Asymmetric periodic traveling wave patterns of two-dimensional Boussinesq systems

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Abstract

We consider a Boussinesq system which describes three-dimensional water waves in a fluid layer with the depth being small with respect to the wave length. We prove the existence of a large family of bifurcating bi-periodic patterns of traveling waves, which are *non-symmetric with respect to the direction of propagation*. The existence of such bifurcating asymmetric bi-periodic traveling waves is still an open problem for the Euler equation (potential flow, without surface tension).

In this study, the lattice of wave vectors is spanned by two vectors $\mathbf{k}_1$ and $\mathbf{k}_2$ of equal or different lengths and the direction of propagation $\mathbf{c}$ of the waves is close to the critical value $\mathbf{c}_0$ which is a solution of the dispersion equation. The wave pattern may be understood at leading order as the superposition of two planar waves of equal or different amplitudes, respectively, with wave vectors $\mathbf{k}_1$ and $\mathbf{k}_2$.

Our class of non-symmetric waves bifurcates from the rest state. The four components of the two basic wave vectors are constrained by the dispersion equation, forming a 3-dimensional set of free parameters. Here we are able to avoid the *small divisor problem* by restricting the study to propagation directions $\mathbf{c}$ such that $(\mathbf{k}_1 \cdot \mathbf{c})/(\mathbf{k}_2 \cdot \mathbf{c})$ is any rational number close to $(\mathbf{k}_1 \cdot \mathbf{c}_0)/(\mathbf{k}_2 \cdot \mathbf{c}_0)$. However, we need to solve a problem of weak differentiability with respect to the propagation direction for the pseudo-inverse of the linear operator. It appears that the above rationality condition influences only mildly the domain of existence of the bifurcating waves.

In the special case where the lattice is generated by wave vectors $\mathbf{k}_1$ and $\mathbf{k}_2$ of equal length, the bisecting direction is the critical propagation direction $\mathbf{c}_0$, the parameter set is two-dimensional and the rationality condition gives bifurcating asymmetric waves which propagate in a direction $\mathbf{c}$ at a small angle with the bisector of $\mathbf{k}_1$ and $\mathbf{k}_2$.

In the last section of the paper, we show examples of wave patterns for $\mathbf{k}_1$ and $\mathbf{k}_2$ of equal or different lengths, with various amplitude ratios along the two basic wave vectors and with various angles between the traveling direction $\mathbf{c}$ and the critical direction $\mathbf{c}_0$.

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1 Introduction

We consider the following Boussinesq system

$$\begin{aligned} \eta_t + \nabla\cdot\mathbf{v} + \nabla\cdot(\eta\mathbf{v}) - \frac{1}{6}\Delta\eta_t &= 0, \\ \mathbf{v}_t + \nabla\eta + \frac{1}{2}\nabla(\mathbf{v}\cdot\mathbf{v}) - \frac{1}{6}\Delta\mathbf{v}_t &= 0, \end{aligned} \tag{1}$$

proposed by Bona, Colin, Lannes [2], describing small-amplitude gravity waves of an ideal, incompressible liquid layer, with small depth relative to a characteristic wave length. Here, the horizontal coordinate $\mathbf{x}$ and time t are scaled by h_0 and $\sqrt{h_0/g}$, with g being the acceleration of gravity and h_0 being the average water depth. The elevation of waves $\eta(\mathbf{x}, t)$ and the horizontal velocity $\mathbf{v}(\mathbf{x}, t)$ at level $\sqrt{2/3}h_0$ of the depth of the undisturbed fluid, are scaled by h_0 and $\sqrt{gh_0}$ respectively. The derivation of (1) is similar to its one-dimensional version, which is given in detail in [1].

We are interested in traveling waves of constant velocity $\mathbf{c}$ which have a periodic horizontal pattern in $\mathbf{x} \in \mathbb{R}^2$. In the paper [6] we considered diamond patterns Γ spanned by wave vectors $\mathbf{k}_1$ and $\mathbf{k}_2$ having the same length and we proved the existence of bifurcating symmetric solutions, where the amplitudes ε_1 and ε_2 along the basic wave vectors are equal, propagating in the direction of the bisector of the wave vectors. We managed to apply the Lyapunov-Schmidt method to the system above, which is impossible for the full Euler equations without surface tension, due to a small divisor problem (see [7]).

In the present work we consider asymmetric waves experimentally produced by Hammack et al in [5]. *Assuming the presence of surface tension*, asymmetric waves were theoretically predicted from the full Euler equation by Craig and Nicholls in [3] (numerically sketched on page 631) using Lyapunov-Schmidt reduction, and by Groves and Haragus in [4] with the theory of spatial dynamics. As in [3] and [4] these waves may result from a choice of pattern Γ spanned by two wave vectors $\mathbf{k}_1$ and $\mathbf{k}_2$ having different lengths. They may also result from a pattern Γ spanned by two wave vectors $\mathbf{k}_1$ and $\mathbf{k}_2$ having

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the same length, but with *different amplitudes* ε_1 and ε_2 along these basic wave vectors. In absence of surface tension, the above methods cannot apply, in particular because of a small divisor problem.

In the present model, we don't need to add surface tension due to a *fundamental factorization property* of the dispersion relation of the Boussinesq system (1). We are able to find a good estimate of the inverse operator (see Lemma 8) provided that we restrict the study to *propagation directions* $\mathbf{c}$ *where the ratio* $(\mathbf{k}_1 \cdot \mathbf{c})/(\mathbf{k}_2 \cdot \mathbf{c})$ *is any rational number* r/s *close to the ratio* $(\mathbf{k}_1 \cdot \mathbf{c}_0)/(\mathbf{k}_2 \cdot \mathbf{c}_0)$, where $\mathbf{c}_0$ is the propagation velocity given by the dispersion relation $\Delta(\mathbf{k}_j, \mathbf{c}_0) = 0$. This allows us to avoid the small divisor problem and use an adapted Lyapunov-Schmidt type method, despite of the lack of regularity with respect to the angle parameter (between $\mathbf{c}$ and $\mathbf{c}_0$) in the pseudo-inverse of the linearized operator. *This rationality condition influences mildly the domain of existence* of the bifurcating waves in allowing an existence domain of the order $(\ln s)^{-1}$. Our main result is Theorem 11, which can be roughly summed up as follows:

Theorem 1. *Choose basic wave vectors* $(\mathbf{k}_1, \mathbf{k}_2)$ *in the form of (7) which satisfy the non-degeneracy condition (40), such that the dispersion relation* $\Delta(\mathbf{k}, \mathbf{c}_0) = 0$ *defined in (18) with* $\mathbf{c}_0 = c_0(1, 0)$ *has* $\pm\mathbf{k}_j$, $j = 1, 2$ *as the only solutions* $\mathbf{k} = \pm\mathbf{k}_j$, $j = 1, 2$, *in* Γ *(i.e., we have now only 3 free parameters). Then choose the bifurcation parameter* $\mathbf{c}$ *such that the ratio*

$$\frac{\mathbf{k}_1 \cdot \mathbf{c}}{\mathbf{k}_2 \cdot \mathbf{c}} = \frac{r}{s} \in \mathbb{Q}^+ \tag{2}$$

is close enough to $\frac{\mathbf{k}_1 \cdot \mathbf{c}_0}{\mathbf{k}_2 \cdot \mathbf{c}_0}$. *Fix* $\sigma \in \mathbb{N}$ *large enough and assume* $1 \leq s \leq \sigma$. *Then, there is a family of bifurcating bi-periodic traveling waves,* $U = (\eta, \mathbf{v})$ *which are solutions of (1), are in general non-symmetric with respect to the propagation direction* $\mathbf{c}$, *and are of the form*

$$U = \sum_{1 \leq j+l+m+q \leq n} A^j \overline{A}^l B^m \overline{B}^q U_{jlmq} + o((|A| + |B|)^n)$$

with

$$A = \varepsilon_1 e^{i\mathbf{k}_1 \cdot \mathbf{y}}, \quad B = \varepsilon_2 e^{i\mathbf{k}_2 \cdot \mathbf{y}}.$$

The bifurcation parameter $\mathbf{c} = \frac{c_0}{1+\mu}(1, w)$ *is linked with the amplitudes* ε_1 *and* ε_2 *by*

$$\mu = \alpha_1 \varepsilon_1^2 + \alpha_2 \varepsilon_2^2 + O(\varepsilon_1^2 + \varepsilon_2^2)^2,$$
$$w = \beta_1 \varepsilon_1^2 + \beta_2 \varepsilon_2^2 + O(\varepsilon_1^2 + \varepsilon_2^2)^2.$$

The "rational" restriction (2) implies a "rational type of" restriction on amplitudes $(\varepsilon_1, \varepsilon_2)$ *which are uniformly bounded by* $O\{(|\mu|/\ln \sigma)^{1/2}\}$ *with* $|\mu| << (\ln \sigma)^{-1}$.

Remark 2. *In the phases of* A *and* B, $\mathbf{y}$ *corresponds to an arbitrary horizontal shift for the solution.*

Remark 3. *The "rational" restriction (2) concerns only w (not μ).*
Remark 4. *The U_{jlmq} are bi-periodic functions of $\mathbf{x} - \mathbf{c}t$. For $j + l + m + q$ less than or equal to 2, the functions U_{jlmq} and the coefficients α_i, β_i, $i = 1, 2$ are explicitly given in the Appendix.*

In the case when the waves propagate in the critical direction $\mathbf{c}_0$ the rationality restriction only concerns the ratio $\frac{\mathbf{k}_1 \cdot \mathbf{c}_0}{\mathbf{k}_2 \cdot \mathbf{c}_0}$. The result also applies when the lattice is built with wave vectors $\mathbf{k}_1$ and $\mathbf{k}_2$ of equal length, with the bisector direction as the critical propagation direction $\mathbf{c}_0$. In such a case, the free parameter set is two-dimensional and the rationality condition gives bifurcating asymmetric waves which propagate in a direction making a small angle with the bisector of $\mathbf{k}_1$ and $\mathbf{k}_2$. The factorization property of the dispersion relation mentioned above is *specific to the Boussinesq system (1)*, while the corresponding problem for the free surface of a potential flow in absence of surface tension (Euler equations) is still open.

We show in section 5 several patterns of traveling asymmetric waves computed with the *explicit expression of the free surface* elevation for the terms of order 1 and 2 in amplitudes $(\varepsilon_1, \varepsilon_2)$.

2 Formulation of the problem

We are looking for solutions of System (1) of the form of 2-dimensional traveling waves, i.e., η and $\mathbf{v}$ are functions of $\widetilde{\mathbf{x}} = \mathbf{x} - \mathbf{c}t$, where $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$, and $\mathbf{c}$ is the velocity of the traveling wave which plays the role of a two-dimensional bifurcation parameter. For these solutions, system (1) reads

$$\begin{aligned} \nabla \cdot (\mathbf{v} + \eta \mathbf{v}) - \mathbf{c} \cdot \nabla(\eta - \frac{1}{6}\Delta\eta) &= 0, \\ \nabla(\eta + \frac{1}{2}(\mathbf{v} \cdot \mathbf{v})) - \mathbf{c} \cdot \nabla(\mathbf{v} - \frac{1}{6}\Delta\mathbf{v}) &= \mathbf{0}, \end{aligned} \tag{3}$$

where we assume the flow is potential, i.e.,

$$\operatorname{curl}(\mathbf{v}) = 0, \tag{4}$$

which is shown to be consistent with Euler equations in [6]. We consider the *periodic solutions* with Fourier expansions of the form (for simplicity of notation, $\mathbf{x}$ is used for $\widetilde{\mathbf{x}}$)

$$\begin{aligned} \eta(\mathbf{x}) &= \sum_{\mathbf{k} \in \Gamma} \eta_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}, \\ \mathbf{v}(\mathbf{x}) &= \sum_{\mathbf{k} \in \Gamma} \mathbf{v}_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}, \end{aligned} \tag{5}$$

where Γ is a lattice in the plane defined by two non-colinear vectors $\mathbf{k}_1$ and $\mathbf{k}_2$. This means that $\mathbf{k} \in \mathbf{\Gamma}$, where

$$\mathbf{k} = (k_1, k_2) = n_1\mathbf{k}_1 + n_2\mathbf{k}_2, \quad n_1, n_2 \in \mathbb{Z}. \tag{6}$$

Because of (4) we have

$$\mathbf{v}_\mathbf{k} \times \mathbf{k} = \mathbf{0}.$$

For simplicity, we require $\mathbf{v}_\mathbf{0} = \mathbf{0}$ and $\eta_\mathbf{0} = 0$, so the averages of the elevation η and of the horizontal velocity are set to be zero. One might treat the nonzero case as in the case of the symmetric doubly periodic wave pattern (c.f. [6]). This would introduce 3 additional parameters which do not change the results qualitatively.

Let us define the basis $\{\mathbf{k}_1, \mathbf{k}_2\}$ of the lattice Γ by

$$\mathbf{k}_1 = l_1(1, \tau_1), \quad \mathbf{k}_2 = l_2(1, -\tau_2), \quad l_j, \tau_j > 0, \quad j = 1, 2 \tag{7}$$

where $\tau_j = \tan\theta_j$. We then have for $\mathbf{k} = (k_1, k_2) = n_1\mathbf{k}_1 + n_2\mathbf{k}_2$

$$k_1 = n_1 l_1 + n_2 l_2, \quad k_2 = n_1\tau_1 l_1 - n_2\tau_2 l_2. \tag{8}$$

The lattice Γ *forms a diamond pattern if* $\mathbf{k}_1$ and $\mathbf{k}_2$ *are symmetric with respect to the* x_1-axis, making an angle $\pm\theta$ with this axis. In such a case,

$$l_1 = l_2 \stackrel{def}{=} l, \quad \tau_1 = \tau_2 \stackrel{def}{=} \tau, \quad \theta_1 = \theta_2 \stackrel{def}{=} \theta.$$

Now we define the Sobolev space of bi-periodic functions which are square integrable with their p first derivatives over a period parallelogram:

$$H^p_{\natural\natural} \stackrel{def}{=} \left\{ u = \sum_{\mathbf{k}\in\Gamma} u_k e^{i\mathbf{k}\cdot\mathbf{x}} \in H^p\{\mathbb{R}^2/\Gamma'\} \right\},$$

where Γ' is the lattice of periods dual of Γ defined by

$$\Gamma' = \left\{ n_1\lambda_1 + n_2\lambda_2 \in \mathbb{R}^2;\ \lambda_j \cdot \mathbf{k}_n = 2\pi\delta_{jn},\ j, n \in \{1, 2\}, (n_1, n_2) \in \mathbb{Z}^2 \right\}. \tag{9}$$

We equip $H^p_{\natural\natural}$ with the classical Hermitian product $\langle \cdot, \cdot \rangle_{H^p}$. Note that any $u \in H^p_{\natural\natural}$ is invariant under the shift

$$\sigma : \mathbf{x} \mapsto \mathbf{x} + \lambda_j.$$

We notice that l_j *has to be chosen small enough* for the consistency of the Boussinesq model, in which the horizontal wave lengths $|\lambda_j|$ should be large with respect to 1 (which is the depth of the fluid layer at rest). Moreover, in the final assumptions we also assume that the parallelogram built with the vectors λ_1 and λ_2 is *not too flat* (see conditions on τ_j and l_j in Definition 7) . The basic function space in our study is

$$G_p \stackrel{def}{=} \{U = (\eta, \mathbf{v}) \in H^p_{\natural\natural}\}^3 \cap \{\mathrm{curl}(\mathbf{v}) = \mathbf{0}\} \cap \{\eta_{\mathbf{0}} = 0,\ \mathbf{v}_{\mathbf{0}} = \mathbf{0}\},$$

and System (3) can be reformulated in the form

$$\mathcal{L}_{\mathbf{c}} U + \mathcal{G}\mathcal{N}(U, U) = \mathbf{0}, \tag{10}$$

where

$$\mathcal{L}_{\mathbf{c}} U = \begin{pmatrix} \nabla \cdot \mathbf{v} - \mathbf{c} \cdot \nabla(\eta - \frac{1}{6}\Delta\eta) \\ \nabla\eta - \mathbf{c} \cdot \nabla(\mathbf{v} - \frac{1}{6}\Delta\mathbf{v}) \end{pmatrix}, \tag{11}$$

$$\mathcal{N}(U, U) = (\frac{1}{2}(\mathbf{v} \cdot \mathbf{v}), \eta\mathbf{v}), \quad \mathcal{G}(g, \mathbf{f}) = (\nabla \cdot \mathbf{f}, \nabla g).$$

It is clear that the linear maps

$$\mathcal{L}_{\mathbf{c}} : G_p \to G_{p-3},\ p \geq 3; \quad \mathcal{G} : G_p \to G_{p-1},\ p \geq 1$$

are bounded and the quadratic map

$$\mathcal{N} : G_p \to G_p, \quad p \geq 2$$

is bounded ($p \geq 2$ is necessary for having the product of two functions of $H^p_{\natural\natural}$ in $H^p_{\natural\natural}$). Moreover we have, for any U_1 and $U_2 \in G_p$,

$$\begin{aligned} \langle \mathcal{L}_{\mathbf{c}} U_1, U_2 \rangle_{H^0} &= -\langle U_1, \mathcal{L}_{\mathbf{c}} U_2 \rangle_{H^0}, \quad p \geq 3 \\ \langle \mathcal{G} U_1, U_2 \rangle_{H^0} &= -\langle U_1, \mathcal{G} U_2 \rangle_{H^0}, \quad p \geq 1, \end{aligned} \tag{12}$$

after integration by parts.

System (10) possesses important symmetries. We define their representations by the following bounded linear operators $\mathcal{T}_{\mathbf{v}}$ and $\mathcal{S}_0$:

$$(\mathcal{T}_{\mathbf{y}} U)(\mathbf{x}) = U(\mathbf{x} + \mathbf{y}), \quad (\mathcal{S}_0 U)(\mathbf{x}) = (\eta(-\mathbf{x}), \mathbf{v}(-\mathbf{x})).$$

It is clear that the following commutation properties hold

$$\begin{aligned} &\mathcal{T}_{\mathbf{y}}\mathcal{L}_{\mathbf{c}} = \mathcal{L}_{\mathbf{c}}\mathcal{T}_{\mathbf{y}}, \ \ \mathcal{T}_{\mathbf{y}}\mathcal{N}(U, U) = \mathcal{N}(\mathcal{T}_{\mathbf{y}}U, \mathcal{T}_{\mathbf{y}}U),\ \mathcal{T}_{\mathbf{y}}\mathcal{G} = \mathcal{G}\mathcal{T}_{\mathbf{y}}, \\ &\mathcal{S}_0\mathcal{L}_{\mathbf{c}} = -\mathcal{L}_{\mathbf{c}}\mathcal{S}_0, \ \ \mathcal{S}_0\mathcal{N}(U, U) = \mathcal{N}(\mathcal{S}_0 U, \mathcal{S}_0 U),\ \mathcal{S}_0\mathcal{G} = -\mathcal{G}\mathcal{S}_0. \end{aligned} \tag{13}$$

The first set of properties results from the invariance of the original system under the translations of the plane, while the second set comes from the reversibility of the original system.

If *the lattice* Γ *has a diamond structure*, we have an additional symmetry. Define S_1 by

$$(\mathcal{S}_1 U)(\mathbf{x}) = (\eta(\hat{\mathbf{x}}), \hat{\mathbf{v}}(\hat{\mathbf{x}})),$$

where $\hat{\mathbf{x}} = (x_1, -x_2)$ is the symmetric vector of $\mathbf{x}$ with respect to the x_1-axis. It is clear that in the case when the velocity $\mathbf{c}$ of the wave is colinear to the x_1-axis, we have the following additional commutation properties

$$\mathcal{S}_1\mathcal{L}_{\mathbf{c}} = \mathcal{L}_{\mathbf{c}}\mathcal{S}_1, \quad \mathcal{S}_1\mathcal{N}(U,U) = \mathcal{N}(\mathcal{S}_1U, \mathcal{S}_1U), \;\; \mathcal{S}_1\mathcal{G} = \mathcal{G}\mathcal{S}_1. \tag{14}$$

3 Study of the linearized operator

3.1 Inversion of the linear operator

To use the Lyapunov-Schmidt method, it is fundamental to study the linear system

$$\mathcal{L}_{\mathbf{c}}U = P, \tag{15}$$

where $P = (q, \mathbf{p}) \in G_l$ ($l \geq 0$) is given and we are looking for $U = (\eta, \mathbf{v}) \in G_l$. For the periodic vector function $\mathbf{p}$ and the periodic scalar function q with Fourier series

$$\begin{aligned} \mathbf{p}(\mathbf{x}, t) &= \sum_{\mathbf{k}\in\Gamma} \mathbf{p}_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}, \;\; \mathbf{p}_{\mathbf{0}} = \mathbf{0}, \;\; \mathbf{p}_{\mathbf{k}} \times \mathbf{k} = \mathbf{0}, \\ q(\mathbf{x}, t) &= \sum_{\mathbf{k}\in\Gamma} q_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}, \quad q_{\mathbf{0}} = 0, \end{aligned} \tag{16}$$

System (15) leads to

$$\begin{aligned} -(1 + \frac{1}{6}|\mathbf{k}|^2)(\mathbf{c}\cdot\mathbf{k})\eta_{\mathbf{k}} + \mathbf{k}\cdot\mathbf{v}_{\mathbf{k}} &= -\, iq_{\mathbf{k}}, \\ \mathbf{k}\eta_{\mathbf{k}} - (1 + \frac{1}{6}|\mathbf{k}|^2)(\mathbf{c}\cdot\mathbf{k})\mathbf{v}_{\mathbf{k}} &= -\, i\mathbf{p}_{\mathbf{k}}, \end{aligned} \tag{17}$$

where $\mathbf{k} \in \Gamma$. Define

$$\Delta(\mathbf{k}, \mathbf{c}) = (1 + \frac{1}{6}|\mathbf{k}|^2)^2(\mathbf{c}\cdot\mathbf{k})^2 - |\mathbf{k}|^2. \tag{18}$$

The linearized operator $\mathcal{L}_{\mathbf{c}}$ has a nontrivial kernel in G_l if there exists a pair $(\mathbf{k}_0, \mathbf{c}_0)$ satisfying

$$\Delta(\mathbf{k}_0, \mathbf{c}_0) = 0 \text{ and } \mathbf{k}_0 \neq 0. \tag{19}$$

The solution of (17) can be written as follows.

- When $\Delta(\mathbf{k}, \mathbf{c}) \neq 0$, the solution reads

$$\begin{aligned} \eta_{\mathbf{k}} =& i\,\frac{(1 + \frac{1}{6}|\mathbf{k}|^2)(\mathbf{c}\cdot\mathbf{k})q_{\mathbf{k}} + \mathbf{k}\cdot\mathbf{p}_{\mathbf{k}}}{\Delta(\mathbf{k}, \mathbf{c})}, \\ \mathbf{v}_{\mathbf{k}} =& i\,\frac{(1 + \frac{1}{6}|\mathbf{k}|^2)(\mathbf{c}\cdot\mathbf{k})\mathbf{p}_{\mathbf{k}} + q_{\mathbf{k}}\mathbf{k}}{\Delta(\mathbf{k}, \mathbf{c})}, \end{aligned} \tag{20}$$

where we notice that

$$\mathrm{curl}(\mathbf{v}_k e^{i\mathbf{k}\cdot\mathbf{x}}) = 0.$$

- When $\mathbf{k} = 0$,then $\mathbf{v_0} = \eta_\mathbf{0} = 0$.
- When $\Delta(\mathbf{k},\mathbf{c}) = 0$, $\mathbf{k} \neq 0$, and if $(\mathbf{p_k}, q_\mathbf{k})$ satisfies the compatibility condition

$$\mathrm{sgn}(\mathbf{k}\cdot\mathbf{c})\mathbf{k}\cdot\mathbf{p_k} + |\mathbf{k}|q_\mathbf{k} = 0, \tag{21}$$

the solution reads

$$\begin{aligned} \eta_\mathbf{k} &= i\,\mathrm{sgn}(\mathbf{k}\cdot\mathbf{c})\frac{q_\mathbf{k}}{|\mathbf{k}|} + |\mathbf{k}|\beta, \\ \mathbf{v_k} &= \mathrm{sgn}(\mathbf{k}\cdot\mathbf{c})\mathbf{k}\beta, \end{aligned} \tag{22}$$

where β is an arbitrary constant in $\mathbb{C}$.

3.2 Kernel of $\mathcal{L}_{\mathbf{c}_0}$

To obtain bifurcating solutions we need to have a nontrivial kernel for the operator $\mathcal{L}_\mathbf{c}$ for some critical values of the parameters. Hence we need to study the set

$$\{\mathbf{k}\in\Gamma; \Delta(\mathbf{k},\mathbf{c}) = 0\}$$

for a given velocity $\mathbf{c}$. Without loss of generality, we can assume that $\mathbf{c} = \mathbf{c}_0 = c_0(1,0)$ and the basic wave vectors $\mathbf{k}_1$ and $\mathbf{k}_2$ are solutions of

$$\Delta(\mathbf{k}_j, \mathbf{c}_0) = 0, \quad j = 1,2. \tag{23}$$

This means that

$$c_0^2 = \frac{1+\tau_j^2}{\{1+\frac{l_j^2}{6}(1+\tau_j^2)\}^2}, \quad j=1,2, \tag{24}$$

i.e.,

$$\frac{1}{c_0^2} = \left(\cos\theta_1 + \frac{l_1^2}{6\cos\theta_1}\right)^2 = \left(\cos\theta_2 + \frac{l_2^2}{6\cos\theta_2}\right)^2, \quad 0<\theta_j<\pi/2, \tag{25}$$

which leads to the relationship (automatically satisfied when we choose a diamond lattice Γ)

$$6(\cos\theta_1 - \cos\theta_2) = \frac{l_2^2}{\cos\theta_2} - \frac{l_1^2}{\cos\theta_1}. \tag{26}$$

Therefore, for fixed angles θ_1, θ_2, the *point* (l_1, l_2) *(close to 0) needs to belong to a hyperbola in the plane.* The critical set in the 4-dimensional space $(\tau_1, \tau_2, l_1, l_2)$ is a 3-dimensional hypersurface restricted to the quadrant $\tau_1, \tau_2, l_1, l_2 > 0$. When Γ is a diamond lattice, we only have two parameters (τ, l) for the critical set.

Replacing $\mathbf{k}$ by $n_1\mathbf{k}_1 + n_2\mathbf{k}_2$ in the equation $\Delta(\mathbf{k}, \mathbf{c}_0) = 0$, we obtain

$$\left(1 + \frac{1}{6}|n_1\mathbf{k}_1 + n_2\mathbf{k}_2|^2\right) |\mathbf{c}_0 \cdot (n_1\mathbf{k}_1 + n_2\mathbf{k}_2)| = |n_1\mathbf{k}_1 + n_2\mathbf{k}_2|, \tag{27}$$

or, more explicitly,

$$\begin{aligned} 0 = &\left(1 + \frac{1}{6}\{(n_1 l_1 + n_2 l_2)^2 + (n_1\tau_1 l_1 - n_2\tau_2 l_2)^2\}\right)^2 c_0^2 (n_1 l_1 + n_2 l_2)^2 + \\ &- \{(n_1 l_1 + n_2 l_2)^2 + (n_1\tau_1 l_1 - n_2\tau_2 l_2)^2\}. \end{aligned} \tag{28}$$

We already know that

$$(n_1, n_2) = (\pm 1, 0), (0, \pm 1),$$

are solutions of (27). Next we want to determine the number of solutions (n_1, n_2) of (27).

When the equalities (24) hold, the critical set in the 4-dimensional space of parameters $(\tau_1, \tau_2, l_1, l_2)$ is a 3-dimensional hypersurface. When c_0 is considered as a function of τ_1 and l_1, then for a fixed pair (n_1, n_2), the equation (28) represents a 2-dimensional submanifold: express for instance (τ_1, τ_2) as a function of (l_1, l_2). The set of relations (28) is countable for all $(n_1, n_2) \in \mathbb{Z}^2$. This yields a countable set of 2-dimensional submanifolds of the 3-dimensional critical hypersurface. Therefore, there is a full measure set of choice of parameters $(\tau_1, \tau_2, l_1, l_2)$ on the 3-dimensional hypersurface such that none of relations (28) is satisfied, except for $(n_1, n_2) = (\pm 1, 0)$ and $(n_1, n_2) = (0, \pm 1)$. Hence, a general choice of parameters provides no solution of (27) except for $\pm\mathbf{k}_1$ and $\pm\mathbf{k}_2$. The consequence is that *the dimension of* $\ker \mathcal{L}_{\mathbf{c}_0}$ *is 4, in general.*
Remark 5. *In* case of resonance*, which means that the dispersion equation* $\Delta(\mathbf{k}, \mathbf{c}_0) = 0$ *has more than the 4 solutions* $\pm\mathbf{k}_1$ *and* $\pm\mathbf{k}_2$*, the kernel of* $\mathcal{L}_{\mathbf{c}_0}$ *is finite dimensional as we shall see in the next two subsections. Hence for the Boussinesq system (1), there is no possibility to have a "complete resonance" (i.e. with an infinite - dimensional kernel) as it might occur in the corresponding problem governed by the Euler equations (see [7]). In the present paper we do not consider resonant situations.*

3.3 Inverse of $\mathcal{L}_{\mathbf{c}_0}$ *when* l_1/l_2 *is rational*

Let us assume that the scalars l_1 and l_2 are such that

$$\frac{l_1}{l_2} = \frac{r_0}{s_0} \in \mathbb{Q}^+, \tag{29}$$

where r_0 and $s_0 \in \mathbb{N}$ are relatively prime. When $\mathbf{c}_0 \cdot \mathbf{k} \neq 0$, this assumption gives a lower bound for $\mathbf{c}_0 \cdot \mathbf{k}$. Indeed, we have

$$\mathbf{c}_0 \cdot \mathbf{k} = c_0(n_1 l_1 + n_2 l_2) = \frac{c_0 l_2}{s_0}(n_1 r_0 + n_2 s_0),$$

i.e.,

$$|\mathbf{c}_0 \cdot \mathbf{k}| \geq \frac{c_0 l_2}{s_0}$$

for any $(n_1, n_2) \neq 0$ in $\mathbb{Z}^2$ with $\mathbf{c}_0 \cdot \mathbf{k} \neq 0$. It is then clear that, for $|\mathbf{k}| > K$ where

$$K = \frac{9 s_0}{c_0 l_2},$$

and for any $(n_1, n_2) \neq 0$ in $\mathbb{Z}^2$ (even when $\mathbf{c}_0 \cdot \mathbf{k} = 0$),

$$\left|(1 + \frac{1}{6}|\mathbf{k}|^2)|\mathbf{c}_0 \cdot \mathbf{k}| - |\mathbf{k}|\right| > \frac{1}{2}|\mathbf{k}|, \tag{30}$$

which provides a lower bound for $|\Delta(\mathbf{k}, \mathbf{c}_0)|$. Notice that when Γ is a diamond lattice, we have $l_1 = l_2 = l$ and $s_0 = 1$.

Let us now remark that

$$|\mathbf{k}|^2 = (n_1 l_1 + n_2 l_2)^2 + (n_1 \tau_1 l_1 - n_2 \tau_2 l_2)^2$$

is a positive definite quadratic form of (n_1, n_2), hence the following inequality ($d_1 > 0$)

$$d_1^2(n_1^2 + n_2^2) \leq |\mathbf{k}|^2 \leq d_0^2(n_1^2 + n_2^2) \tag{31}$$

holds, where

$$\begin{aligned} d_1^2 =& \frac{1}{2}\left((1+\tau_1^2)l_1^2 + (1+\tau_2^2)l_2^2\right) - \frac{1}{2}\sqrt{\Delta}, \\ \Delta =& \left((1+\tau_1^2)l_1^2 + (1+\tau_2^2)l_2^2\right)^2 - 4 l_1^2 l_2^2 (\tau_1 + \tau_2)^2. \end{aligned} \tag{32}$$

For $a > 0$, we have $a - \sqrt{a^2 - b^2} > b^2/2a$, hence

$$d_1 > \frac{l_1 l_2 (\tau_1 + \tau_2)}{((1+\tau_1^2)l_1^2 + (1+\tau_2^2)l_2^2)^{1/2}} = \frac{|\mathbf{k}_1 \times \mathbf{k}_2|}{(\mathbf{k}_1^2 + \mathbf{k}_2^2)^{1/2}} \tag{33}$$

holds. This shows that for $\mathbf{k} \in \Gamma$, the condition $|\mathbf{k}| \leq K$ leads to the condition

$$(n_1^2 + n_2^2)^{1/2} \leq \frac{K}{d_1},$$

where d_1 satisfies (33), which means that there is only a finite number of "bad" (n_1, n_2). Hence, in general, the parameters (l_j, τ_j) are not among the finite number of "bad" (resonant) curves defined by conditions (28) and (29) on the 3-dimensional manifold given by (24). We are now able to prove the following

Lemma 6. *Let* $\mathbf{c} = c_0(1,0)$, $\frac{l_1}{l_2} = \frac{r_0}{s_0} \in \mathbb{Q}^+$, $(c_0, l_1, l_2, \tau_1, \tau_2)$ *satisfy*

$$c_0^2 = \frac{1+\tau_1^2}{\{1+\frac{l_1^2}{6}(1+\tau_1^2)\}^2} = \frac{1+\tau_2^2}{\{1+\frac{l_2^2}{6}(1+\tau_2^2)\}^2},$$

such that $\pm\mathbf{k}_j, j = 1,2$ *are the only solutions of the dispersion relation* $\Delta(\mathbf{k}, \mathbf{c}_0) = 0$ *with* $\mathbf{k} \in \Gamma$. *Then, for any given*

$$P = (q, \mathbf{p}) \in G_p, \quad p \geq 0,$$

satisfying the compatibility conditions

$$\langle P, \xi_{\pm\mathbf{k}_j}\rangle_{H^0} = 0, \ j = 1,2, \tag{34}$$

the general solution $U = (\eta, \mathbf{v}) \in G_{p+1}$ *of the system*

$$\mathcal{L}_{\mathbf{c}_0} U = P,$$

is given by

$$U = \widetilde{\mathcal{L}}_{\mathbf{c}_0}^{-1} P + A\xi_{\mathbf{k}_1} + \overline{A}\xi_{-\mathbf{k}_1} + B\xi_{\mathbf{k}_2} + \overline{B}\xi_{-\mathbf{k}_2}, \tag{35}$$

where

$$\xi_{\pm\mathbf{k}_j} = (\sqrt{1+\tau_j^2}, 1, (-1)^{j+1}\tau_j)e^{\pm i\mathbf{k}_j\cdot\mathbf{x}}, \tag{36}$$

A, B *are complex numbers, and* $\widetilde{\mathcal{L}}_{\mathbf{c}_0}^{-1}$ *is the bounded linear operator:* $G_p \to G_{p+1} \cap \{\ker \mathcal{L}_{\mathbf{c}_0}\}_{H^0}^{\perp}$ *for* $p \geq 0$. *In addition there is a positive* ρ *such that*

$$||\widetilde{\mathcal{L}}_{\mathbf{c}_0}^{-1}\mathcal{G}||_{\mathcal{L}(G_p)} \leq \rho. \tag{37}$$

Proof: Assume that (c_0, l_j, τ_j), $j = 1,2$, are such that $\pm\mathbf{k}_j$, $j = 1,2$, are the only nontrivial solutions in Γ of (27) (this is the general case) and let us define the eigenvectors $\xi_{\pm\mathbf{k}_j}$ of $\mathcal{L}_{\mathbf{c}_0}$ by (36), where $\mathbf{c}_0 = (c_0, 0)$. Then we observe that with the Hermitian scalar product in $\{H^0_{\natural\natural}\}^3$ the compatibility condition (21) is equivalent to (34). Moreover, using the symmetries, we have

$$\mathcal{T}_{\mathbf{y}}\xi_{\pm\mathbf{k}_j} = \xi_{\pm\mathbf{k}_j}e^{\pm i\mathbf{k}_j\cdot\mathbf{y}}, \ \mathcal{S}_0\xi_{\pm\mathbf{k}_j} = \overline{\xi}_{\pm\mathbf{k}_j} = \xi_{\mp\mathbf{k}_j}. \tag{38}$$

In the case *when the lattice* Γ *has a diamond structure*, we have in addition the following symmetry property

$$\mathcal{S}_1\xi_{\pm\mathbf{k}_1} = \xi_{\pm\mathbf{k}_2}. \tag{39}$$

The above calculations (the proof of the estimates is made in [6]) show that we are able to define an operator $\widetilde{\mathcal{L}}_{\mathbf{c}_0}^{-1}$, which is the pseudo-inverse of $\mathcal{L}_{\mathbf{c}_0}$, mapping G_p into G_{p+1} for any $p \geq 0$, provided the compatibility condition (21) is satisfied. The componentwise definition of

$$U = \widetilde{\mathcal{L}}_{\mathbf{c}_0}^{-1} P$$

reads

$$\{\widetilde{\mathcal{L}}_{\mathbf{c}_0}^{-1} P\}_{\mathbf{k}} = U_{\mathbf{k}} = (\eta_{\mathbf{k}}, \mathbf{v}_{\mathbf{k}}),$$

where

- $\{\widetilde{\mathcal{L}}_{\mathbf{c}_0}^{-1} P\}_{\mathbf{k}} = (\eta_{\mathbf{k}}, \mathbf{v}_{\mathbf{k}})$ is given by (20) for $\Delta(\mathbf{k}, \mathbf{c}_0) \neq 0$, i.e. for $\mathbf{k} \neq \pm\mathbf{k}_1, \pm\mathbf{k}_2$, and $\mathbf{0}$,
- $\{\widetilde{\mathcal{L}}_{\mathbf{c}_0}^{-1} P\}_{\mathbf{0}} = 0$, for $\mathbf{k} = (0, 0)$,
- for $\mathbf{k} = \pm\mathbf{k}_j$ we set (see (22))

$$\{\widetilde{\mathcal{L}}_{\mathbf{c}_0}^{-1} P\}_{\pm\mathbf{k}_j} = \Big(\pm \frac{i}{2} \frac{q_{\pm\mathbf{k}_j}}{|\mathbf{k}_j|}, -\frac{i}{2} \frac{\pm\mathbf{k}_j q_{\pm\mathbf{k}_j}}{|\mathbf{k}_j|^2} \Big),$$

  so that $\widetilde{\mathcal{L}}_{\mathbf{c}_0}^{-1} P$ is orthogonal, in $\{H^0_{\natural\natural}\}^3$, to the four-dimensional space $E = \operatorname{span}\{\xi_{\pm\mathbf{k}_j}; j = 1, 2\}$, i.e.

$$\langle \widetilde{\mathcal{L}}_{\mathbf{c}_0}^{-1} P, \xi_{\pm\mathbf{k}_j} \rangle_{H^0} = 0, \quad j = 1, 2.$$

Notice that *our pseudo-inverse operator* $\widetilde{\mathcal{L}}_{\mathbf{c}_0}^{-1}$ *is defined even if* $P = (q, \mathbf{p})$ *does not satisfy the compatibility condition* (21).

3.4 Inverse of the perturbed operator $\mathcal{L}_{\mathbf{c}_0} + w\mathcal{L}^{(1)}$

In what follows we need to consider the perturbed operator $\mathcal{L}_{c_0(1,w)} = \mathcal{L}_{\mathbf{c}_0} + w\mathcal{L}^{(1)}$ for w close to 0, where

$$\mathcal{L}^{(1)} U = -c_0 \frac{\partial}{\partial x_2} (I - \frac{1}{6} \Delta) U.$$

Taking $w \neq 0$ (which plays the role of an angular bifurcation parameter) means that we intend to find traveling waves moving not exactly in the direction of the x_1-axis. We shall see that this is linked with the ratio of amplitudes ε_1 and ε_2 of the waves along the basic wave vectors $\mathbf{k}_1$ and $\mathbf{k}_2$. The perturbation $w\mathcal{L}^{(1)}$ appears to be singular as it leads to a *small divisor problem* when we invert $\mathcal{L}_{c_0(1,w)}$, contrary to the inversion of $\mathcal{L}_{\mathbf{c}_0}$ with the assumption (29). Indeed, the $\Delta(\mathbf{k}, \mathbf{c})$ in the denominators of (20) may become very small for large $|\mathbf{k}|$. In what follows, we control the smallness of $\Delta(\mathbf{k}, \mathbf{c})$ in assuming again a rationality condition. Let us first define a non-flatness condition of the parallelograms generated by the vectors $\mathbf{k}_1$ and $\mathbf{k}_2$.

Definition 7. *We say that* $(\mathbf{k}_1, \mathbf{k}_2)$ *satisfies the* δ*-non-flatness condition if for a fixed* $\delta \in (0, 1)$,

$$\begin{aligned} &\delta < \tau_j < \delta^{-1}, \quad j = 1, 2 \\ &\delta < \frac{l_1}{l_2} < \delta^{-1}, \ l_2 < \delta. \end{aligned} \tag{40}$$

This condition also insures that the parallelograms of the dual lattice Γ' built with the vectors λ_1 and λ_2 are not flat and their size is large with respect to 1 (which is the scale of the depth of the fluid layer).

We now show the following

Lemma 8. *Let* $\mathbf{c} = c_0(1, w)$, $\delta \in (0, 1)$, *and choose basic wave vectors* $(\mathbf{k}_1, \mathbf{k}_2)$ *which satisfy the* δ*-non-flatness condition, such that the dispersion relation* $\Delta(\mathbf{k}, \mathbf{c}_0) = 0$ *has the only solutions* $\mathbf{k} = \pm\mathbf{k}_j$, $j = 1, 2$, *in* Γ. *Then choose* $|w| \leq \frac{\delta}{5}$ *and the ratio*

$$\frac{\mathbf{k}_1 \cdot \mathbf{c}}{\mathbf{k}_2 \cdot \mathbf{c}} = \frac{r}{s} \in \mathbb{Q}^+, \tag{41}$$

with $r, s \in \mathbb{N}$ *being relatively prime. Then, except for* τ_2 *in a small neighborhood of a finite set* $\tau_2(\tau_1, l_1, l_2)$ *of cardinality at most* $O(\ln s)$, *the linear operator* $\mathcal{L}_{\mathbf{c}}$ *has a bounded inverse in the orthogonal complement of* $\ker \mathcal{L}_{\mathbf{c}_0}$ *in* G_0, *with the estimate*

$$||\widetilde{\mathcal{L}}_{\mathbf{c}}^{-1}\mathcal{G}||_{\mathcal{L}(G_l)} \leq c(s), \quad l \geq 0, \tag{42}$$

where $c(s)$ *is bounded by* $\gamma \ln s$. *Moreover, for any* $q \geq 0$

$$\widetilde{\mathcal{L}}_{\mathbf{c}}^{-1} = \widetilde{\mathcal{L}}_{\mathbf{c}_0}^{-1} + \sum_{1 \leq n \leq q} (-w)^n (\widetilde{\mathcal{L}}_{\mathbf{c}_0}^{-1}\mathcal{L}^{(1)})^n \widetilde{\mathcal{L}}_{\mathbf{c}_0}^{-1} + \mathcal{R}_q(w), \tag{43}$$

$$||\mathcal{R}_q(w)||_{\mathcal{L}(G_l, G_{l-2(q+1)+1})} \leq |w|^{q+1}\gamma^{q+1}c(s),$$

where the linear operator $\widetilde{\mathcal{L}}_{\mathbf{c}}^{-1}$ *is computed in* $\{\ker \mathcal{L}_{\mathbf{c}_0}\}^{\perp}_{H^0}$, $(\widetilde{\mathcal{L}}_{\mathbf{c}_0}^{-1}\mathcal{L}^{(1)})^n\widetilde{\mathcal{L}}_{\mathbf{c}_0}^{-1} \in \mathcal{L}(G_l, G_{l-2n+1})$, *and* $\gamma > 0$ *is independent of* s.

Remark 9. *We notice that the operator* $\widetilde{\mathcal{L}}_{\mathbf{c}}^{-1}\mathcal{G}$ *is bounded. This is just what is needed to apply the Lyapunov-Schmidt method, since the nonlinear terms take the form* $\mathcal{GN}(U, U)$, *where* $\mathcal{N}$ *is a bounded quadratic operator.*

Remark 10. *We observe that the operator* $\widetilde{\mathcal{L}}_{\mathbf{c}}^{-1}$ *in* $\mathcal{L}(G_l, G_{l+1})$ *is weakly differentiable in* w *at 0. Formula (43) gives precisely the loss of regularity of the successive derivatives in* w *at the origin (the loss is 2 at each increasing order). The difficulty introduced by this non-smoothness is in fact not a problem for the 4-dimensional bifurcation equation.*

Proof: First, for any $\mathbf{k} = n_1\mathbf{k}_1 + n_2\mathbf{k}_2$, $n_j \in \mathbb{Z}$, we have by (41)

$$\frac{\mathbf{k}_1 \cdot \mathbf{c}}{\mathbf{k}_2 \cdot \mathbf{c}} = \frac{l_1(1 + \tau_1 w)}{l_2(1 - \tau_2 w)} = \frac{r}{s} \in \mathbb{Q}^+.$$

Hence

$$\mathbf{c} \cdot \mathbf{k} = c_0 l_2(1 - \tau_2 w)(n_1\frac{r}{s} + n_2)$$

and if $\mathbf{c} \cdot \mathbf{k} \neq 0$, we have

$$|\mathbf{c} \cdot \mathbf{k}| \geq \frac{c_0 d}{s}, \tag{44}$$

where d satisfies

$$d \leq l_2|1 - \tau_2 w|.$$

In choosing w such that $|w| \leq \frac{\delta}{5}$ we can take $d = \frac{4l_2}{5}$. Notice that if $\mathbf{c}\cdot\mathbf{k} = 0$, (20) yields

$$|\eta_{\mathbf{k}}| + |\mathbf{v}_{\mathbf{k}}| \leq \frac{1}{|\mathbf{k}|}(|q_{\mathbf{k}}| + |\mathbf{p}_{\mathbf{k}}|). \tag{45}$$

Now, if $\mathbf{c}\cdot\mathbf{k} \neq 0$, we have

$$\left(1 + \frac{1}{6}|\mathbf{k}|^2\right)|\mathbf{c}\cdot\mathbf{k}| - |\mathbf{k}| \geq |\mathbf{k}|\left\{\frac{|\mathbf{k}|c_0 d}{6s} - 1\right\}$$

and for $|\mathbf{k}| \geq \frac{7s}{c_0 d}$ we obtain

$$\left(1 + \frac{1}{6}|\mathbf{k}|^2\right)|\mathbf{c}\cdot\mathbf{k}| - |\mathbf{k}| \geq \frac{|\mathbf{k}|}{6}.$$

We then use the fundamental factorization of $\Delta(\mathbf{k},\mathbf{c})$:

$$\begin{aligned}\Delta(\mathbf{k},\mathbf{c}) &= \left\{(1 + \frac{1}{6}|\mathbf{k}|^2)|\mathbf{c}\cdot\mathbf{k}| - |\mathbf{k}|\right\}\left\{(1 + \frac{1}{6}|\mathbf{k}|^2)|\mathbf{c}\cdot\mathbf{k}| + |\mathbf{k}|\right\} \\ &\geq \frac{|\mathbf{k}|}{6}\left\{(1 + \frac{1}{6}|\mathbf{k}|^2)|\mathbf{c}\cdot\mathbf{k}| + |\mathbf{k}|\right\},\end{aligned}$$

and (20) leads to the estimate

$$|\eta_{\mathbf{k}}| + |\mathbf{v}_{\mathbf{k}}| \leq \frac{6}{|\mathbf{k}|}(|q_{\mathbf{k}}| + |\mathbf{p}_{\mathbf{k}}|). \tag{46}$$

We observe that if $|\mathbf{k}||\mathbf{c}\cdot\mathbf{k}| > 7$, (46) holds.

It remains to study the region $\mathcal{R}$ of the plane (n_1, n_2) where

$$|\mathbf{k}| \leq \frac{7s}{c_0 d}, \quad |\mathbf{k}||\mathbf{c}\cdot\mathbf{k}| \leq 7, \;\; \Delta(\mathbf{k},\mathbf{c}) \neq 0, \;\text{ and } \mathbf{c}\cdot\mathbf{k} \neq 0. \tag{47}$$

Using here the estimate (31), we observe that the region $\mathcal{R}$ is included in the region $\mathcal{A}$ defined by

$$\mathcal{A} \leq \left\{(n_1, n_2) \in \mathbb{Z}^2; n_1^2 + n_2^2 < \left(\frac{7s}{c_0 d d_1}\right)^2, |n_2 + \frac{r}{s}n_1| \leq \frac{7}{c_0 d d_1\sqrt{n_1^2 + n_2^2}}\right\}.$$

The area of $\mathcal{A}$ in the plane (n_1, n_2) can be computed with polar coordinates. We set

$$\begin{aligned}&n_1 = \rho\cos\theta, \quad n_2 = \rho\sin\theta, \\ &\rho \leq \min\left\{\left(\frac{7\cos\theta_0}{c_0 d d_1}\right)^{1/2}|\sin(\theta - \theta_0)|^{-1/2}, \frac{7s}{c_0 d d_1}\right\}\end{aligned}$$

where

$$\tan\theta_0 = -r/s, \quad \theta_0 \in (-\pi/2, 0).$$

Estimating $2\int_\phi^{\pi/2}\rho^2(\theta)d\theta+2\phi(\frac{7s}{c_0dd_1})^2$ for large s, for $\rho^2(\theta)=\left(\frac{7\cos\theta_0}{c_0dd_1}\right)|\sin\theta|^{-1}$ and $\sin\phi=\frac{c_0dd_1\cos\theta_0}{7s^2}$, yields

$$\begin{aligned}\mathrm{Area}(\mathcal{A}) &= \frac{14\cos\theta_0}{c_0dd_1}\ln(\frac{1}{\tan\phi/2})+\frac{98s^2}{(c_0dd_1)^2}\sin^{-1}(\frac{c_0dd_1\cos\theta_0}{7s^2})\\ &\sim \frac{28\cos\theta_0}{c_0dd_1}\ln s.\end{aligned}$$

We notice that, by construction, r/s is close to l_1/l_2, hence $\cos\theta_0$ is close to $\frac{l_2}{\sqrt{l_1^2+l_2^2}}$ and the following estimate holds

$$\frac{\cos\theta_0}{dd_1}\le\frac{5\left((1+\tau_1^2)l_1^2+(1+\tau_2^2)l_2^2\right)^{1/2}}{4l_1l_2(l_1^2+l_2^2)^{1/2}(\tau_1+\tau_2)}.$$

For $\tau_2<\delta^{-1}$ the estimate for c_0 (see (24)) is independent of τ_2 (but depends on l_2 and δ), which shows that $\mathrm{Area}(\mathcal{A})\le\gamma_0(\ln s)$ with γ_0 independent of s. Hence the number of points (n_1,n_2) lying in $\mathcal{A}$ is of order $\ln s$.

In what follows, it is useful to notice that for

$$|\mathbf{k}|^2>\frac{7l_2d_0^2}{l_1c_0dd_1},$$

we have

$$n_2k_2<0. \tag{48}$$

To see this, we look at the intersection of the curve (in polar coordinates)

$$\rho^2=\frac{7\cos\theta_0}{c_0dd_1}|\sin(\theta-\theta_0)|^{-1}$$

which bounds the region $\mathcal{A}$, with the n_1-axis ($\theta=0$). The points of this curve with $\theta_0<\theta<0$ are such that $n_1>0$, $n_2<0$. This shows that for points in the region of $\mathcal{A}$ such that

$$n_1^2+n_2^2>\frac{7}{c_0dd_1}\frac{s}{r},$$

n_1 and n_2 have opposite signs. Then in order to obtain (48) we use (31), and observe that r/s is close to l_1/l_2, and $k_2=n_1l_1\tau_1-n_2l_2\tau_2$ has the sign of n_1.

Now, the equation

$$\left(1+\frac{1}{6}|\mathbf{k}|^2\right)|\mathbf{c}\cdot\mathbf{k}|-|\mathbf{k}|=0$$

is equivalent to (28) with c_0^2 replaced by its expression (24) as a function of τ_2 and l_2, which makes for every "bad" pair (n_1,n_2) a polynomial equation of degree 8 in τ_2. Hence we cannot have more than 8 roots $\tau_2>0$ for every "bad" pair (n_1, n_2). This makes a finite set of "bad" values for $\tau_2 = \tau_2^{(p)}(\tau_1, l_1, l_2)$ of cardinality $O(\ln s)$. We then need to exclude small neighborhoods of these roots for controlling the size of the inverse of $(1 + \frac{1}{6}|\mathbf{k}|^2)|\mathbf{c}\cdot\mathbf{k}| - |\mathbf{k}|$. Let us exclude $O(\ln s)$ neighborhoods of these specific values of τ_2. For having still remaining good values for the $(\tau_2)'s$, we may choose, for each (n_1, n_2), neighborhoods of exclusions of size $O(\nu/\ln s)$ around every such root τ_2, with $\nu << 1$. Let us show that outside these neighborhoods we have

$$\left|\left(1 + \frac{1}{6}|\mathbf{k}|^2\right)|\mathbf{c}\cdot\mathbf{k}| - |\mathbf{k}|\right| \geq \frac{c|\mathbf{k}|}{\ln s}, \text{ for large } s. \tag{49}$$

To show this, it is sufficient to show that the derivative of

$$g(\tau_2) = \left(1 + \frac{1}{6}|\mathbf{k}|^2\right)|\mathbf{c}\cdot\mathbf{k}| - |\mathbf{k}|$$

with respect to τ_2 at any root τ_0 of (28) satisfies $|g'(\tau_0)| > c|\mathbf{k}|$ for some c independent of s. Indeed, an elementary computation gives

$$\frac{\partial_{\tau_2}|\mathbf{c}\cdot\mathbf{k}|}{|\mathbf{c}\cdot\mathbf{k}|}\Big|_{\tau_2=\tau_0} = -\frac{w}{1-\tau_0 w} + \tau_0\frac{6 - l_2^2(1+\tau_0^2)}{(1+\tau_0^2)(6 + l_2^2(1+\tau_0^2)}$$

hence

$$g'(\tau_0) = |\mathbf{k}|\left\{\frac{-n_2 l_2 k_2}{|\mathbf{k}|^2}\left(\frac{|\mathbf{k}|^2 - 6}{|\mathbf{k}|^2 + 6}\right) - \frac{w}{1-\tau_0 w} + \tau_0\frac{6 - l_2^2(1+\tau_0^2)}{(1+\tau_0^2)(6 + l_2^2(1+\tau_0^2)}\right\}.$$

For

$$|\mathbf{k}| > M, \quad M = \max\left\{\frac{7 l_2 d_0^2}{l_1 c_0 d d_1}, \sqrt{6}\right\}$$

the inequality (48) shows that the first term on the right hand side is positive. Moreover, for $\tau_2 < \delta^{-1}$, and $|w| < \delta/5$, we have

$$\left|\frac{w}{1-\tau_2 w}\right| < \frac{\delta}{4}.$$

Taking l_2 small enough, such that

$$l_2 < 1, \quad l_2\tau_0 < 1$$

and remarking that $\tau_2 < \delta^{-1}$, we see that this condition holds as soon as

$$l_2 < \delta < 1. \tag{50}$$

We obtain $l_2^2(1+\tau_0^2) < 2$, hence

$$\frac{6 - l_2^2(1+\tau_0^2)}{(6 + l_2^2(1+\tau_0^2)} > \frac{1}{2},$$

and we conclude, (since $\delta < \tau_2 < 1/\delta$), that

$$\tau_0 \frac{6 - l_2^2(1+\tau_0^2)}{(1+\tau_0^2)(6 + l_2^2(1+\tau_0^2))} > \frac{\delta}{2(1+\delta^2)}$$

which is independent of s. We notice that

$$4 > 2(1+\delta^2)$$

hence

$$g'(\tau_0) > |\mathbf{k}| \left\{ \frac{\delta}{2(1+\delta^2)} - \frac{\delta}{4} \right\} = c|\mathbf{k}|, \quad c > 0.$$

In the region $\mathcal{R}$ where

$$|\mathbf{k}| \leq M,$$

the number of corresponding points of the plane (n_1, n_2) is bounded by a finite number independent of s. For avoiding the corresponding bad values of τ_2 near the corresponding roots, we just need to avoid a fixed (independent of s) small ν neighborhood of this finite number of roots, since the minimal value of $|g'(\tau_0)|$ at these roots is independent of s.

This ends the proof of the fact that in choosing τ_2 outside a small open set included in (δ, δ^{-1}) and for $|\mathbf{k}| \leq \frac{7s}{c_0 d}$ we obtain (49). Finally, we find a constant $\gamma > 0$ independent of s such that

$$|\eta_{\mathbf{k}}| + |\mathbf{v}_{\mathbf{k}}| \leq \frac{\gamma \ln s}{|\mathbf{k}|}(|q_{\mathbf{k}}| + |\mathbf{p}_{\mathbf{k}}|). \tag{51}$$

Now, collecting (45), (46) (,51) we obtain an estimate valid for all $\mathbf{k}$ such that $\mathbf{k} \neq \pm\mathbf{k}_1, \pm\mathbf{k}_2$

$$\left| \left(1 + \frac{1}{6}|\mathbf{k}|^2\right) |\mathbf{c} \cdot \mathbf{k}| - |\mathbf{k}| \right| \geq \frac{|\mathbf{k}|}{c(s)},$$

and the required estimate (42) follows for $\widetilde{\mathcal{L}}_{\mathbf{c}}^{-1}\mathcal{G}$. Property (12) and

$$\mathcal{G}\xi_{\pm\mathbf{k}_j} = \pm i l_j \sqrt{1+\tau_j^2}\,\xi_{\pm\mathbf{k}_j} \tag{52}$$

imply that the subspace $\{\ker \mathcal{L}_{\mathbf{c}_0}\}^{\perp}_{H^0}$ is mapped into itself by $\mathcal{G}$. Notice that the dependency in s of the bound of the linear operator $\widetilde{\mathcal{L}}_{\mathbf{c}}^{-1}\mathcal{G}$ is delicate to control, since the dangerous values of (n_1, n_2) (for which we may have roots of (28)) are large ones, and not so frequent in the set $\mathcal{A}$.

For obtaining the precise loss of differentiability indicated by (43), we first observe that the subspace $\{\ker \mathcal{L}_{\mathbf{c}_0}\}^{\perp}_{H^0}$ is stable under $\mathcal{L}^{(1)}$ since we have property (12) and

$$\mathcal{L}^{(1)}\xi_{\pm\mathbf{k}_j} = \pm i(-1)^j l_j \tau_j \sqrt{1+\tau_j^2}\,\xi_{\pm\mathbf{k}_j}. \tag{53}$$

Then, for $F \in \{\ker \mathcal{L}_{\mathbf{c}_0}\}^{\perp}_{H^0}$, the equation

$$\mathcal{L}_{\mathbf{c}}U = (\mathcal{L}_{\mathbf{c}_0} + w\mathcal{L}^{(1)})U = F$$

leads to

$$U = \widetilde{\mathcal{L}}_{\mathbf{c}_0}^{-1} F + U_1, \quad \mathcal{L}_{\mathbf{c}} U_1 = -w\mathcal{L}^{(1)} \widetilde{\mathcal{L}}_{\mathbf{c}_0}^{-1} F,$$

which leads to (43) for $q = 0$. Writing now

$$U_1 = -w\widetilde{\mathcal{L}}_{\mathbf{c}_0}^{-1} \mathcal{L}^{(1)} \widetilde{\mathcal{L}}_{\mathbf{c}_0}^{-1} F + U_2, \quad \mathcal{L}_{\mathbf{c}} U_2 = w^2 \mathcal{L}^{(1)} \widetilde{\mathcal{L}}_{\mathbf{c}_0}^{-1} \mathcal{L}^{(1)} \widetilde{\mathcal{L}}_{\mathbf{c}_0}^{-1} F,$$

leads to (43) for $q = 1$. Then the result (43) follows for any q and Lemma 8 is proved.

4 Bifurcation equations

Let us introduce the set of two parameters (μ, w):

$$\mathbf{c} = \frac{c_0}{1+\mu}(1, w)$$

and notice that

$$\mathcal{L}_{\mathbf{c}} U = \frac{1}{1+\mu}\left(\mathcal{L}_{\mathbf{c}_0} U + w\mathcal{L}^{(1)} U + \mu \mathcal{G} U\right),$$

which allows us to rewrite Equation (10) as

$$\mathcal{L}_{\mathbf{c}_0} U + \mu \mathcal{G} U + (1+\mu)\mathcal{G}\mathcal{N}(U, U) + w\mathcal{L}^{(1)} U = 0. \tag{54}$$

Notice that this choice of parameters might be questionable. However it has the benefit that all the bad (and interesting) singularities are concentrated only in the linear term $w\mathcal{L}^{(1)}U$. All other terms are very nice for a Lyapunov-Schmidt method (thanks to Lemma 6) and may be treated in a standard way as in [6], which immediately gives the result of the forthcoming theorem for $w = 0$. Our purpose is to show the following more general result.

Theorem 11. *Let $\delta \in (0, 1)$ and choose basic wave vectors $(\mathbf{k}_1, \mathbf{k}_2)$ which satisfy the δ-non-flatness condition, such that the dispersion relation $\Delta(\mathbf{k}, \mathbf{c}_0) = 0$ has the only solutions $\mathbf{k} = \pm\mathbf{k}_j$, $j = 1, 2$, in Γ. Then choose $\mathbf{c} = \frac{c_0}{1+\mu}(1, w)$ such that $|w| \leq \frac{\delta}{5}$ and the ratio*

$$\frac{\mathbf{k}_1 \cdot \mathbf{c}}{\mathbf{k}_2 \cdot \mathbf{c}} = \frac{r}{s} \in \mathbb{Q}^+, \tag{55}$$

where $r, s \in \mathbb{N}$ are relatively prime, is close enough to $\frac{\mathbf{k}_1 \cdot \mathbf{c}_0}{\mathbf{k}_2 \cdot \mathbf{c}_0}$. Fix $\sigma \in \mathbb{N}$ large enough and assume $1 \leq s \leq \sigma$. Choose values of $\tau_2 \in (\delta, \delta^{-1})$, except in a small neighborhood of a finite set $\tau_2^{(m)}(\tau_1, l_1, l_2)$ of cardinality at most $O(\ln \sigma)$. Then, for any $p \geq 5$, there is a family of bifurcating bi-periodic traveling waves, $U = (\eta, \mathbf{v})$ which are solutions of (3) in G_p, are in general non-symmetric with *respect to the propagation direction* $\mathbf{c}$*, and are of the form*

$$U = \sum_{1\le j+l+m+q\le n} A^j \overline{A}^l B^m \overline{B}^q U_{jlmq} + o((|A|+|B|)^n) \tag{56}$$

with

$$A = \varepsilon_1 e^{i\mathbf{k}_1\cdot\mathbf{y}}, \quad B = \varepsilon_2 e^{i\mathbf{k}_2\cdot\mathbf{y}},$$

where $\mathbf{y}$ *corresponds to an arbitrary horizontal shift,*

$$\begin{aligned} \mu &= \alpha_1\varepsilon_1^2 + \alpha_2\varepsilon_2^2 + O(\varepsilon_1^2+\varepsilon_2^2)^2, \\ w &= \beta_1\varepsilon_1^2 + \beta_2\varepsilon_2^2 + O(\varepsilon_1^2+\varepsilon_2^2)^2, \end{aligned} \tag{57}$$

where the "rational" restriction (55) on w *implies a restriction on amplitudes* $(\varepsilon_1, \varepsilon_2)$ *which are uniformly bounded by* $O\{(|\mu|/\ln\sigma)^{1/2}\}$ *with* $|\mu| << (\ln\sigma)^{-1}$.

Remark 12. *If we forget about the translation invariance of the set of solutions, we notice that we have a basic 3-dimensional set of free parameters with* $(\mathbf{k}_1, \mathbf{k}_2)$ *subjected to the dispersion relation, with the bifurcation parameters* (μ, w) *or equivalently the amplitudes* $(\varepsilon_1, \varepsilon_2)$*. However, we should notice that the rationality condition (55) only allows a reduced choice for* w *of measure zero in* $\mathbb{R}$*.*

Remark 13. *If we fix the order of regularity* p*, we need to stop the expansion (56) at order* n *such that* $p-(2(n-2)\geq 2$*, i.e.* $n \leq 1+p/2$*. This is due to the loss of regularity for increasing powers in* w *for the expansion of* $\mathcal{V}(A, \overline{A}, B, \overline{B}, \mu, w)$ *defined below.*

Remark 14. *With the calculations presented in the appendix, the explicit expression for the orders* 1 *and* 2 *in* ε_1 *and* ε_2 *of the solution* U *is*

$$\begin{aligned} U =& A\xi_{\mathbf{k}_1} + \overline{A}\xi_{-\mathbf{k}_1} + B\xi_{\mathbf{k}_2} + \overline{B}\xi_{-\mathbf{k}_2} + \zeta_{2,0}(A^2 e^{2i\mathbf{k}_1\cdot\mathbf{x}} + \overline{A}^2 e^{-2i\mathbf{k}_1\cdot\mathbf{x}}) + \\ &+ \zeta_{0,2}(B^2 e^{2i\mathbf{k}_2\cdot\mathbf{x}} + \overline{B}^2 e^{-2i\mathbf{k}_2\cdot\mathbf{x}}) + \zeta_{1,1}(AB e^{i(\mathbf{k}_1+\mathbf{k}_2)\cdot\mathbf{x}} + \overline{AB} e^{-i(\mathbf{k}_1+\mathbf{k}_2)\cdot\mathbf{x}}) + \\ &+ \zeta_{1,-1}(A\overline{B} e^{i(\mathbf{k}_1-\mathbf{k}_2)\cdot\mathbf{x}} + \overline{A}B e^{-i(\mathbf{k}_1-\mathbf{k}_2)\cdot\mathbf{x}}) + h.o.t., \end{aligned}$$

where $A, B \in \mathbb{C}$*, and* $\xi_{\pm\mathbf{k}_j}$*,* $\zeta_{l,n}$ *are defined in (36), (71), (72). The coefficients* α_j *and* β_j *appearing in the expansions of* μ *and* w *are given by (68). These explicit expressions allow us to make numerical computations and show pictures at the end of the paper.*

Proof of the theorem: First we decompose $U \in G_p$ as

$$U = X + V$$

where

$$\begin{aligned} &X = A\xi_{\mathbf{k}_1} + \overline{A}\xi_{-\mathbf{k}_1} + B\xi_{\mathbf{k}_2} + \overline{B}\xi_{-\mathbf{k}_2} \in E, \\ &\langle V, \xi_{\pm\mathbf{k}_j}\rangle_{H^0} = 0, \quad j = 1, 2. \end{aligned}$$

Observe that $E \subset G_p$ for all $p \geq 0$. The above decomposition is unique for any $p \geq 0$, hence the mapping $U \mapsto V$ defines a *projection* $\mathcal{Q}$ *from* G_p *to*

$G_p \cap \{\ker \mathcal{L}_{\mathbf{c}_0}\}^{\perp}_{H^0}$, which is orthogonal for $p = 0$. Now, we note that

$$\mathcal{Q}\mathcal{G}X = 0, \quad \mathcal{Q}\mathcal{G}V = \mathcal{G}V, \quad p \geq 1,$$
$$\mathcal{Q}\mathcal{L}^{(1)}X = 0, \quad \mathcal{Q}\mathcal{L}^{(1)}V = \mathcal{L}^{(1)}V, \quad p \geq 3.$$

Assuming $U \in G_p$, $p \geq 3$, it follows from (54) that

$$\mathcal{L}_{c_0(1,w)}V + \mu\mathcal{G}V + (1+\mu)\mathcal{Q}\mathcal{G}\mathcal{N}(X+V, X+V) = 0, \tag{58}$$
$$\langle \mu\mathcal{G}X + w\mathcal{L}^{(1)}X + (1+\mu)\mathcal{G}\mathcal{N}(X+V, X+V), \xi_{\pm\mathbf{k}_j}\rangle = 0, \ j = 1, 2. \tag{59}$$

We notice that (58) may be solved by the implicit function theorem in $G_p \cap \{\ker \mathcal{L}_{\mathbf{c}_0}\}^{\perp}_{H^0}$, for any $p \geq 3$, with respect to V. Indeed, Equation (58) is of the form

$$\mathcal{L}_{c_0(1,w)}V + \mathcal{F}(X, V, \mu) = 0$$

in G_{p-3}, where $\mathcal{F}$ is analytic in its arguments as a function from $E \times \Big(G_p \cap \{\ker \mathcal{L}_{\mathbf{c}_0}\}^{\perp}_{H^0}\Big) \times \mathbb{R}$ into $G_{p-1} \cap \{\ker \mathcal{L}_{\mathbf{c}_0}\}^{\perp}_{H^0}$, and satisfies

$$\mathcal{F}(0, 0, \mu) = 0, \quad D_V\mathcal{F}(0, 0, 0) = 0.$$

Due to Lemma 8, the operator $\mathcal{L}_{c_0(1,w)}$ has a bounded inverse from $G_{p-1} \cap \{\ker \mathcal{L}_{\mathbf{c}_0}\}^{\perp}_{H^0}$ to $G_p \cap \{\ker \mathcal{L}_{\mathbf{c}_0}\}^{\perp}_{H^0}$, and *this bound is uniform in* w, *provided that* w satisfies the rationality condition (41), $(\mathbf{k}_1, \mathbf{k}_2)$ the non-flatness condition, *and* s *is bounded by some fixed* σ. Due to the bound of $\{\widetilde{\mathcal{L}}_{c_0(1,w)}\}^{-1}$ found in Lemma 8 we need to assume that

$$|\mu| \ln \sigma << 1, \quad ||X|| \ln \sigma << 1, \tag{60}$$

which implies

$$||V||^2_{G_p} \ln \sigma \ \text{ and } |\mu| ||V||_{G_p} \ln \sigma << ||V||_{G_p}$$

and finally

$$||V|| = O(||X||^2 \ln \sigma). \tag{61}$$

Therefore, for A, B close enough to 0, w satisfying (41), and $(\mathbf{k}_1, \mathbf{k}_2)$ satisfying the non-flatness condition and $s \leq \sigma$, we obtain

$$V = \mathcal{V}(A, \overline{A}, B, \overline{B}, \mu, w) \in G_p \cap \{\ker \mathcal{L}_{\mathbf{c}_0}\}^{\perp}_{H^0}$$

which is analytic in $(A, \overline{A}, B, \overline{B}, \mu)$, the *dependency in* w *being more subtle.* In fact $\mathcal{V}(A, \overline{A}, B, \overline{B}, \mu, w)$ is in $G_p \cap \{\ker \mathcal{L}_{\mathbf{c}_0}\}^{\perp}_{H^0}$ with $p \geq 3$, and has an asymptotic expansion in powers of w in the neighborhood of 0. To prove this, let us define

$$\mathcal{V}_0 = \mathcal{V}(A, \overline{A}, B, \overline{B}, \mu, 0), \quad \mathcal{V}_1 = \mathcal{V}(A, \overline{A}, B, \overline{B}, \mu, w) - \mathcal{V}_0.$$

Then $\mathcal{V}_1$ satisfies

$$\begin{aligned} 0 =& \mathcal{L}_{c_0(1,w)}\mathcal{V}_1 + w\mathcal{L}^{(1)}\mathcal{V}_0 + \mu\mathcal{G}\mathcal{V}_1 + 2(1+\mu)\mathcal{Q}\mathcal{G}\mathcal{N}(X+\mathcal{V}_0, \mathcal{V}_1) \\ &+ (1+\mu)\mathcal{Q}\mathcal{G}\mathcal{N}(\mathcal{V}_1, \mathcal{V}_1). \end{aligned} \tag{62}$$

Since $w\mathcal{L}^{(1)}\mathcal{V}_0 \in G_{p-3} \cap \{\ker \mathcal{L}_{\mathbf{c}_0}\}^{\perp}_{H^0}$, with a small enough norm, we can solve Equation (62) with respect to $\mathcal{V}_1$ in $G_{p-2} \cap \{\ker \mathcal{L}_{\mathbf{c}_0}\}^{\perp}_{H^0}$, provided that $p \geq 5$. Denoting by $\mathcal{V}_{10}$ the value of the solution $\mathcal{V}_1$ when one replaces $\mathcal{L}_{c_0(1,w)}$ by $\mathcal{L}_{\mathbf{c}_0}$, we can set $\mathcal{V}_1 = \mathcal{V}_{10} + \mathcal{V}_2$ and obtain $\mathcal{V}_2$ by the implicit function theorem in $G_{p-4} \cap \{\ker \mathcal{L}_{\mathbf{c}_0}\}^{\perp}_{H^0}$, and so on. Now we have estimates of the form

$$\begin{aligned} ||\mathcal{V}_0||_{G_p} &\leq \gamma c(\sigma)||X||^2, \\ ||\mathcal{V}_1||_{G_{p-2}} &\leq \gamma c(\sigma)|w|||X||^2, \\ ||\mathcal{V}_2||_{G_{p-4}} &\leq \gamma c(\sigma)|w|^2||X||^2, \end{aligned}$$

and so on. This proves the assertion on the asymptotic expansion in powers of w (not converging in general) for $\mathcal{V}(A, \overline{A}, B, \overline{B}, \mu, w)$ in any space $G_p \cap \{\ker \mathcal{L}_{\mathbf{c}_0}\}^{\perp}_{H^0}$, $p \geq 3$ (the choice of p is arbitrary, but we need to stop the expansion at some order to insure the existence of the solution in some space G_p, as indicated in the Remark 13.

Now, using the symmetry properties (13) of the basic equation (54), (38), and also

$$\mathcal{T}_{\mathbf{y}}\mathcal{Q} = \mathcal{Q}\mathcal{T}_{\mathbf{y}}, \ \mathcal{S}_0\mathcal{Q} = \mathcal{Q}\mathcal{S}_0,$$

we show that the uniqueness of $\mathcal{V}$ leads to the following properties:

$$\begin{aligned} \mathcal{T}_{\mathbf{y}}\mathcal{V}(A, \overline{A}, B, \overline{B}, \mu, w) =& \mathcal{V}(Ae^{i\mathbf{k}_1\cdot\mathbf{y}}, \overline{A}e^{-i\mathbf{k}_1\cdot\mathbf{y}}, Be^{i\mathbf{k}_2\cdot\mathbf{y}}, \overline{B}e^{-i\mathbf{k}_2\cdot\mathbf{y}}, \mu, w), \\ \mathcal{S}_0\mathcal{V}(A, \overline{A}, B, \overline{B}, \mu, w) =& \mathcal{V}(\overline{A}, A, \overline{B}, B, \mu, w). \end{aligned} \tag{63}$$

More precisely, we have in any $G_p \cap \{\ker \mathcal{L}_{\mathbf{c}_0}\}^{\perp}_{H^0}$, $p \geq 3$

$$\begin{aligned} \mathcal{V}(A, \overline{A}, B, \overline{B}, \mu, w) =& -\widetilde{\mathcal{L}}_{\mathbf{c}_0}^{-1}\mathcal{Q}\mathcal{G}\mathcal{N}(X, X) \\ &+ O((|\mu| + |w|)||X||^2 + ||X||^3). \end{aligned} \tag{64}$$

Now replacing V by $\mathcal{V}(A, \overline{A}, B, \overline{B}, \mu, w)$ in (59) which consists of four equations, we obtain in fact 2 complex equations, with their complex conjugates, of the form

$$h_1(A, \overline{A}, B, \overline{B}, \mu, w) = 0, \quad h_2(A, \overline{A}, B, \overline{B}, \mu, w) = 0,$$

where h_1 is obtained by replacing $\mathbf{k}_1$ in (59) and h_2 by $\mathbf{k}_2$, and h_j, $j = 1, 2$, is analytic in $(A, \overline{A}, B, \overline{B}, \mu)$ and in C^l at the origin with respect to w (l is arbitrary). The symmetry properties (13), (38), and (63) lead, for any $\mathbf{y} \in \mathbb{R}^2$, to the following relationships

$$
\begin{aligned}
h_1(Ae^{i\mathbf{k}_1\cdot\mathbf{y}}, \overline{A}e^{-i\mathbf{k}_1\cdot\mathbf{y}}, Be^{i\mathbf{k}_2\cdot\mathbf{y}}, \overline{B}e^{-i\mathbf{k}_2\cdot\mathbf{y}}, \mu, w) =& e^{i\mathbf{k}_1\cdot\mathbf{y}} h_1(A, \overline{A}, B, \overline{B}, \mu, w),\\
h_2(Ae^{i\mathbf{k}_1\cdot\mathbf{y}}, \overline{A}e^{-i\mathbf{k}_1\cdot\mathbf{y}}, Be^{i\mathbf{k}_2\cdot\mathbf{y}}, \overline{B}e^{-i\mathbf{k}_2\cdot\mathbf{y}}, \mu, w) =& e^{i\mathbf{k}_2\cdot\mathbf{y}} h_2(A, \overline{A}, B, \overline{B}, \mu, w),\\
h_1(\overline{A}, A, \overline{B}, B, \mu, w) =& -\overline{h_1}(A, \overline{A}, B, \overline{B}, \mu, w).
\end{aligned}
$$

It results that

$$
\begin{aligned}
h_1(A, \overline{A}, B, \overline{B}, \mu, w) = iAg_1(|A|^2, |B|^2, \mu, w),\\
h_2(A, \overline{A}, B, \overline{B}, \mu, w) = iBg_2(|A|^2, |B|^2, \mu, w),
\end{aligned}
$$

where g_1 and g_2 are *real valued* smooth functions of their arguments, since the h_j are smooth. When $B = 0$ (or $A = 0$) one obtains plane waves with basic wave vector $\mathbf{k}_1$ (or $\mathbf{k}_2$), and the direction of propagation being somewhat arbitrarily provided it is not orthogonal to $\mathbf{k}_1$ (or $\mathbf{k}_2$). When $AB \neq 0$, one obtains the bi-periodic traveling waves, which are the main object of our study. To conclude their existence, we need to solve the real system of two equations:

$$
\begin{aligned}
g_1(|A|^2, |B|^2, \mu, w) =& 0,\\
g_2(|A|^2, |B|^2, \mu, w) =& 0.
\end{aligned}
\tag{65}
$$

In the case when the lattice Γ has a diamond structure and the x_1-axis is chosen such that $\mathbf{k}_1$ and $\mathbf{k}_2$ are symmetric with respect to this axis, we have the additional symmetry properties (14) and (39) which, thanks to the uniqueness of $\mathcal{V}$ and for $w = 0$ (i.e. when $\mathbf{c}$ is in the x_1-direction), leads to

$$
\mathcal{S}_1\mathcal{V}(A, \overline{A}, B, \overline{B}, \mu, 0) = \mathcal{V}(B, \overline{B}, A, \overline{A}, \mu, 0).
$$

This implies

$$
h_1(B, \overline{B}, A, \overline{A}, \mu, 0) = h_2(A, \overline{A}, B, \overline{B}, \mu, 0),
$$

hence

$$
g_1(|B|^2, |A|^2, \mu, 0) = g_2(|A|^2, |B|^2, \mu, 0). \tag{66}
$$

The computations in the general case, detailed in the Appendix, lead to

$$
g_j = 2l_j(1+\tau_j^2)^{3/2}\Omega\left\{\mu + (-1)^j w\tau_j + a_j|A|^2 + b_j|B|^2 + h.o.t.\right\}, \tag{67}
$$

where the coefficients are explicitly given in the Appendix. This leads to

$$
\begin{aligned}
\mu =& -\frac{a_1+a_2}{2}\varepsilon_1^2 - \frac{b_1+b_2}{2}\varepsilon_2^2 + O(\varepsilon_1^2+\varepsilon_2^2)^2,\\
w(\tau_1+\tau_2) =& (a_1-a_2)\varepsilon_1^2 + (b_1-b_2)\varepsilon_2^2 + O(\varepsilon_1^2+\varepsilon_2^2)^2.
\end{aligned}
\tag{68}
$$

From the bounds (60) and (61), one has

$$
\varepsilon_1 + \varepsilon_2 = O(|\mu|^{1/2}(\ln\sigma)^{-1/2}), \tag{69}
$$

which finishes the proof of Theorem 11.
Remark 15. *A particular case of Theorem 11 is when* $\frac{l_1}{l_2} = \frac{r}{s} \in \mathbb{Q}$. *This corresponds to* $w = 0$, *i.e., waves traveling in the* x_1 *direction.*

5 Plotting the free surface

We can now plot the traveling surfaces in the (z_1, z_2) plane, where z_2 is the traveling direction and points downward, i.e.,

$$x_1 = \frac{wz_1 + z_2}{\sqrt{1+w^2}}, \quad x_2 = \frac{-z_1 + wz_2}{\sqrt{1+w^2}}.$$

In all figures the crests are light and troughs are dark.

By choosing the waves of the bifurcating family with

$$A = \varepsilon_1, \quad B = \varepsilon_2,$$

the elevation η of the waves indicated in the pictures is computed with terms up to degree 2 in $(\varepsilon_1, \varepsilon_2)$:

$$\begin{aligned}\eta \approx\ & 2\varepsilon_1\sqrt{1+\tau_1^2}\cos(\mathbf{k}_1\cdot\mathbf{x}) + 2\varepsilon_2\sqrt{1+\tau_2^2}\cos(\mathbf{k}_2\cdot\mathbf{x}) \\ & + 2\varepsilon_1^2(\zeta_{2,0})_1\cos(2\mathbf{k}_1\cdot\mathbf{x}) + 2\varepsilon_2^2(\zeta_{0,2})_1\cos(2\mathbf{k}_2\cdot\mathbf{x}) \\ & + 2\varepsilon_1\varepsilon_2(\zeta_{1,1})_1\cos((\mathbf{k}_1+\mathbf{k}_2)\cdot\mathbf{x}) + 2\varepsilon_1\varepsilon_2(\zeta_{1,-1})_1\cos((\mathbf{k}_1-\mathbf{k}_2)\cdot\mathbf{x}).\end{aligned}$$

For fixed values of l_1, τ_1, τ_2, we compute l_2 with formula (24) and once ε_1 and ε_2 are fixed, we compute w with (68). When $\tau_1 = \tau_2 = \tau$ and $l_1 = l_2$, the lattice Γ is symmetric. Figure 1 shows the influence of the ratio $\varepsilon_1/\varepsilon_2$ when the lattice Γ is symmetric. When $\varepsilon_2/\varepsilon_1 = 1$, the wave pattern is symmetric with respect to the propagation direction (here the vertical direction). Figures 2, 3 and 4 also show cases with a symmetric lattice Γ for different values of τ and compare the *asymmetric pattern* with $\varepsilon_2/\varepsilon_1 = 0.5$ with the symmetric one with $\varepsilon_2/\varepsilon_1 = 1$. Figures 5 and 6 show cases with a non-symmetric lattice Γ. Figure 7 provides two examples of waves where $w \approx 0$ i.e., once ε_1 is fixed, we compute ε_2 with (68) in such a way that $w = 0$ at leading order. Notice that in view of Theorem 11, these solutions exist for l_1/l_2 rational. But in our computed examples this ratio may not be rational, so we take r/s to be a rational approximation of l_1/l_2 in such a way that w is very close to 0.

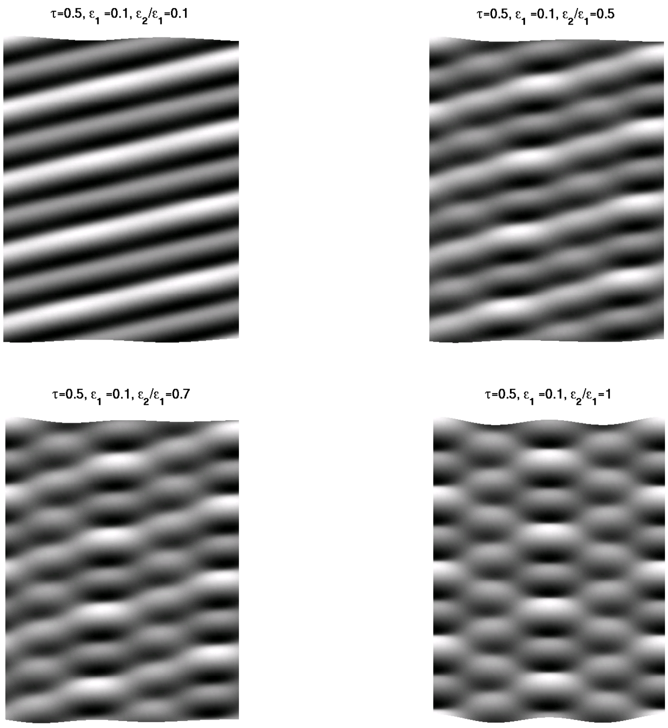

Fig. 1. Γ symmetric, $\tau = 0.5$, $l_1 = l_2 = 0.25$, $\varepsilon_1 = 0.1$, i) $\varepsilon_2/\varepsilon_1 = 0.1$, ii) $\varepsilon_2/\varepsilon_1 = 0.5$, iii) $\varepsilon_2/\varepsilon_1 = 0.7$ (asymmetric waves), iv) $\varepsilon_2/\varepsilon_1 = 1$ (symmetric waves). The direction of propagation of the waves is the vertical axis, point downward. Crests are light and troughs are dark.

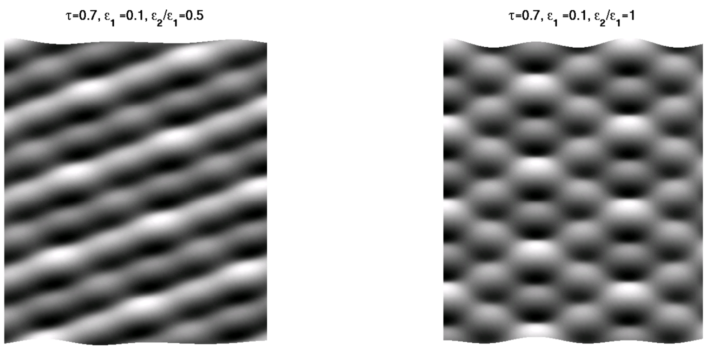

Fig. 2. Γ symmetric, $\tau = 0.7$, $l_1 = l_2 = 0.25$, $\varepsilon_1 = 0.1$, i) $\varepsilon_2/\varepsilon_1 = 0.5$ (asymmetric waves), ii) $\varepsilon_2/\varepsilon_1 = 1$ (symmetric waves).

6 Appendix - Computation of the coefficients

This Appendix is devoted to the computation of the principal part of the system (67), leading to the existence of non-symmetric traveling waves for

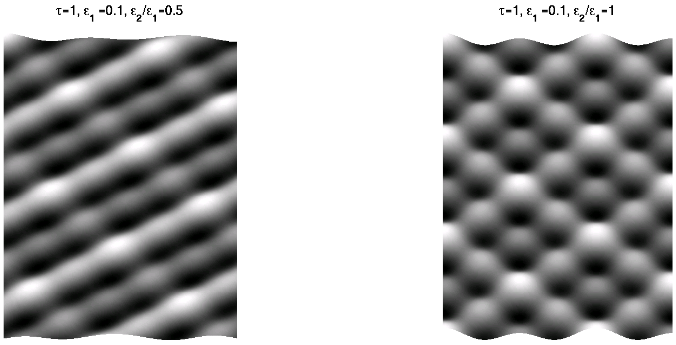

Fig. 3. Γ symmetric, $\tau = 1$, $l_1 = l_2 = 0.25$, $\varepsilon_1 = 0.1$, i) $\varepsilon_2/\varepsilon_1 = 0.5$ (asymmetric waves), ii) $\varepsilon_2/\varepsilon_1 = 1$ (symmetric waves).

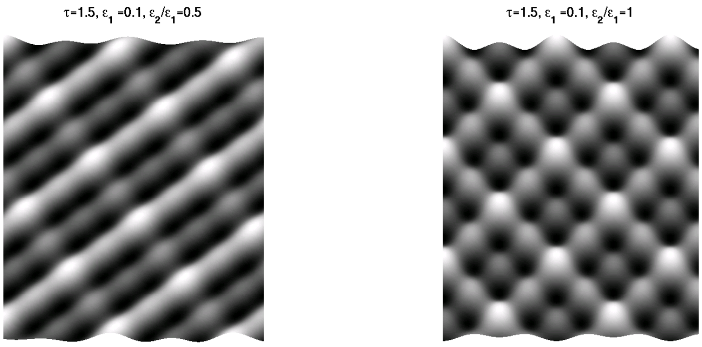

Fig. 4. Γ symmetric, $\tau = 1.5$, $l_1 = l_2 = 0.25$, $\varepsilon_1 = 0.1$, i) $\varepsilon_2/\varepsilon_1 = 0.5$ (asymmetric waves), ii) $\varepsilon_2/\varepsilon_1 = 1$ (symmetric waves).

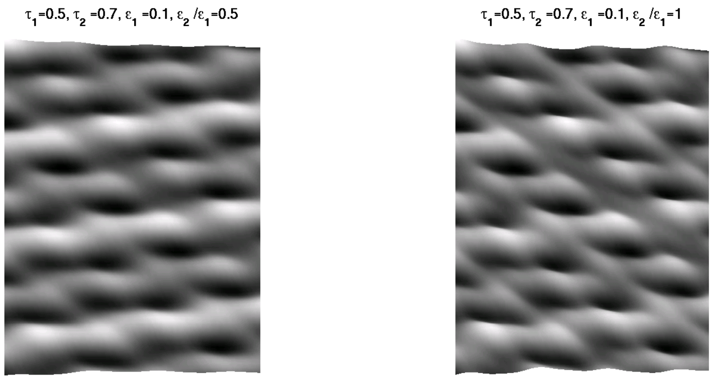

Fig. 5. Γ asymmetric, $\tau_1 = 0.5$, $\tau_2 = 0.7$, $l_1 = 0.25$, $\varepsilon_1 = 0.1$, i) $\varepsilon_2/\varepsilon_1 = 0.5$, ii) $\varepsilon_2/\varepsilon_1 = 1$.

(1). First, Equation (64) with the symmetry properties (63) lead to

$$\begin{aligned} V =&\zeta_{2,0}(A^2 e^{2i\mathbf{k}_1\cdot\mathbf{x}} + \overline{A}^2 e^{-2i\mathbf{k}_1\cdot\mathbf{x}}) + \zeta_{0,2}(B^2 e^{2i\mathbf{k}_2\cdot\mathbf{x}} + \overline{B}^2 e^{-2i\mathbf{k}_2\cdot\mathbf{x}}) + \\ &+ \zeta_{1,1}(ABe^{i(\mathbf{k}_1+\mathbf{k}_2)\cdot\mathbf{x}} + \overline{A}\overline{B}e^{-i(\mathbf{k}_1+\mathbf{k}_2)\cdot\mathbf{x}}) + \\ &+ \zeta_{1,-1}(A\overline{B}e^{i(\mathbf{k}_1-\mathbf{k}_2)\cdot\mathbf{x}} + \overline{A}Be^{-i(\mathbf{k}_1-\mathbf{k}_2)\cdot\mathbf{x}}) + h.o.t. \end{aligned} \tag{70}$$

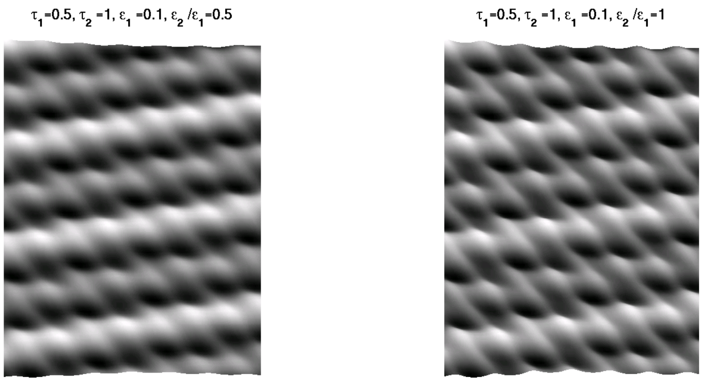

Fig. 6. Γ asymmetric, $\tau_1 = 0.5$, $\tau_2 = 1$, $l_1 = 0.25$, $\varepsilon_1 = 0.1$, i) $\varepsilon_2/\varepsilon_1 = 0.5$, ii) $\varepsilon_2/\varepsilon_1 = 1$.

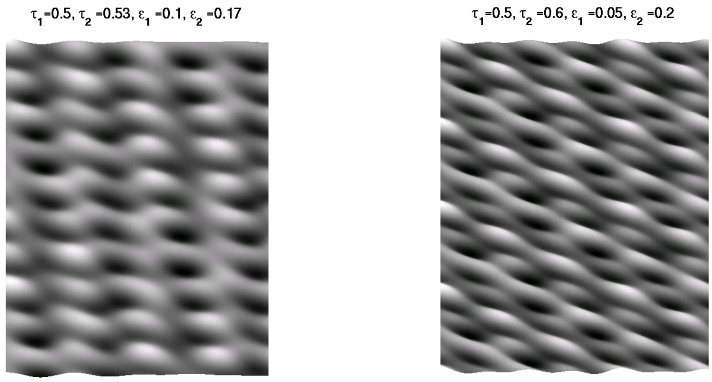

Fig. 7. Γ asymmetric, here $w = 0$, $l_1 = 0.25$, $\varepsilon_1 = 0.1$, i) $\tau_1 = 0.5$, $\tau_2 = 0.53$, $\varepsilon_2 = 0.15$, ii) $\tau_1 = 0.5, \tau_2 = 0.6, \varepsilon_1 = 0.05, \varepsilon_2 = 0.2$.

where

$$\begin{aligned}
\zeta_{2,0} e^{2i\mathbf{k}_1\cdot\mathbf{x}} &= -\widetilde{\mathcal{L}}_{\mathbf{c}_0}^{-1} \mathcal{GN}(\xi_{\mathbf{k}_1}, \xi_{\mathbf{k}_1}),\\
\zeta_{0,2} e^{2i\mathbf{k}_2\cdot\mathbf{x}} &= -\widetilde{\mathcal{L}}_{\mathbf{c}_0}^{-1} \mathcal{GN}(\xi_{\mathbf{k}_2}, \xi_{\mathbf{k}_2}),\\
\zeta_{1,1} e^{i(\mathbf{k}_1+\mathbf{k}_2)\cdot\mathbf{x}} &= -2\widetilde{\mathcal{L}}_{\mathbf{c}_0}^{-1} \mathcal{GN}(\xi_{\mathbf{k}_1}, \xi_{\mathbf{k}_2}),\\
\zeta_{1,-1} e^{i(\mathbf{k}_1-\mathbf{k}_2)\cdot\mathbf{x}} &= -2\widetilde{\mathcal{L}}_{\mathbf{c}_0}^{-1} \mathcal{GN}(\xi_{\mathbf{k}_1}, \xi_{-\mathbf{k}_2}).
\end{aligned}$$

The suppression of the projection $\mathcal{Q}$ comes from the non-resonance of $2\mathbf{k}_1, 2\mathbf{k}_2, \mathbf{k}_1 \pm \mathbf{k}_2$ with $\pm\mathbf{k}_j$ and we also used the fact

$$\mathcal{GN}(\xi_{\mathbf{k}_j}, \xi_{-\mathbf{k}_j}) = 0, \quad j = 1, 2.$$

Straightforward calculations show that

$$\mathcal{GN}(\xi_{\mathbf{k}_1},\xi_{\mathbf{k}_1}) = \begin{pmatrix} 2il_1(1+\tau_1^2)^{3/2} \\ il_1(1+\tau_1^2) \\ i\tau_1 l_1(1+\tau_1^2) \end{pmatrix} e^{2i\mathbf{k}_1\cdot\mathbf{x}},$$

$$\mathcal{GN}(\xi_{\mathbf{k}_2},\xi_{\mathbf{k}_2}) = \begin{pmatrix} 2il_2(1+\tau_2^2)^{3/2} \\ il_2(1+\tau_2^2) \\ -i\tau_2 l_2(1+\tau_2^2) \end{pmatrix} e^{2i\mathbf{k}_2\cdot\mathbf{x}},$$

$$\begin{aligned} 2\mathcal{GN}(\xi_{\mathbf{k}_1},\xi_{\mathbf{k}_2}) = {} & i(1-\tau_1\tau_2) \begin{pmatrix} l_1\sqrt{1+\tau_1^2}+l_2\sqrt{1+\tau_2^2} \\ l_1+l_2 \\ (\tau_1 l_1 - l_2\tau_2) \end{pmatrix} e^{i(\mathbf{k}_1+\mathbf{k}_2)\cdot\mathbf{x}} \\ & + i\sqrt{1+\tau_1^2}\sqrt{1+\tau_2^2} \begin{pmatrix} l_1\sqrt{1+\tau_1^2}+l_2\sqrt{1+\tau_2^2} \\ 0 \\ 0 \end{pmatrix} e^{i(\mathbf{k}_1+\mathbf{k}_2)\cdot\mathbf{x}}, \end{aligned}$$

$$\begin{aligned} 2\mathcal{GN}(\xi_{\mathbf{k}_1},\xi_{-\mathbf{k}_2}) = {} & i(1-\tau_1\tau_2) \begin{pmatrix} l_1\sqrt{1+\tau_1^2}-l_2\sqrt{1+\tau_2^2} \\ l_1-l_2 \\ (\tau_1 l_1 + l_2\tau_2) \end{pmatrix} e^{i(\mathbf{k}_1-\mathbf{k}_2)\cdot\mathbf{x}} \\ & + i\sqrt{1+\tau_1^2}\sqrt{1+\tau_2^2} \begin{pmatrix} l_1\sqrt{1+\tau_1^2}-l_2\sqrt{1+\tau_2^2} \\ 0 \\ 0 \end{pmatrix} e^{i(\mathbf{k}_1-\mathbf{k}_2)\cdot\mathbf{x}}, \end{aligned}$$

and therefore, we find by using (20) that

$$\begin{aligned} \zeta_{2,0} &= \frac{2l_1^2(1+\tau_1^2)}{D_{2,0}} \begin{pmatrix} 2c_0\sqrt{1+\tau_1^2}D_{1,0}+1+\tau_1^2 \\ c_0 D_{1,0}+2\sqrt{1+\tau_1^2} \\ \tau_1(c_0 D_{1,0}+2\sqrt{1+\tau_1^2}) \end{pmatrix}, \\ \zeta_{0,2} &= \frac{2l_2^2(1+\tau_2^2)}{D_{0,2}} \begin{pmatrix} 2c_0\sqrt{1+\tau_2^2}D_{0,1}+1+\tau_2^2 \\ c_0 D_{0,1}+2\sqrt{1+\tau_2^2} \\ -\tau_2(c_0 D_{0,1}+2\sqrt{1+\tau_2^2}) \end{pmatrix}, \end{aligned} \tag{71}$$

$$\zeta_{1,1} = \frac{L_+}{D_{1,1}} \begin{pmatrix} D_+c_0(l_1+l_2) \\ l_1+l_2 \\ \tau_1 l_1 - \tau_2 l_2 \end{pmatrix} + \frac{1-\tau_1\tau_2}{D_{1,1}} \begin{pmatrix} 6D_+ - 1 \\ D_+c_0(l_1+l_2)^2 \\ D_+c_0(l_1+l_2)(\tau_1 l_1 - \tau_2 l_2) \end{pmatrix},$$

$$\begin{aligned} \zeta_{1,-1} = \frac{L_-}{D_{1,-1}} &\begin{pmatrix} D_-c_0(l_1-l_2) \\ l_1-l_2 \\ \tau_1 l_1 + \tau_2 l_2 \end{pmatrix} \\ &+ \frac{1-\tau_1\tau_2}{D_{1,-1}} \begin{pmatrix} 6D_- - 1 \\ D_-c_0(l_1-l_2)^2 \\ D_-c_0(l_1-l_2)(\tau_1 l_1 + \tau_2 l_2) \end{pmatrix}, \end{aligned} \tag{72}$$

where

$$\begin{aligned} L_+ &= \left(1-\tau_1\tau_2+\sqrt{1+\tau_1^2}\sqrt{1+\tau_2^2}\right)\left(l_1\sqrt{1+\tau_1^2}+l_2\sqrt{1+\tau_2^2}\right), \\ L_- &= \left(1-\tau_1\tau_2+\sqrt{1+\tau_1^2}\sqrt{1+\tau_2^2}\right)\left(l_1\sqrt{1+\tau_1^2}-l_2\sqrt{1+\tau_2^2}\right), \\ D_{1,0} &= 1+\frac{2l_1^2}{3}(1+\tau_1^2), \quad D_{0,1} = 1+\frac{2l_2^2}{3}(1+\tau_2^2), \\ D_{2,0} &= 4l_1^2[(D_{1,0})^2c_0^2-(1+\tau_1^2)], \\ D_{0,2} &= 4l_2^2[(D_{0,1})^2c_0^2-(1+\tau_2^2)], \\ D_+ &= 1+\frac{1}{6}[(l_1+l_2)^2+(l_1\tau_1-l_2\tau_2)^2], \\ D_- &= 1+\frac{1}{6}[(l_1-l_2)^2+(l_1\tau_1+l_2\tau_2)^2], \\ D_{1,1} &= c_0^2(l_1+l_2)^2D_+^2-(l_1+l_2)^2-(l_1\tau_1-l_2\tau_2)^2, \\ D_{1,-1} &= c_0^2(l_1-l_2)^2D_-^2-(l_1-l_2)^2-(l_1\tau_1+l_2\tau_2)^2. \end{aligned}$$

Let us now calculate the leading terms in (65). Let us notice that

$$\langle \xi_{\mathbf{k}_j}, \xi_{\mathbf{k}_j} \rangle = 2(1+\tau_j^2)\Omega,$$

where

$$\Omega = \frac{4\pi^2}{l_1 l_2(\tau_1+\tau_2)}.$$

is the *area of the parallelogram* formed with λ_1 and λ_2 (see the definition of the lattice of periods Γ' in (9)). Now, from (52) and (53) we have the following identities

$$\begin{aligned} \mu\langle \mathcal{G}\xi_{\mathbf{k}_j}, \xi_{\mathbf{k}_j} \rangle &= 2i\mu l_j(1+\tau_j^2)^{3/2}\Omega, \\ \langle w\mathcal{L}^{(1)}\xi_{\mathbf{k}_j}, \xi_{\mathbf{k}_j} \rangle &= 2i(-1)^j w\tau_j l_j(1+\tau_j^2)^{3/2}\Omega \end{aligned}$$

and it is clear that with our non-resonance assumption we have

$$\langle \mathcal{G}\mathcal{N}(X,X), \xi_{\mathbf{k}_j}\rangle = 0.$$

For deriving the principal parts of g_1 and g_2 in (65), we obtain (67) with

$$\begin{aligned} 2il_1(1+\tau_1^2)^{3/2}\Omega a_1 =& \langle 2\mathcal{G}\mathcal{N}(\xi_{-\mathbf{k}_1}, \zeta_{2,0}e^{2i\mathbf{k}_1\cdot\mathbf{x}}), \xi_{\mathbf{k}_1}\rangle, \\ 2il_2(1+\tau_2^2)^{3/2}\Omega b_2 =& \langle 2\mathcal{G}\mathcal{N}(\xi_{-\mathbf{k}_2}, \zeta_{0,2}e^{2i\mathbf{k}_2\cdot\mathbf{x}}), \xi_{\mathbf{k}_2}\rangle, \end{aligned} \tag{73}$$

$$\begin{aligned} &2il_1(1+\tau_1^2)^{3/2}\Omega b_1 = \\ &\quad \langle 2\mathcal{G}\left\{\mathcal{N}(\xi_{\mathbf{k}_2}, \zeta_{1,-1}e^{i(\mathbf{k}_1-\mathbf{k}_2)\cdot\mathbf{x}}) + \mathcal{N}(\xi_{-\mathbf{k}_2}, \zeta_{1,1}e^{i(\mathbf{k}_1+\mathbf{k}_2)\cdot\mathbf{x}})\right\}, \xi_{\mathbf{k}_1}\rangle, \\ &2il_2(1+\tau_2^2)^{3/2}\Omega a_2 = \\ &\quad \langle 2\mathcal{G}\left\{\mathcal{N}(\xi_{\mathbf{k}_1}, \zeta_{1,-1}e^{-i(\mathbf{k}_1-\mathbf{k}_2)\cdot\mathbf{x}}) + \mathcal{N}(\xi_{-\mathbf{k}_1}, \zeta_{1,1}e^{i(\mathbf{k}_1+\mathbf{k}_2)\cdot\mathbf{x}})\right\}, \xi_{\mathbf{k}_2}\rangle. \end{aligned} \tag{74}$$

Solving (67) with respect to μ and w and denoting $|A| = \varepsilon_1$, $|B| = \varepsilon_2$ leads to (68).

Notice that the value $w = 0$ leads to asymmetric waves provided that

$$(a_1 - a_2)(b_1 - b_2) < 0.$$

This particular case gives (the propagation direction is the x_1-axis)

$$\begin{aligned} \varepsilon_2^2 &= \frac{a_1 - a_2}{b_2 - b_1}\varepsilon_1^2 + O(\varepsilon_1^4), \\ \mu &= -\frac{\varepsilon_1^2}{2(b_2 - b_1)}\{(a_1 + a_2)(b_2 - b_1) + (a_1 - a_2)(b_1 + b_2)\} + O(\varepsilon_1^4). \end{aligned}$$

If *the lattice* Γ *has a diamond structure* and we choose the x_1-axis such that $\mathbf{k}_1$ and $\mathbf{k}_2$ are symmetric with respect to this axis, the additional symmetry (66) implies

$$a_1 = b_2, \; a_2 = b_1,$$

and

$$\begin{aligned} \mu =& -\frac{a_1 + a_2}{2}(\varepsilon_1^2 + \varepsilon_2^2) + O(\varepsilon_1^2 + \varepsilon_2^2)^2, \\ w\tau =& (\varepsilon_1^2 - \varepsilon_2^2)\left\{\frac{(a_1 - a_2)}{2} + O(\varepsilon_1^2 + \varepsilon_2^2)\right\}, \end{aligned}$$

where only rational values of the small parameter $w\tau = (r-s)/(r+s)$, $s \leq \sigma$ are allowed, which leads to a restricted choice for the amplitudes ε_1 and ε_2. The special choice $\varepsilon_1 = \varepsilon_2$ gives the symmetrical waves propagating in the x_1-direction as described in [6].

It remains to compute the coefficients a_j and b_j. Since

$$
\begin{aligned}
\langle 2\mathcal{G}\mathcal{N}(\xi_{-\mathbf{k}_1},\zeta_{2,0}e^{2i\mathbf{k}_1\cdot\mathbf{x}}),\xi_{\mathbf{k}_1}\rangle =&\frac{2il_1^3(1+\tau_1^2)^3\Omega}{D_{2,0}}(4c_0D_{1,0}+5\sqrt{1+\tau_1^2}),\\
\langle 2\mathcal{G}\mathcal{N}(\xi_{-\mathbf{k}_2},\zeta_{0,2}e^{2i\mathbf{k}_2\cdot\mathbf{x}}),\xi_{\mathbf{k}_2}\rangle =&\frac{2il_2^3(1+\tau_2^2)^3\Omega}{D_{0,2}}(4c_0D_{0,1}+5\sqrt{1+\tau_2^2}),\\
\langle 2\mathcal{G}\mathcal{N}(\xi_{\mathbf{k}_2},\zeta_{1,-1}e^{i(\mathbf{k}_1-\mathbf{k}_2)\cdot\mathbf{x}}),\xi_{\mathbf{k}_1}\rangle =&\frac{il_1(1+\tau_1^2)^{1/2}\Omega}{D_{1.-1}}\{L_-^2+\\
&+2L_-(1-\tau_1\tau_2)D_-c_0(l_1-l_2)+6(1-\tau_1\tau_2)^2(D_--1)\},\\
\langle 2\mathcal{G}\mathcal{N}(\xi_{-\mathbf{k}_2},\zeta_{1,1}e^{i(\mathbf{k}_1+\mathbf{k}_2)\cdot\mathbf{x}}),\xi_{\mathbf{k}_1}\rangle =&\frac{il_1(1+\tau_1^2)^{1/2}\Omega}{D_{1.1}}\{L_+^2+\\
&+2L_+(1-\tau_1\tau_2)D_+c_0(l_1+l_2)+6(1-\tau_1\tau_2)^2(D_+-1)\},\\
\langle 2\mathcal{G}\mathcal{N}(\xi_{\mathbf{k}_1},\zeta_{1,-1}e^{i(\mathbf{k}_2-\mathbf{k}_1)\cdot\mathbf{x}}),\xi_{\mathbf{k}_2}\rangle =&\frac{il_2(1+\tau_2^2)^{1/2}\Omega}{D_{1.-1}}\{L_-^2+\\
&+2L_-(1-\tau_1\tau_2)D_-c_0(l_1-l_2)+6(1-\tau_1\tau_2)^2(D_--1)\},\\
\langle 2\mathcal{G}\mathcal{N}(\xi_{-\mathbf{k}_1},\zeta_{1,1}e^{i(\mathbf{k}_1+\mathbf{k}_2)\cdot\mathbf{x}}),\xi_{\mathbf{k}_2}\rangle =&\frac{il_2(1+\tau_2^2)^{1/2}\Omega}{D_{1.1}}\{L_+^2+\\
&+2L_+(1-\tau_1\tau_2)D_+c_0(l_1+l_2)+6(1-\tau_1\tau_2)^2(D_+-1)\},
\end{aligned}
$$

we obtain

$$
\begin{aligned}
a_1 =&\frac{l_1^2(1+\tau_1^2)^{3/2}}{D_{2,0}}(4c_0D_{1,0}+5\sqrt{1+\tau_1^2}),\\
b_2 =&\frac{l_2^2(1+\tau_2^2)^{3/2}}{D_{0,2}}(4c_0D_{0,1}+5\sqrt{1+\tau_2^2}),\\
a_2 =&\frac{1}{2(1+\tau_2^2)}\left\{\frac{L_+^2}{D_{1,1}}+\frac{L_-^2}{D_{1,-1}}+2(1-\tau_1\tau_2)c_0(\frac{L_+D_+}{D_{1,1}}(l_1+l_2)+\right.\\
&\left.+\frac{L_-D_-}{D_{1,-1}}(l_1-l_2))+6(1-\tau_1\tau_2)^2(\frac{D_+-1}{D_{1,1}}+\frac{D_--1}{D_{1,-1}})\right\},\\
b_1 =&\frac{1}{2(1+\tau_1^2)}\left\{\frac{L_+^2}{D_{1,1}}+\frac{L_-^2}{D_{1,-1}}+2(1-\tau_1\tau_2)c_0(\frac{L_+D_+}{D_{1,1}}(l_1+l_2)+\right.\\
&\left.+\frac{L_-D_-}{D_{1,-1}}(l_1-l_2))+6(1-\tau_1\tau_2)^2(\frac{D_+-1}{D_{1,1}}+\frac{D_--1}{D_{1,-1}})\right\}.
\end{aligned}
$$

In the *case when the lattice* Γ *has a diamond structure*, and we choose the x_1-axis such that $\mathbf{k}_1$ and $\mathbf{k}_2$ are symmetric with respect to this axis, these formulas become

$$
\begin{aligned}
a_1 = b_2 &= \frac{l^2(1+\tau^2)^{3/2}}{D_{2,0}}(4c_0 D_{1,0} + 5\sqrt{1+\tau^2}), \\
a_2 = b_1 &= \frac{1}{2(c_0^2 D_+^2 - 1)}\left\{4c_0 D_+ \frac{(1-\tau^2)}{\sqrt{1+\tau^2}} + \frac{5+2\tau^2+\tau^4}{1+\tau^2}\right\} - \frac{(1-\tau^2)^2}{2(1+\tau^2)},
\end{aligned}
$$

where

$$
\begin{aligned}
D_{1,0} = D_{0,1} &= 1 + \frac{2l^2}{3}(1+\tau^2), \\
D_{2,0} = D_{0,2} &= 4l^2(c_0^2 D_{1,0}^2 - (1+\tau^2)), \\
D_+ &= 1 + \frac{2l^2}{3}.
\end{aligned}
$$

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