,.))}\mu(dz_0)dz_2\dots dz_{2n}=\\
&\qquad\int_{\mathbb{R}\times(\mathbb{R}^2)^n}\varphi(z_{2n}^{(2)})\prod_{0\leq k\leq n-1}\left[\psi_{2k+2}^{\Delta}(z_{2k+2}^{(1)},z_{2k+2}^{(2)})\psi_{2k+1}^{\Delta}(z_{2k}^{(2)},z_{2k+2}^{(1)})\widetilde{Q}^2(z_{2k},dz_{2k+2})\right]\\
&\qquad\qquad\times\frac{1}{\mu(\psi_{2n|0}^{\Delta}(0,.))}\mu(dz_0)dz_2\dots dz_{2n}\,,
\end{aligned}$$

which proves the desired result (recall Equation (3.7)). ☐

## 6. Stability results

In this section, the observations are non longer fixed.

6.1. **Stability of the truncated filter.** We show here that a product of coefficients $\tau_{.}$ decays geometrically in expectation (see the Lemma below). These coefficients are the contraction coefficients of the operators $S_{.}^{U}$, $S_{.}^{U,(p)}$, which are related to the truncated filter through Proposition 5.6. This is why we say that the result below means the stability of the truncated filter.

We set, for all $t$ in $\mathbb{R}$, $k\geq1$,

$$\tau(t,\Delta)=1-(\epsilon'(t,\Delta)\epsilon(t,\Delta))^2\,,$$
$$\tau_k=1-(\epsilon'_k\epsilon_{k-1})^2\,.$$

We set, for $L>0$,

$$\widetilde{\alpha}(L)=\frac{96C\sqrt{\tau}}{L\sqrt{\pi}}\exp\left(-\frac{1}{2}\left(\frac{L}{6C\sqrt{2\tau}}\right)^2\right)\,.$$

We fix $L>0$ such that

$$L\geq3m_0+3CM\tau^2\text{ and }\widetilde{\alpha}(L)\leq\frac{1}{4}\,. \tag{6.1}$$

We set

$$\rho = \frac{\tau(L,\Delta)+\sqrt{\tau(L,\Delta)^2+4\widetilde{\alpha}(L)(1-\tau(L,\Delta))}}{2}\,.$$

**Lemma 6.1.** *For $0\leq k\leq n-1$, we have*

$$\mathbb{E}(\tau_{2n+1}\tau_{2n-1}\dots\tau_{2k+3}|\mathcal{F}_{0:(2k+1)\tau})\leq\left(1-\frac{\epsilon(L,\Delta)^2\epsilon'(L,\Delta)^2}{2}\right)^{n-k-2}\,,$$

$$\mathbb{E}(\tau_{2n}\tau_{2n-2}\dots\tau_{2k+2}|\mathcal{F}_{0:2k\tau})\leq\left(1-\frac{\epsilon(L,\Delta)^2\epsilon'(L,\Delta)^2}{2}\right)^{n-k-2}\,.$$

*Proof.* We only write the proof of the second Equation above (the proof of the other equation is very similar). We take $L>0$ and we set

$$\theta_{2k} = \begin{cases}\tau(L,\Delta) & \text{if } |m_{2k}-m_{2k-1}|<L \text{ and } |m_{2k-1}-m_{2k-2}|<L\\ 1 & \text{otherwise.}\end{cases}$$

For all $k$, we have $\tau_{2k}\leq\theta_{2k}$. For any $k\geq 1$, $|m_k-m_{k-1}|$ is a function of $Y_{(k-2)_+\tau:k\tau}$. So, for all $k$, $\theta_{2k}$ is a function of $Y_{(2k-3)_+\tau:2k\tau}$ We fix $k\geq 0$ and we define, for $n\geq 0$,

$$e_{2n|2k+2} = \begin{cases}\mathbb{E}(\theta_{2n}\theta_{2n-2}\dots\theta_{2k+2}|\mathcal{F}_{2k\tau}) & \text{if } k\leq n-1\,,\\ 1 & \text{otherwise.}\end{cases}$$

We suppose now that $n\geq k+2$. We then have

$$e_{2n|2k+2}=\mathbb{E}(\mathbb{E}(\theta_{2n}\theta_{2n-2}|\mathcal{F}_{(2n-3)\tau})\theta_{2n-4}\dots\theta_{2k+2}|\mathcal{F}_{2k\tau})$$

and

$$\begin{aligned}
&\mathbb{E}(\theta_{2n}\theta_{2n-2}|\mathcal{F}_{(2n-3)\tau})\\
&= \mathbb{E}(\theta_{2n-2}(1-1_{[0,L)}(D_{2n})1_{[0,L)}(D_{2n-1}))+\tau(L,\Delta)\theta_{2n-2}1_{[0,L)}(D_{2n})1_{[0,L)}(D_{2n-1})|\mathcal{F}_{(2n-3)\tau})\\
&\quad= \mathbb{E}(\theta_{2n-2}\tau(L,\Delta)+(1-\tau(L,\Delta))\theta_{2n-2}(1-1_{[0,L)}(D_{2n})1_{[0,L)}(D_{2n-1}))|\mathcal{F}_{(2n-3)\tau})\\
&\quad\leq \tau(L,\Delta)\mathbb{E}(\theta_{2n-2}|\mathcal{F}_{(2n-3)\tau})+(1-\tau(L,\Delta))[\mathbb{P}(|m_{2n}-m_{2n-1}|\geq L|\mathcal{F}_{(2n-3)\tau})\\
&\qquad\qquad+\mathbb{P}(|m_{2n-1}-m_{2n-2}|\geq L|\mathcal{F}_{(2n-3)\tau})]\,.
\end{aligned}$$

Using Equation (4.17), we get

$$\begin{aligned}
&\mathbb{E}(\theta_{2n}\theta_{2n-2}|\mathcal{F}_{(2n-3)\tau})\leq\tau(L,\Delta)\mathbb{E}(\theta_{2n-2}|\mathcal{F}_{(2n-3)\tau})\\
&\quad+(1-\tau(L,\Delta))\left(\mathbb{P}\left(C\mathcal{V}_{(2n-2)\tau,2n\tau}\geq\frac{L}{3}|\mathcal{F}_{(2n-3)\tau}\right)+\mathbb{P}\left(C\mathcal{W}_{(2n-2)\tau,2n\tau}\geq\frac{L}{3}|\mathcal{F}_{(2n-3)\tau}\right)\right)\\
&+(1-\tau(L,\Delta))\left(\mathbb{P}\left(C\mathcal{V}_{(2n-3)\tau,(2n-1)\tau}\geq\frac{L}{3}|\mathcal{F}_{(2n-3)\tau}\right)+\mathbb{P}\left(C\mathcal{W}_{(2n-3)\tau,(2n-1)\tau}\geq\frac{L}{3}|\mathcal{F}_{(2n-3)\tau}\right)\right)\\
&\leq\tau(L,\Delta)\mathbb{E}(\theta_{2n-2}|\mathcal{F}_{(2n-3)\tau})+4(1-\tau(L,\Delta))\mathbb{P}\left(C\mathcal{V}_{0,2\tau}\geq\frac{L}{3}\right)\\
&\qquad\text{(like in Equations (4.28), (4.29))}\\
&\leq\tau(L,\Delta)\mathbb{E}(\theta_{2n-2}|\mathcal{F}_{(2n-3)\tau})+8(1-\tau(L,\Delta))\mathbb{P}\left(2C|W_{2\tau}|\geq\frac{L}{3}\right)\\
&\qquad\text{(using Equation (4.30))}\\
&\leq\tau(L,\Delta)\mathbb{E}(\theta_{2n-2}|\mathcal{F}_{(2n-3)\tau})+(1-\tau(L,\Delta))\widetilde{\alpha}(L)\,.
\end{aligned}$$

The constant $\rho$ is the positive root of the polynomial $X^2-\tau(L,\Delta)X-(1-\tau(L,\Delta))\widetilde{\alpha}(L)$. So we have

$$1>\rho=\tau(L,\Delta)+\frac{1}{\rho}(1-\tau(L,\Delta))\widetilde{\alpha}(L)\geq\tau(L,\Delta)+(1-\tau(L,\Delta))\widetilde{\alpha}(L)\,.$$

So, we have

$$e_{2n|2k+2}\leq\tau(L,\Delta)e_{2n-2|2k+2}+(1-\tau(L,\Delta))\widetilde{\alpha}(L)e_{2n-4|2k+2}\leq\rho\times\sup(e_{2n-2|2k+2},e_{2n-4|2k+2})\,.$$

Suppose now that $k$ is fixed. We have

$$e_{2k+2|2k+2} \leq 1 \,,\; e_{2k+4|2k+2} \leq 1 \,.$$

So, by recurrence,

$$e_{2n|2k+2} \leq \rho^{(n-k-2)} \,.$$

As $\widetilde{\alpha}(L) \leq 1/4$, we have

$$\rho \leq \frac{1}{2}(\tau(L,\Delta) + \sqrt{\tau(L,\Delta)^2 + 1 - \tau(L,\Delta)}) \leq \frac{\tau(L,\Delta)+1}{2} = 1 - \frac{(\epsilon(L,\Delta)^2\epsilon'(L,\Delta))^2}{2} \,.$$

□

**Lemma 6.2.** *For $n \geq 1$ and $k$ in $\{1,2,\dots,n-1\}$, for all $\mu$, $\mu'$ in $\mathcal{P}(\mathbb{R})$,*

$$\begin{aligned}&\left\|\overline{R}^{\Delta}_n\overline{R}^{\Delta}_{n-1}\dots\overline{R}^{\Delta}_{k+1}(\mu) - \overline{R}^{\Delta}_n\overline{R}^{\Delta}_{n-1}\dots\overline{R}^{\Delta}_{k+1}(\mu')\right\| \\ &\qquad\leq \prod_{i=1}^{\lfloor (n-k)/2\rfloor}(1-\epsilon^2_{k+2i-1}(\epsilon'_{k+2i})^2) \times 4\inf\left(1, \frac{\|\mu-\mu'\|}{(\epsilon'_{k+2})^2\epsilon^4_{k+1}}\right) \,.\end{aligned}$$

*Proof.* We write the proof in the case where $n$ and $k$ are even. If $k$ was even and $n$ was odd, we would have to use the operators $S^{(p)U}_{\dots}$. If $k$ was odd, the proof would be very similar but would require to introduce new and heavy notations.

By Proposition 5.6, Remark 5.7 and Equation (5.7), we have, for all $\mu$ in $\mathcal{P}(\mathbb{R})$ and all test function $\varphi$ in $\mathcal{C}^+_b(\mathbb{R})$,

$$\begin{aligned}\left(\overline{R}^{\Delta}_n\overline{R}^{\Delta}_{n-1}\dots\overline{R}^{\Delta}_{k+1}(\mu)\right)(\varphi) &\propto \int \varphi(u^{(1)}_{n+1})\left(\prod_{i=1}^{(n-k)/2}\psi^{\Delta}_{k+2i-1}(u_{k+2i-1})\right) \\ &\quad\times\left(\prod_{i=1}^{(n-k-2)/2} S^U_{n|k+2i+1}(u_{k+2i-1}, du_{k+2i+1})\right) \\ &\quad\times S^U_{n|n+1}(u_{n-1}, du^{(1)}_{n+1})\left(\int_{z'\in\mathbb{R}} S^{\Delta}_{n|k}((0,z^{(2)}_k),(du^{(2)}_{k+1},dz'))\right)(\psi^{\Delta}_{n|k}(0,.)\bullet\mu)(dz^{(2)}_k) \,,\end{aligned}$$

where we integrate over $z^{(2)}_k \in \mathbb{R}$, $u_{k+1}, u_{k+3}, \dots, u_{n-1} \in \mathbb{R}^2$, $u^{(1)}_{n+1} \in \mathbb{R}$.

By Proposition 5.4, we know that $S^U_{n|k+2i+1}$ is $(\epsilon^2_{k+2i-1}(\epsilon'_{k+2i})^2)$-mixing for all $i$ in $\{1,2,\dots,1+(n-k)/2\}$. We now apply Proposition 3.1 with the $S^U_{n|k+2i+1}$ playing the roles of the $\mathfrak{Q}_{\dots}$ and the $\psi^{\Delta}_{k+2i-1}$ playing the roles of the $\Psi_{\dots}$. By Equations (3.8), (3.9), we then have, for all $\mu$ and $\mu'$ in $\mathcal{P}(\mathbb{R})$,

$$\begin{aligned}&\left\|\overline{R}^{\Delta}_n\overline{R}^{\Delta}_{n-1}\dots\overline{R}^{\Delta}_{k+1}(\mu) - \overline{R}^{\Delta}_n\overline{R}^{\Delta}_{n-1}\dots\overline{R}^{\Delta}_{k+1}(\mu')\right\| \\ &\qquad\leq \prod_{i=1}^{(n-k)/2}(1-\epsilon^2_{k+2i+1}(\epsilon'_{k+2i})^2)\times 2\inf\left(1, \frac{\|\psi^{\Delta}_{n|k}(0,.)\bullet\mu - \psi^{\Delta}_{n|k}(0,.)\bullet\mu'\|}{\epsilon^2_{k+1}(\epsilon'_{k+2})^2}\right) \,.\end{aligned}$$

By Equations (5.4), (3.8), (3.9), we have

$$\begin{aligned}\|\psi^{\Delta}_{n|k}(0,.)\bullet\mu - \psi^{\Delta}_{n|k}(0,.)\bullet\mu'\| &\leq 2\inf\left(1, \frac{\|\psi^{\Delta}_{n|k}(0,.)\|_\infty}{\langle\mu,\psi^{\Delta}_{n|k}(0,.)\rangle}\|\mu-\mu'\|\right) \\ &\leq 2\inf\left(1, \frac{\|\mu-\mu'\|}{\epsilon^2_{k+1}}\right) \,.\end{aligned}$$

From which we get the result. □

6.2. **Approximation of the optimal filter by the truncated filter.** We recall that "$\underset{\Delta}{\preceq}$" is defined in Definition 2.5.

**Proposition 6.3.** *There exists $\tau_\infty$ such that, if $\tau \geq \tau_\infty$, we have*

$$\sup_{n\geq 0} \log(\mathbb{E}(\|\pi_{n\tau} - \pi^{\Delta}_{n\tau}\|)) \underset{\Delta,c}{\preceq} -\frac{\grave{\Delta}^2}{h}\,; \tag{6.2}$$

*and, if we suppose that $(\pi'_t)_{t\geq 0}$ is defined as in Corollary 4.5, then*

$$\sup_{n\geq 0} \log(\mathbb{E}(\|\pi'_{n\tau} - (\pi')^{\Delta}_{n\tau}\|)) \underset{\Delta,c}{\preceq} -\frac{\grave{\Delta}^2}{h}\,. \tag{6.3}$$

*Proof.* We write the proof only for Equation (6.2), the proof for Equation (6.3) being very similar. We have

$$\|\pi_{n\tau} - \pi^{\Delta}_{n\tau}\| \leq \|\pi_{n\tau} - \overline{R}^{\Delta}_n(\pi_{(n-1)\tau})\| + \sum_{1\leq k\leq n-1} \|\overline{R}^{\Delta}_{n:k+1}(\pi_{k\tau}) - \overline{R}^{\Delta}_{n:k+1}(\overline{R}^{\Delta}_k(\pi_{(k-1)\tau}))\|\,. \tag{6.4}$$

Let us fix $k \in \{1, 2, \dots, n-1\}$. From Lemma 6.2, we get

$$\begin{aligned}
&\mathbb{E}(\|\overline{R}^{\Delta}_{n:k+1}(\pi_{k\tau}) - \overline{R}^{\Delta}_{n:k+1}(\overline{R}^{\Delta}_k(\pi_{(k-1)\tau}))\|) \\
&\quad\leq \mathbb{E}\left(\mathbb{E}\left(\left.\prod_{2\leq i\leq \lfloor\frac{n-k}{2}\rfloor} (1 - (\epsilon^2_{k+2i-1}(\epsilon'_{k+2i})^2))\right| \mathcal{F}_{(k+1)\tau}\right) \times 2\inf\left(1, \frac{\|\pi_{k\tau} - \overline{R}^{\Delta}_k(\pi_{(k-1)\tau})\|}{(\epsilon'_{k+2})^2\epsilon^4_{k+1}}\right)\right)\,,
\end{aligned} \tag{6.5}$$

with the convention that a product over indexes in the null set is equal to one. From Lemma 6.1, we get

$$\begin{aligned}
\mathbb{E}(\|\overline{R}^{\Delta}_{n:k+1}(\pi_{k\tau}) - \overline{R}^{\Delta}_{n:k+1}(\overline{R}^{\Delta}_k(\pi_{(k-1)\tau}))\|) &\leq \left(1 - \frac{(\epsilon(L,\Delta)^2\epsilon'(L,\Delta))^2}{2}\right)^{(\lfloor\frac{n-k}{2}\rfloor - 3)_+} \\
&\quad \times 2\mathbb{E}\left(\inf\left(1, \frac{\|\pi_{k\tau} - \overline{R}^{\Delta}_k(\pi_{(k-1)\tau})\|}{(\epsilon'_{k+2})^2{\epsilon^4}_{k+1}}\right)\right)\,.
\end{aligned} \tag{6.6}$$

As in [OR05], p. 434, we can bound

$$\inf\left(1, \frac{\|\pi_{n\tau} - \overline{R}^{\Delta}_k(\pi_{(k-1)\tau})\|}{(\epsilon'_{k+2})^2\epsilon^4_{k+1}}\right) \leq \inf\left(1, \frac{T(\Delta)}{(\epsilon'_{k+2})^4\epsilon^8_{k+1}}\right) + \inf\left(1, \frac{\|\pi_{n\tau} - \overline{R}^{\Delta}_k(\pi_{(k-1)\tau})\|^2}{T(\Delta)}\right)\,. \tag{6.7}$$

We have, if $\Delta$ satisfies the assumption of Proposition 4.6,

$$\begin{aligned}
&\mathbb{E}\left(\inf\left(1, \frac{\|\pi_{k\tau} - \overline{R}^{\Delta}_k(\pi_{(k-1)\tau})\|^2}{T(\Delta)}\right)\right) \\
&\qquad = \mathbb{E}\left(\frac{\|\pi_{k\tau} - \overline{R}^{\Delta}_k(\pi_{(k-1)\tau})\|^2}{T(\Delta)} 1_{[0,1]}\left(\frac{\|\pi_{k\tau} - \overline{R}^{\Delta}_k(\pi_{(k-1)\tau})\|^2}{T(\Delta)}\right)\right) \\
&\qquad\quad + \mathbb{P}\left(\frac{\|\pi_{k\tau} - \overline{R}^{\Delta}_k(\pi_{(k-1)\tau})\|^2}{T(\Delta)} > 1\right) \\
&\qquad \leq \frac{2}{\sqrt{T(\Delta)}} \mathbb{E}(\|\pi_{k\tau} - \overline{R}^{\Delta}_k(\pi_{(k-1)\tau})\|) \\
&\qquad\qquad\qquad (\text{using Prop. 4.6}) \preceq \sqrt{T(\Delta)}\,.
\end{aligned} \tag{6.8}$$

We look now at the term $\inf(1, T(\Delta)(\epsilon'_{k+2})^{-4}\epsilon^{-8}_{k+1})$. Using Equations (2.24), (2.29), (2.30), (2.32) and the remarks below Hypothesis 2, we have, for all $k$,

$$T(\Delta) \underset{\Delta,c}{\preceq} \frac{h^{-1}\tau^{\iota-\frac{1}{2}}}{\Delta}\exp\left(-\frac{1}{2}\left(\frac{\tau^{\frac{1}{2}-\iota}\Delta}{6C\sqrt{2}}\right)^2\right) + \left(M+\sqrt{h}\right)\frac{\sqrt{h}e^{63M^2\tau+\frac{9\tau M}{2}+640M^2}}{\Delta\theta^{3/2}}\exp\left(-\frac{1}{4B_2}d(\Delta)^2\right)C_1'(h,\tau)\,, \tag{6.9}$$

$$\epsilon(D,\Delta)^{-1} = \exp\left(\frac{\left(D+\frac{\Delta}{B_2(1+p_{2,1})}\right)^2}{2\tau} - \frac{\left(\left(D-\frac{\Delta}{B_2(1+p_{2,1})}\right)_+\right)^2}{2\tau}\right)e^{2M\left(D+2\frac{\Delta}{B_2(1+p_{2,1})}\right)+\left(\tau+\frac{\tau^2}{2}\right)M}$$

$$= \begin{cases} \exp\left(\frac{2D\Delta}{\tau B_2(1+p_{2,1})}+2M\left(D+2\frac{\Delta}{B_2(1+p_{2,1})}\right)+\left(\tau+\frac{\tau^2}{2}\right)M\right) & \text{if } D \geq \frac{\Delta}{B_2(1+p_{2,1})}\,,\\ \exp\left(\frac{\left(D+\frac{\Delta}{B_2(1+p_{2,1})}\right)^2}{2\tau}+2M\left(D+2\frac{\Delta}{B_2(1+p_{2,1})}\right)+\left(\tau+\frac{\tau^2}{2}\right)M\right) & \text{otherwise}\,. \end{cases} \tag{6.10}$$

$$(\epsilon'(D,\Delta))^{-2} = \exp\Bigg[B_2p_{2,1}^2\left(\frac{\Delta}{B_2(1+p_{2,1})}+D\right)^2+2B_2\Delta p_{2,1}\left(\frac{\Delta}{B_2(1+p_{2,1})}+D\right) +4M\left(\frac{\Delta}{B_2(1+p_{2,1})}+D\right)+2\tau\left(M+\frac{M^2}{2}\right)\Bigg]\,. \tag{6.11}$$

We note that the above expressions are nondecreasing functions of $D$. From Equation (4.17), we get, for $j=k+1,\,k+2$.

$$D_j \leq C(\tau^2M+\mathcal{V}_{(k-1)\tau,(k+2)\tau}+\mathcal{W}_{(k-1)\tau,(k+2)\tau})\,. \tag{6.12}$$

The variables $\mathcal{V}_{(k-1)\tau,(k+3)\tau}$ and $\mathcal{W}_{(k-1)\tau,(k+3)\tau}$ are independent and can be controlled as in Equations (4.28), (4.29). So we can bound

$$\forall x\in\mathbb{R}\,,\ \mathbb{P}(\mathcal{V}_{(k-1)\tau,(k+3)\tau}+\mathcal{W}_{(k-1)\tau,(k+3)\tau}\geq x)\leq 2\mathbb{P}(\mathcal{V}_{0,4\tau}\geq\frac{x}{2})\leq 4\mathbb{P}(2|W_{4\tau}|\geq x)\,.$$

So

$$\mathbb{E}\left(\inf\left(1,\frac{T(\Delta)}{(\epsilon'_{k+2})^4\epsilon_{k+1}^8}\right)\right)\leq\int_0^{+\infty}\inf\left\{1,T(\Delta)\epsilon(C\tau^2M+Cz,\Delta)^{-8} \times\epsilon'(C\tau^2M+Cz,\Delta)^{-4}\right\}\frac{8\exp\left(-\frac{z^2}{2(16\tau)}\right)}{\sqrt{2\pi\times16\tau}}dz\,. \tag{6.13}$$

We have

$$\int_0^{\left(\frac{\Delta}{CB_2(1+p_{2,1})}-\tau^2M\right)_+}\epsilon(C\tau^2M+Cz,\Delta)^{-8}\epsilon'(C\tau^2M+Cz,\Delta)^{-4}\frac{8\exp\left(-\frac{z^2}{2(16\tau)}\right)}{\sqrt{2\pi\times16\tau}}dz \tag{6.14}$$

$$\leq 8\epsilon\left(\frac{\Delta}{B_2(1+p_{2,1})},\Delta\right)^{-8}\epsilon'\left(\frac{\Delta}{B_2(1+p_{2,1})},\Delta\right)^{-4}$$

$$= 8\exp\Bigg[\frac{16}{\tau}\left(\frac{\Delta}{B_2(1+p_{2,1})}\right)^2+48\left(\frac{M\Delta}{B_2(1+p_{2,1})}\right)+8\left(\tau+\frac{\tau^2}{2}\right)M+8B_2p_{2,1}^2\left(\frac{\Delta}{B_2(1+p_{2,1})}\right)^2 +4B_2\Delta p_{2,1}\left(\frac{\Delta}{B_2(1+p_{2,1})}\right)+16M\left(\frac{\Delta}{B_2(1+p_{2,1})}\right)+4\tau\left(M+\frac{M^2}{2}\right)\Bigg]\,.$$

From Subsection 2.4, we get

$$p_{2,1}=O\left(\frac{1}{\theta}\right)\,,\ B_2\underset{\theta\to+\infty}{\longrightarrow}\frac{h}{2}\,.$$

So there exists $\tau_0$, such that, for $\tau\geq\tau_0$,

$$\log\left(\int_0^{\left(\frac{\Delta}{CB_2(1+p_{2,1})}-\tau^2M\right)_+}\inf\{1,T(\Delta)\epsilon(C\tau^2M+Cz,\Delta)^{-8}\times\epsilon'(C\tau^2M+Cz,\Delta)^{-4}\}\frac{8\exp\left(-\frac{z^2}{2(16\tau)}\right)}{\sqrt{2\pi\times16\tau}}dz\right)\underset{\Delta,c}{\preceq}-\frac{\Delta^2}{h}\,. \tag{6.15}$$

We now want to bound

$$\int_{\left(\frac{\Delta}{CB_2(1+p_{2,1})}-\tau^2M\right)_+}^{+\infty}\inf\{1,T(\Delta)\epsilon(C\tau^2M+Cz,\Delta)^{-8}\times\epsilon'(C\tau^2M+Cz,\Delta)^{-4}\}\frac{8\exp\left(-\frac{z^2}{2(16\tau)}\right)}{\sqrt{2\pi\times16\tau}}dz\,.$$

Let us set, for $z\geq0$,

$$\begin{aligned}\Phi(z)=T(\Delta)\times\exp&\left(\frac{16(C\tau^2M+Cz)\Delta}{\tau B_2(1+p_{2,1})}+16M\left(C\tau^2M+Cz+\frac{2\Delta}{B_2(1+p_{2,1})}\right)+8\left(\tau+\frac{\tau^2}{2}\right)M\right)\\ \times\exp&\left[2B_2p_{2,1}^2\left(\frac{\Delta}{B_2(1+p_{2,1})}+C\tau^2M+Cz\right)^2+\frac{4\Delta p_{2,1}}{B_2}\left(\frac{\Delta}{B_2(1+p_{2,1})}+C\tau^2M+Cz\right)\right.\\ &\left.+8M\left(\frac{\Delta}{B_2(1+p_{2,1})}+C\tau^2M+Cz\right)+4\tau\left(M+\frac{M^2}{2}\right)\right]\,.\end{aligned}$$

The function $\Phi$ is increasing. Let us set

$$z_0=\inf\{z\,:\,\Phi(z)\geq1\}\,.$$

We take $\epsilon\in(0,1/2)$. We want to show that $z_0\geq\Delta\tau^{1-\epsilon}$ (at least for $\Delta$ big enough). We look at $\Phi(\Delta\tau^{1-\epsilon})$. There exists $\tau_1$ such that, for $\tau\geq\tau_1$,

$$\frac{1}{2}\left(\frac{\tau^{\frac12-\iota}}{6C\sqrt2}\right)^2>\frac{1}{4B_2}\left(\frac{1}{1+p_{2,1}}-6\theta^{1-\iota}B_2p_{2,1}\right)^2,$$

and then

$$\log(T(\Delta))\underset{\Delta\to+\infty}{\sim}-\frac{d(\Delta)^2}{4B_2}\,.$$

There exists $\tau_2$ such that, for $\tau\geq\tau_2$, we have

$$\begin{aligned}&\frac{16(C\tau^2M+C\tau^{1-\epsilon}\Delta)\Delta}{\tau B_2(1+p_{2,1})}+16M\left(C\tau^2M+C\tau^{1-\epsilon}\Delta+\frac{2\Delta}{B_2(1+p_{2,1})}\right)8\left(\tau+\frac{\tau^2}{2}\right)M\\ &+2B_2p_{2,1}^2\left(\frac{\Delta}{B_2(1+p_{2,1})}+C\tau^2M+C\tau^{1-\epsilon}\Delta\right)^2+\frac{4\Delta p_{2,1}}{B_2}\left(\frac{\Delta}{B_2(1+p_{2,1})}+C\tau^2M+C\tau^{1-\epsilon}\Delta\right)\\ &\qquad+8M\left(\frac{\Delta}{B_2(1+p_{2,1})}+C\tau^2M+C\tau^{1-\epsilon}\Delta\right)+4\tau\left(M+\frac{M^2}{2}\right)\underset{\Delta,c}{\preceq}\frac{\Delta^2}{h\tau^\epsilon}\,.\end{aligned}$$

So, if $\tau\geq\sup(\tau_1,\tau_2)$,

$$\Phi(\Delta\tau^{1-\epsilon})\underset{\Delta\to+\infty}{\longrightarrow}0\,,$$

and so, for all $\tau$, $h$, there exists $\Delta_1(\tau,h)$ such that, if $\Delta\geq\Delta_1(\tau,h)$.

$$z_0\geq\Delta\tau^{1-\epsilon}\,.$$

There exists $\tau_3$ such that, for $\tau\geq\tau_3$, $\int_{\mathbb{R}}\Phi(z)\exp\left(-\frac{z^2}{2(16\tau)}\right)dz$ is finite. We can then compute

$$\begin{aligned}\int_{\mathbb{R}}\Phi(z)\frac{e^{-\frac{z^2}{2(16\tau)}}}{\sqrt{2\pi\times16\tau}}dz=T(\Delta)\times\exp&\left(\frac{16(C\tau^2M)\Delta}{\tau B_2(1+p_{2,1})}+16M\left(C\tau^2M+\frac{2\Delta}{B_2(1+p_{2,1})}\right)+8\left(\tau+\frac{\tau^2}{2}\right)M\right)\\ \times\exp&\left[2B_2p_{2,1}^2\left(\frac{\Delta}{B_2(1+p_{2,1})}+C\tau^2M\right)^2+\frac{4\Delta p_{2,1}}{B_2}\left(\frac{\Delta}{B_2(1+p_{2,1})}+C\tau^2M\right)\right.\end{aligned}$$

$$+8M\left(\frac{\Delta}{B_2(1+p_{2,1})}+C\tau^2M\right)+4\tau\left(M+\frac{M^2}{2}\right)\Bigg]$$
$$\times\int_{\mathbb{R}}\exp\left(\frac{16C\Delta}{\tau B_2(1+p_{2,1})}z+16MCz+4B_2p_{2,1}^2\left(\frac{\Delta}{B_2(1+p_{2,1})}+C\tau^2M\right)Cz\right.$$
$$\left.+2B_2p_{2,1}^2C^2z^2+\frac{\Delta p_{2,1}}{B_2}.4Cz\right)$$
$$\times\frac{\exp\left(-\frac{z^2}{2(16\tau)}\right)}{\sqrt{2\pi(16\tau)}}dz\,.$$

The integral above ($\int_{\mathbb{R}}\dots dz$) is equal to

$$\sqrt{\frac{1}{16\tau}\left(\frac{1}{16\tau}-4B_2p_{2,1}^2C^2\right)^{-1}}\exp\left(\frac{1}{2}\left(\frac{1}{16\tau}-4B_2p_{2,1}^2C^2\right)^{-1}\right.$$
$$\left.\left(\frac{16C\Delta}{\tau B_2(1+p_{2,1})}+16MC+4B_2p_{2,1}^2\left(\frac{\Delta}{B_2(1+p_{2,1})}+C\tau^2M\right)C+2MC+4\frac{\Delta p_{2,1}}{B_2}\right)^2\right)$$

From Equations (6.9), (4.18), we get

$$\log(T(\Delta))\underset{\Delta,c}{\preceq}-\frac{\Delta^2}{h}\,. \tag{6.16}$$

So there exists $\tau_4$ such that, $\tau\geq\tau_4$,

$$\log\left(\int_{\mathbb{R}}\Phi(z)\frac{e^{-\frac{z^2}{2(16\tau)}}}{\sqrt{2\pi\times16\tau}}dz\right)\underset{\Delta,c}{\preceq}-\frac{\Delta^2}{h}\,.$$

So, if $\tau\geq\sup(\tau_1,\tau_2,\tau_3,\tau_4)$, we get, using again Equation (4.30)

$$\int_{\left(\frac{\Delta}{CB_2(1+p_{2,1})}-\tau^2M\right)_+}^{+\infty}\inf\{1,T(\Delta)\epsilon(C\tau^2M+Cz,\Delta)^{-12}\times\epsilon'(C\tau^2M+Cz,\Delta)^{-4}\} \tag{6.17}$$
$$\times\frac{8\exp\left(-\frac{z^2}{2(16\tau)}\right)}{\sqrt{2\pi\times16\tau}}dz$$
$$\preceq\int_0^{z_0}\Phi(z)\frac{e^{-\frac{z^2}{2(16\tau)}}}{\sqrt{2\pi\times16\tau}}dz+\int_{z_0}^{+\infty}\frac{e^{-\frac{z^2}{2(16\tau)}}}{\sqrt{2\pi\times16\tau}}dz$$
$$\leq\int_{\mathbb{R}}\Phi(z)\frac{e^{-\frac{z^2}{2(16\tau)}}}{\sqrt{2\pi\times16\tau}}dz+\frac{e^{-\frac{z_0^2}{2(16\tau)}}}{z_0\sqrt{2\pi\times16\tau}}\times4\sqrt{\tau},$$

and

$$\log\left(\int_{-\infty}^{+\infty}\Phi(z)\frac{e^{-\frac{z^2}{2(16\tau)}}}{\sqrt{2\pi\times16\tau}}dz+\frac{e^{-\frac{z_0^2}{2(16\tau)}}}{\sqrt{2\pi}}\times4\sqrt{\tau}\right)\underset{\Delta,c}{\preceq}-\frac{\Delta^2}{h}\,. \tag{6.18}$$

In the remaining of the proof, we will suppose $\tau\geq\sup(\tau_1,\dots,\tau_4)$. Looking at Equation (6.4), we see that we can now bound all the terms on its right-hand side. We have

$$\mathbb{E}(\|\pi_{n\tau}-\overline{R}_n^{\Delta}(\pi_{(n-1)\tau})\|)\preceq T(\Delta)\,,$$

by Proposition 4.6. For $k$ in $\{1,\dots,n-1\}$, we have bounded

$$\mathbb{E}(\|\overline{R}_{n:k+1}^{\Delta}(\pi_{k\Delta})-\overline{R}_{n:k+1}^{\Delta}(\overline{R}_k^{\Delta}(\pi_{(k-1)\Delta})\|)$$

$$\leq \left(1-\frac{\epsilon(L,\Delta)^2\epsilon'(L,\Delta)^2}{2}\right)^{\left(\lfloor\frac{n-k}{2}\rfloor-3\right)_+}\times 2\mathbb{E}\left(\inf\left(1,\frac{\|\pi_{k\tau}-\overline{R}_k^{\Delta}(\pi_{(k-1)\tau})\|}{(\epsilon'_{k+2})^2\epsilon_{k+1}^4}\right)\right).$$

And the last expectation can be bounded by the sum of the following expectations :

$$\mathbb{E}\left(\inf\left(1,\frac{\|\pi_{k\tau}-\overline{R}_k^{\Delta}(\pi_{(k-1)\tau})\|^2}{T(\Delta)}\right)\right)\preceq\sqrt{T(\Delta)}\,,$$

$$\mathbb{E}\left(\inf\left(1,\frac{T(\Delta)}{(\epsilon'_{k+2})^4\epsilon_{k+1}^8}\right)\right)\preceq\exp\left(-\widehat{B}_1\frac{\Delta^2}{h}\right)\text{ for }\Delta\geq\Delta_0(\tau)\,,$$

for some constant $\widehat{B}_1$, where the bounds come from Equations (6.8), (6.13), (6.15), (6.17), (6.18) (we also use Lemma 2.6). The constant $\widehat{B}_1$ above is universal and $\Delta_0$ is continuous in $\tau$. So we get, for all $\Delta\geq\Delta_0(\tau)$, using Equation (6.16),

$$\mathbb{E}(\|\pi_{n\tau}-\pi_{n\tau}^{\Delta}\|)\preceq\exp\left(-\widehat{C}_1\frac{\Delta^2}{h}\right)\sum_{k\geq 0}\left(1-\frac{\epsilon(L,\Delta)^4\epsilon'(L,\Delta)^2}{2}\right)^{\left(\lfloor\frac{n-k}{2}\rfloor-3\right)_+},$$

(for some universal constant $\widehat{C}_1$) from which we get

$$\sup_{n\geq 0}\log\mathbb{E}(\|\pi_{n\tau}-\pi_{n\tau}^{\Delta}\|)\underset{\Delta,c}{\preceq}-\log(\epsilon(L,\Delta)\epsilon'(L,\Delta))-\frac{\Delta^2}{h}\,.$$

Looking at Equations (6.10), (6.11), we see there exists $\tau_5$, such that, for $\tau>\tau_5$,

$$\sup_{n\geq 0}\log\mathbb{E}(\|\pi_{n\tau}-\pi_{n\tau}^{\Delta}\|)\underset{\Delta,c}{\preceq}-\frac{\Delta^2}{h}\,.$$

□

6.3. **Stability of the optimal filter.**

**Theorem 6.4.** *Suppose we are under the same assumptions as in Proposition 6.3. Then there exists* $\nu_0>0$ *such that*

$$\|\pi_t-\pi'_t\|=O(t^{-\nu_0})\,,$$

*when* $t\to+\infty$.

*Proof.* We decompose, for all $n$,

$$\|\pi_{n\tau}-\pi'_{n\tau}\|\leq\|\pi_{n\tau}-\overline{R}_n^{\Delta}\dots\overline{R}_1^{\Delta}(\pi_0)\|+\|\overline{R}_n^{\Delta}\dots\overline{R}_1^{\Delta}(\pi_0)-\overline{R}_n^{\Delta}\dots\overline{R}_1^{\Delta}(\pi'_0)\|$$
$$+\|\overline{R}_n^{\Delta}\dots\overline{R}_1^{\Delta}(\pi_0)-\pi'_{n\tau}\|\,.$$

Let $\tau_\infty$, $\Delta_0$ be the parameters defined in Proposition 6.3. Recall that the operators $(R_n)_{n\geq 0}$, $(R_n^{\Delta})_{n\geq 0}$ depend on $\tau$. Suppose that $L$ is such that (as in Equation (6.1))

$$L\geq 3m_0+3C(2\tau_\infty)^2\,,\ \widetilde{\alpha}(L)\leq\frac{1}{4}\,.$$

Then, as in Equation (6.6), we have, for all $\tau\in[\tau_\infty,2\tau_\infty]$, for all $n\geq 0$,

$$\mathbb{E}(\|\overline{R}_n^{\Delta}\dots\overline{R}_1^{\Delta}(\pi_0)-\overline{R}_n^{\Delta}\dots\overline{R}_1^{\Delta}(\pi'_0)\|)$$
$$\leq\left(1-\frac{\epsilon(L,\Delta)^2\epsilon'(L,\Delta)^2}{2}\right)^{\left(\lfloor\frac{n}{2}\rfloor-3\right)_+}\times 2\mathbb{E}\left(\inf\left(1,\frac{\|\pi_0-\pi'_0\|}{(\epsilon'_2)^2\epsilon_1^4}\right)\right)$$
$$\leq 2\left(1-\frac{\epsilon(L,\Delta)^2\epsilon'(L,\Delta)^2}{2}\right)^{\left(\lfloor\frac{n}{2}\rfloor-3\right)_+}.$$

We have

$$\log(\epsilon(L,\Delta)\epsilon'(L,\Delta))\underset{\Delta,c}{\preceq}-\left(\frac{\Delta^2}{\tau B_2^2(1+p_{2,1})^2}+\frac{p_{2,1}^2\Delta^2}{B_2(1+p_{2,1})^2}+\frac{p_{2,1}\Delta^2}{(1+p_{2,1})}\right).$$

We now take a sequence $\Delta_n = \sqrt{\nu \log(n)}$, for some $\nu > 0$. There exist a constants $b_1$ and an integer $n_0$ such that, for all $\tau \in [\tau_\infty, 2\tau_\infty]$, for $n > n_0$,

$$\left(1 - \frac{(\epsilon(L,\Delta_n)\epsilon'(L,\Delta_n))^2}{2}\right)^{\left(\lfloor \frac{n}{2} \rfloor - 3\right)_+} \leq \exp\left[-\frac{1}{2}\left(\left\lfloor \frac{n}{2} \right\rfloor - 3\right)_+ \times \exp\left(-b_1 \Delta_n^2 \left(\frac{1}{\tau B_2^2(1+p_{2,1})^2} + \frac{p_{2,1}^2}{B_2(1+p_{2,1})^2} + \frac{p_{2,1}}{(1+p_{2,1})}\right)\right)\right] = \exp\left[-\frac{1}{2}\left(\left\lfloor \frac{n}{2} \right\rfloor - 3\right)_+ n^{-\nu'}\right], \tag{6.19}$$

with

$$\nu' = b_1 \nu \left(\frac{1}{\tau B_2^2(1+p_{2,1})^2} + \frac{p_{2,1}^2}{B_2(1+p_{2,1})^2} + \frac{p_{2,1}}{(1+p_{2,1})}\right).$$

By Proposition 6.3, we know there exists a constants $b_1'$ and a integer $n_0'$ such that, for all $\tau \in [\tau_\infty, 2\tau_\infty]$ and $n \geq n_0'$,

$$\sup(\mathbb{E}(\|\pi_{n\tau} - \overline{R}_n^{\Delta_n} \dots \overline{R}_1^{\Delta_n}(\pi_0)\|, \mathbb{E}(\|\pi'_{n\tau} - \overline{R}_n^{\Delta_n} \dots \overline{R}_1^{\Delta_n}(\pi'_0)\|)) \quad \leq \quad \exp\left(-b_1' \frac{\Delta_n^2}{h}\right) \leq n^{-\nu''},$$

with $\nu'' = \frac{b_1' \nu}{h}$. Let us set $\epsilon \in (0,1)$. We choose

$$\nu = \frac{(1-\epsilon)}{b_1}\left(\frac{1}{\tau B_2^2(1+p_{2,1})^2} + \frac{p_{2,1}^2}{B_2(1+p_{2,1})^2} + \frac{p_{2,1}}{(1+p_{2,1})}\right)^{-1},$$

which leads to $\nu' = 1 - \epsilon$. We set $\nu_0 = \nu''$. For any $t \geq \tau_\infty$, if we set $n = \lfloor t/\tau_\infty \rfloor$, then $t = n\tau$ with $\tau \in [\tau_\infty, 2\tau_\infty]$, and so :

$$\mathbb{E}(\|\pi_t - \pi'_t\|) \leq 2n^{-\nu''} + \exp\left(-\frac{1}{2}\left(\left\lfloor \frac{n}{2} \right\rfloor - 3\right)_+ n^{-\nu'}\right),$$

$$\mathbb{E}(\|\pi_t - \pi'_t\|) = O(t^{-\nu''}).$$

□

*Remark* 6.5. One could seek to obtain a sharper bound in the above Theorem by choosing another sequence $(\Delta_n)_{n\geq 0}$. Up to some logarithmic terms, the bound would still be a power of $t$.

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## References

[Ata98] Rami Atar. Exponential stability for nonlinear filtering of diffusion processes in a noncompact domain. *Ann. Probab.*, 26(4):1552–1574, 1998.

[AZ97] Rami Atar and Ofer Zeitouni. Exponential stability for nonlinear filtering. *Ann. Inst. H. Poincaré Probab. Statist.*, 33(6):697–725, 1997.

[BC09] Alan Bain and Dan Crisan. *Fundamentals of stochastic filtering*, volume 60 of *Stochastic Modelling and Applied Probability*. Springer, New York, 2009.

[Ben81] V. E. Beneš. Exact finite-dimensional filters for certain diffusions with nonlinear drift. *Stochastics*, 5(1-2):65–92, 1981.

[CR11] D. Crisan and B. Rozovskiĭ. Introduction. In *The Oxford handbook of nonlinear filtering*, pages 1–15. Oxford Univ. Press, Oxford, 2011.

[DG01] Pierre Del Moral and Alice Guionnet. On the stability of interacting processes with applications to filtering and genetic algorithms. *Ann. Inst. H. Poincaré Probab. Statist.*, 37(2):155–194, 2001.

[LO03] François LeGland and Nadia Oudjane. A robustification approach to stability and to uniform particle approximation of nonlinear filters: the example of pseudo-mixing signals. *Stochastic Process. Appl.*, 106(2):279–316, 2003.

[MY08] Roger Mansuy and Marc Yor. *Aspects of Brownian motion*. Universitext. Springer-Verlag, Berlin, 2008.

[Oco99] D. L. Ocone. Asymptotic stability of Beneš filters. *Stochastic Anal. Appl.*, 17(6):1053–1074, 1999.

[OP96] Daniel Ocone and Etienne Pardoux. Asymptotic stability of the optimal filter with respect to its initial condition. *SIAM J. Control Optim.*, 34(1):226–243, 1996.

[OR05] Nadia Oudjane and Sylvain Rubenthaler. Stability and uniform particle approximation of nonlinear filters in case of non ergodic signals. *Stoch. Anal. Appl.*, 23(3):421–448, 2005.

[Oud00] Nadia Oudjane. *Stabilité et approximations particulaires en filtrage non linéaire application au pistage.* PhD thesis, 2000. Thèse de doctorat Mathématiques et application Rennes 1.

[Sta05] Wilhelm Stannat. Stability of the filter equation for a time-dependent signal on $\mathbb{R}^d$. *Appl. Math. Optim.*, 52(1):39–71, 2005.

[Sta06] Wilhelm Stannat. Stability of the optimal filter via pointwise gradient estimates. In *Stochastic partial differential equations and applications—VII*, volume 245 of *Lect. Notes Pure Appl. Math.*, pages 281–293. Chapman & Hall/CRC, Boca Raton, FL, 2006.

[Sta08] Wilhelm Stannat. On the stability of Feynman-Kac propagators. In *Seminar on Stochastic Analysis, Random Fields and Applications V*, volume 59 of *Progr. Probab.*, pages 345–362. Birkhäuser, Basel, 2008.

[WR15] Inc. Wolfram Research. Mathematica, 2015.

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