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Boele Braaksma[*], Gérard Iooss[†]and Laurent Stolovitch [‡]

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Abstract

We consider the steady Swift - Hohenberg partial differential equation. It is a one-parameter family of PDE on the plane, modeling for example Rayleigh - Bénard convection. For values of the parameter near its critical value, we look for small solutions, quasiperiodic in all directions of the plane and which are invariant under rotations of angle π/q, $q \geq 4$. We solve an unusual small divisor problem, and prove the existence of solutions for small parameter values.

Contents

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1   Introduction

In the present paper we study the existence of a special kind of **stationary solutions** (i.e. independent of t), bifurcating from 0 (i.e. tending towards zero when the parameter λ tends towards 0), called **quasipatterns** of the 2-dimensional Swift-Hohenberg PDE

$$\frac{\partial u}{\partial t} = \lambda u - (1 + \Delta)^2 u - u^3 \tag{1}$$

where u is the unknown real-valued function on some subset of $\mathbb{R}^+ \times \mathbb{R}^2$, $\Delta := \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right)$ and λ is a parameter. These are two-dimensional patterns that have no translation symmetry and are quasiperiodic in any spatial direction.

Mathematical existence of quasipatterns is one of the outstanding problems in pattern formation theory. To our knowledge, hereafter is the first proof of existence of such quasipatterns of a PDE. Quasipatterns were discovered in nonlinear pattern-forming systems in the Faraday wave experiment [BCM92, EF94], in which a layer of fluid is subjected to vertical oscillations. Since their discovery, they have also been found in nonlinear optical systems, shaken convection and in liquid crystals (see references in [AG12]) . In spite of the lack of translation symmetry (in contrast to periodic patterns), the solutions are π/q-rotation invariant for some integer q (most often observed, $2q$ is 8, 10 or 12).

In many of these experiments, the domain is large compared to the size of the pattern, and the boundaries appear to have little effect. Furthermore, the pattern is usually formed in two directions (x_1 and x_2), while the third direction (z) plays little role. Mathematical models of the experiments are therefore often posed with two unbounded directions, and the basic symmetry of the problem is $E(2)$, the Euclidean group of rotations, translations and reflections of the (x_1, x_2) plane.

The above model equation is the simplest pattern-forming PDE, and is extremely successful for describing primary bifurcations (the first symmetry breaking) of hydrodynamical instability problems such as the Rayleigh - Bénard convection. Its essential properties are that

i) the system is invariant under the group $E(2)$;

ii) the instability occurs for a certain critical value of the parameter (here $\lambda = 0$) for which critical modes are given by wave vectors sitting on a circle of *non zero radius* (here the unit circle);

iii) the linear part is selfadjoint and contains the main derivatives.

In contrast to periodic patterns, quasipatterns do not fit into any spatially periodic domain and have Fourier expansions with wavevectors that live on a *quasilattice* (defined below). At the onset of pattern formation, the critical modes have zero growth rate but there are other modes on the quasilattice that have growth rates arbitrarily close to zero, and techniques that are used for periodic patterns cannot be applied. These small growth rates appear as *small divisors*, as seen below, and correspond at criticality ($\lambda = 0$) to the
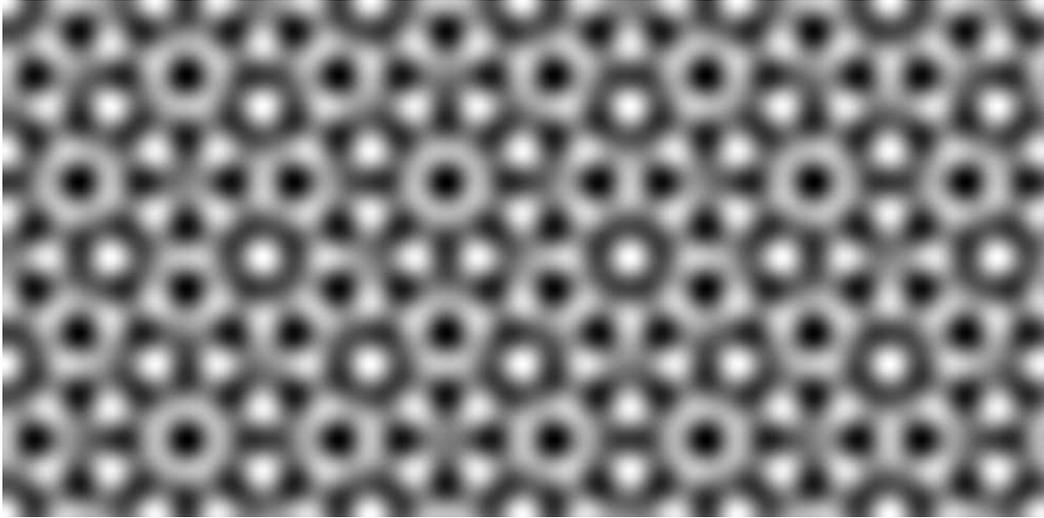
Figure 1: Example 8-fold quasipattern after [IR10]. This is an approximate solution of the steady Swift–Hohenberg equation (2) with $\lambda = 0.1$, computed by using Newton iteration to find an equilibrium solution truncated to wavenumbers satisfying $|\mathbf{k}| \leq \sqrt{5}$ and to the quasilattice Γ_{27} obtained with $N_{\mathbf{k}} \leq 27$.

fact that for the linearized operator at the origin (here $-(1 + \Delta)^2$), **the 0 eigenvalue is not isolated in the spectrum, appearing as part of the continuous spectrum**.

If a formal computation in powers of $\sqrt{\lambda}$ is performed in this case without regard to its validity, this results in a possibly divergent power series in the parameter, and this approach does not lead to the existence of quasipattern solutions, but instead to approximate solutions up to an exponentially small error [IR10].

In this work, **we prove the existence of quasipattern solutions of the steady Swift-Hohenberg equations**. Our result rests on the article [IR10] by G. Iooss and A.M. Rucklidge which settle the mathematical foundation of the problem such as the formulation of suitable functions spaces. We refer to the articles of Rucklidge [RR03, RS07] and Iooss-Rucklidge[IR10] for physical motivation as well as for the bibliography.

1.1   Main result and sketch of the proof

The problem is to find a special kind of solutions defined on $\mathbb{R}^2$ of the steady Swift-Hohenberg equation

$$(1 + \Delta)^2 U - \lambda U + U^3 = 0. \tag{2}$$

The parameter λ is supposed to be real and small in absolute value. The solutions we are interested in should tend towards zero as the parameter goes to zero.

We study equation (2) for $\lambda > 0$. Namely, let $Q = 2q$ be an even integer and let $\mathbf{k}_j = \exp \frac{i\pi(j-1)}{q}$, $j = 1, \ldots, 2q$ be the $2q$ unit vectors of the plane, identified with roots of unity. Let Γ be the set of linear combinations of vectors $\mathbf{k}_j$ with nonnegative integer coefficients. We look for the existence of a (nonzero) π/q-rotation invariant solution of the

form

$$U(\mathbf{x}) = \sum_{\mathbf{k} \in \Gamma} u^{(\mathbf{k})} e^{i\mathbf{k}.\mathbf{x}}$$

which belongs to a "Sobolev" like space $\mathcal{H}_s$, $s \geq 0$:

$$\|U\|_s^2 := \sum_{\mathbf{k} \in \Gamma} |u^{(\mathbf{k})}|^2 (1 + N_{\mathbf{k}}^2)^s < +\infty.$$

The natural number $N_{\mathbf{k}}$ denotes the minimal length of the linear combinations of the $\mathbf{k}_j$'s needed to reach $\mathbf{k}$.

We then show that such a solution exists indeed, for small positive parameters λ:

Theorem 1. *For any $q \geq 4$ and any $s > q/2$, there exists $\lambda_0 > 0$, such that the steady Swift-Hohenberg equation for $0 < \lambda < \lambda_0$, admits a quasipattern solution in $\mathcal{H}_s$, invariant under rotations of angle π/q. Moreover the asymptotic expansion of this solution is given by the formal series given in [IR10].*

One of the main difficulty is that the linearized operator at $W = 0$, has an unbounded inverse. Indeed, it is easy to show that the eigenvalues of $(1 + \Delta)^2$ in $\mathcal{H}_s$ are $(1 - |\mathbf{k}|^2)^2$ where $\mathbf{k} \in \Gamma$. These numbers accumulate in 0. It creates a *small divisor* problem, such that if $\lambda > 0$ nothing can be said *a priori* about $(1 + \Delta)^2 - \lambda \mathrm{Id}$. We use the first terms of the asymptotic expansion of the solution and change the unknown as $U = U_\epsilon + \epsilon^4 W$ and $\lambda = \epsilon^2(\lambda_2 + \lambda_4 \epsilon^2)$ for some well chosen U_ϵ and positive λ_2. Let $\mathcal{L}_\epsilon$ be the linear part at $W = 0$ of the nonlinear equation so obtained. For $\epsilon = 0$, the operator $\mathcal{L}_0 = (1 + \Delta)^2$ is a positive selfadjoint operator in $\mathcal{H}_s$. It is bounded from $\mathcal{H}_{s+4}$ into $\mathcal{H}_s$, but it is not Fredholm, since its range is not closed. Its spectrum is an *essential spectrum* filling the half line $[0, \infty)$. The set of eigenvalues is dense in the spectrum. The linear operator $\mathcal{L}_\epsilon$ is the sum of $\mathcal{L}_0$ and a bounded operator (multiplication by a small function $O(\epsilon^2)$) selfadjoint in $\mathcal{H}_0$. If the range of $\mathcal{L}_\epsilon$ were closed, a usual way to estimate the inverse of the selfadjoint operator $\mathcal{L}_\epsilon$ in $\mathcal{H}_0$, would be to estimate the distance from 0 to its numerical range (see [Kat95]) (containing the spectrum). Such estimate as

$$\langle \mathcal{L}_\epsilon \mathcal{U}, \mathcal{U} \rangle_0 \geq c\epsilon^2 \|U\|_0^2$$

for a certain constant $c > 0$, cannot be proved here. So we need to study the linear operator in more details.

We show that there exists an orthogonal decomposition (depending on ϵ) of the space $\mathcal{H}_s = E_0 \oplus E_1 \oplus E_2$, $s > q/2$, such that the solution of the equation $\mathcal{L}_\epsilon U = f$ in $\mathcal{H}_s$ can be computed and estimated from its E_2-component U_2. The latter is solution of a linear equation $\mathcal{L}_\epsilon^{(2)} U_2 = \tilde{f}$. The main part (with respect to powers of ϵ) of that operator is an operator Λ_ϵ. When restricted to rotation invariant elements, it has a block-diagonal structure of fixed finite dimensional blocks. Then, it is possible to estimate all eigenvalues of these selfadjoint blocks. These eigenvalues have the form

$$(|\mathbf{k}|^2 - 1)^2 + 3\epsilon^2 + O(\epsilon^4)$$

for $\mathbf{k} \in \Gamma$. One of the main feature is that **they do not accumulate at the origin**, and *despite of the small divisor problem* arising for $\epsilon = 0$, we are able to give an upper bound

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in $\mathcal{H}_0$ of the inverse of Λ_ϵ of order $1/\lambda$ for nonzero λ. Then we extend this estimate in $\mathcal{H}_s$ for $s > q/2$. Finally, we use a variant of the implicit function theorem to conclude to the existence of quasipatterns solutions.

Remark 2. *If the coefficient (3) of ϵ^2 were negative, then the operator could have again small divisors. Then the proof of the existence would have been much more involved. At that point, solving the nonlinear problem in U_2 would have required the use of a version of Nash-Moser theorem such as those developed by J. Bourgain, W. Craig, M. Berti and al. (see for instance [Bou95, Cra00, BBP10, Ber07]). Their main feature is the use of the good separation property of the "singular sites" of the main linear operator. Indeed, we can show that the operator $(1 + \Delta)^2$ does have this property.*

Remark 3. *For $\lambda < 0$ the solution $U = 0$ is isolated in an open ball of radius $\sqrt{|\lambda|}$, as it is easily deduced from the following estimate*

$$||[(1 + \Delta)^2 - \lambda]^{-1}|| \le |\lambda|^{-1}$$

which holds in $\mathcal{H}_s$.

2 Analysis of the main part of the differential

2.1 Setting

In this section, we recall and improve some of the properties of the function spaces we use, as defined in [IR10].

Let $Q = 2q$, $q \ge 4$ being an integer. Let us define the unit *wave vectors* (identifying $\mathbb{C}$ with $\mathbb{R}^2$)

$$\mathbf{k}_j := e^{i\pi \frac{j-1}{q}}, \quad j = 1, \ldots, 2q. \tag{3}$$

We define the **quasilattice** $\Gamma \subset \mathbb{R}^2$ to be the set of points spanned by (nonnegative) integer linear combinations of the k_j's :

$$\mathbf{k}_m = \sum_{j=1}^{2q} m_j \mathbf{k}_j, \quad \mathbf{m} = (m_1, \ldots, m_{2q}) \in \mathbb{N}^{2q}. \tag{4}$$

We have $\mathbf{k}_j = -\mathbf{k}_{j+q}$. Hence, we can write

$$\mathbf{k}_m = \sum_{j=1}^{q} m'_j \mathbf{k}_j,$$

where, $m'_j := m_j - m_{j+q}$ belongs to $\mathbb{Z}$. Thus,

$$|\mathbf{k}_m|^2 = \sum_{i,j=1}^{q} m'_i m'_j < \mathbf{k}_i, \mathbf{k}_j > .$$

We then define, for any $\mathbf{m} \in \mathbb{N}^{2q}$ and $\mathbf{k} \in \Gamma$,

$$|\mathbf{m}| := \sum_j m_j, \quad N_\mathbf{k} := \min\{|\mathbf{m}| : \mathbf{k} = \mathbf{k}_m\}.$$

We have

Lemma 4. *[IR10][Lemma 4.1] For any* $\mathbf{k} \in \Gamma$, *we have :*

-
$$N_{\mathbf{k}+\mathbf{k}'} \leq N_{\mathbf{k}} + N_{\mathbf{k}'}, \quad N_{-\mathbf{k}} = N_{\mathbf{k}} \tag{5}$$

-
$$|\mathbf{k}| \leq N_{\mathbf{k}} \tag{6}$$

-
$$\text{card}\{\mathbf{k} \mid N_{\mathbf{k}} = N\} \leq c_1(q)N^{q-1} \tag{7}$$

for some constant $c_1(q)$ *depending only on* q.

As in [IR10], we use function spaces defined as

$$\mathcal{H}_s = \left\{ W = \sum_{\mathbf{k}\in\Gamma} W^{(\mathbf{k})}e^{i\mathbf{k}\cdot\mathbf{x}}; \ \|W\|_s^2 = \sum_{\mathbf{k}\in\Gamma}(1+N_{\mathbf{k}}{}^2)^s|W^{(\mathbf{k})}|^2 < \infty \right\}, \tag{8}$$

which are Hilbert spaces with the scalar product

$$\langle W, V\rangle_s = \sum_{\mathbf{k}\in\Gamma}(1+N_{\mathbf{k}}{}^2)^s W^{(\mathbf{k})}\overline{V}^{(\mathbf{k})}. \tag{9}$$

Lemma 5. *For* $s > q/2$, *for any* $U \in \mathcal{H}_s$ *and any* $V \in \mathcal{H}_0$, *we have*

$$\|UV\|_0 \leq c_s\|U\|_s\|V\|_0$$

for a certain constant $c_s > 0$.

Proof. Using Cauchy-Schwarz inequality, we have

$$
\begin{aligned}
\|UV\|_0^2 &\leq \sum_{\mathbf{k}\in\Gamma}\left|\sum_{\mathbf{m}\in\Gamma}U^{(\mathbf{m})}V^{(\mathbf{k}-\mathbf{m})}\right|^2 \\
&\leq \sum_{\mathbf{k}\in\Gamma}\left(\sum_{\mathbf{m}\in\Gamma}|U^{(\mathbf{m})}|^2(1+N_{\mathbf{m}}^2)^s\right)\left(\sum_{\mathbf{m}'\in\Gamma}|V^{(\mathbf{k}-\mathbf{m}')}|^2(1+N_{\mathbf{m}'}^2)^{-s}\right) \\
&\leq \|U\|_s^2\left(\sum_{\mathbf{k}\in\Gamma}|V^{(\mathbf{k}-\mathbf{m}')}|^2\right)R_s
\end{aligned}
$$

where $R_s := \sum_{\mathbf{m}'\in\Gamma}(1+N_{\mathbf{m}'}^2)^{-s}$. This last sum converges if $s > q/2$. Indeed, according to [IR10][(24)], $\text{card}\{\mathbf{k} \in \Gamma \mid N_{\mathbf{k}} = N\} \leq c(q)N^{q-1}$ for some constant $c(q)$. Hence $\|UV\|_0^2 \leq \|U\|_s^2\|V\|_0^2 R_s$. ∎

Lemma 6. *(Moser-Nirenberg type inequality) Let* $s, s' > q/2$ *and let* $U, V \in \mathcal{H}_s \cap \mathcal{H}_{s'}$. *Then,*

$$\|UV\|_s \leq C(s,s')(\|U\|_s\|V\|_{s'} + \|U\|_{s'}\|V\|_s) \tag{10}$$

for some positive constant $C(s,s')$ *that depends only on* s *and* s'. *For* $\ell \geq 0$ *and* $s > \ell+q/2$, $\mathcal{H}_s$ *is continuously embedded into* $\mathcal{C}^\ell$

We postpone the proof to the appendix.

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2.2 Formal computation

Let us look for formal solutions of the steady Swift–Hohenberg equation

$$\lambda U - (1 + \Delta)^2 U - U^3 = 0, \tag{11}$$

We characterise the functions of interest by their Fourier coefficients on the *quasilattice* Γ generated by the $2q$ equally spaced unit vectors $\mathbf{k}_j$ (see (4)):

$$U(\mathbf{x}) = \sum_{\mathbf{k} \in \Gamma} u^{(\mathbf{k})} e^{i\mathbf{k}.\mathbf{x}}, \quad \mathbf{x} = (x_1, x_2) \in \mathbb{R}^2.$$

We seek a non trivial solution, bifurcating from 0, parameterized by ϵ, and which is invariant under rotations by π/q. As it is shown for example in [IR10], a formal computation with identification of orders in ϵ leads to

$$U(x_1, x_2) = \epsilon u_0(x_1, x_2) + \epsilon^3 u_1(x_1, x_2) + \dots \qquad \lambda = \epsilon^2 \lambda_2 + \epsilon^4 \lambda_4 + \dots \tag{12}$$

and gives at order $\mathcal{O}(\epsilon)$

$$0 = (1 + \Delta)^2 u_0. \tag{13}$$

We take as our basic solution a quasipattern that is invariant under rotations by π/q:

$$u_0 = \sum_{j=1}^{2q} e^{i\mathbf{k}_j \cdot \mathbf{x}}. \tag{14}$$

At order $\mathcal{O}(\epsilon^3)$ we have

$$\lambda_2 u_0 - u_0^3 = (1 + \Delta)^2 u_1. \tag{15}$$

In order to solve this equation for u_1, we must impose a solvability condition, namely that the coefficients of $e^{i\mathbf{k}_j \cdot \mathbf{x}}$, $j = 1, ..., 2q$ on the left hand side of this equation must be zero. Because of the invariance under rotations by π/q, it is sufficient to cancel the coefficient of $e^{i\mathbf{k}_1 \cdot \mathbf{x}}$. For the computation of the coefficient, we need the following property

Property: *If we have*

$$\mathbf{k}_j + \mathbf{k}_l + \mathbf{k}_r + \mathbf{k}_s = 0 \text{ for } j, l, r, s \in \{1, 2q\}$$

then either $\mathbf{k}_j + \mathbf{k}_l = 0$, *or* $\mathbf{k}_j + \mathbf{k}_r$, *or* $\mathbf{k}_j + \mathbf{k}_s = 0$ *(there are two pairs of opposite unit vectors).*

Proof. Since there are 4 unit vectors on the unit circle, we can assume without restriction, that $\mathbf{k}_j$ and $\mathbf{k}_l$ make an angle $2\theta \leq \pi/2$. Then $|\mathbf{k}_j + \mathbf{k}_l| = 2\cos\theta \geq \sqrt{2}$. It results that $|\mathbf{k}_r + \mathbf{k}_s| = 2\cos\theta$ with $\mathbf{k}_r$ and $\mathbf{k}_s$ symmetric with respect to the direction of the bissectrix of $(\mathbf{k}_j, \mathbf{k}_l)$, making the same angle as $\mathbf{k}_j$ and $\mathbf{k}_l$ with the bissectrix. So $\{\mathbf{k}_r, \mathbf{k}_s\}$ is the symmetric with respect to 0 of $\{\mathbf{k}_j, \mathbf{k}_l\}$. ∎

This yields

$$\lambda_2 = 3(2q - 1) \tag{16}$$

which is strictly positive. Moreover

$$u_1 = \sum_{\mathbf{k} \in \Gamma, N_{\mathbf{k}} \neq 1, N_{\mathbf{k}} \leq 3} \alpha_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{x}}, \tag{17}$$

$$\alpha_{3\mathbf{k}_j} = -1/64, \quad \alpha_{2\mathbf{k}_j + \mathbf{k}_l} = -\frac{3}{(1 - |2\mathbf{k}_j + \mathbf{k}_l|^2)^2}, \quad \mathbf{k}_j + \mathbf{k}_l \neq 0,$$

$$\alpha_{\mathbf{k}_j + \mathbf{k}_l + \mathbf{k}_r} = -\frac{6}{(1 - |\mathbf{k}_j + \mathbf{k}_l + \mathbf{k}_r|^2)^2}, \quad j \neq l \neq r \neq j,$$

$$\mathbf{k}_j + \mathbf{k}_l \neq 0, \mathbf{k}_j + \mathbf{k}_r \neq 0, \mathbf{k}_r + \mathbf{k}_l \neq 0.$$

We notice that for any $\mathbf{k}$, $\alpha_{\mathbf{k}} < 0$ in u_1. At order $\mathcal{O}(\epsilon^5)$ we have

$$\lambda_4 u_0 + \lambda_2 u_1 - 3u_0^2 u_1 = (1 + \Delta)^2 u_2. \tag{18}$$

The solvability condition gives λ_4 equal to the coefficient of $e^{i\mathbf{k}_1 \cdot \mathbf{x}}$ in $3u_0^2 u_1$. From the expression (17) of u_1

$$\lambda_4 = \sum_{\mathbf{k}_j + \mathbf{k}_l + \mathbf{k} = \mathbf{k}_1, \ N_{\mathbf{k}} = 3} \alpha_{\mathbf{k}}$$

where all coefficients $\alpha_{\mathbf{k}}$ are negative, it results that

$$\lambda_4 < 0. \tag{19}$$

2.3 Formulation of the problem

Let us define the new unknown function W in rewriting (12) as:

$$\begin{aligned} U &= U_\epsilon + \epsilon^4 W \\ U_\epsilon &= \epsilon u_0 + \epsilon^3 u_1 + \epsilon^5 u_2 \\ \lambda_\epsilon &= \epsilon^2 \lambda_2 + \epsilon^4 \lambda_4 \end{aligned} \tag{20}$$

where u_0, u_1, u_2, λ_2, λ_4, are as above. Given a particular (small) positive value of λ, we get ϵ^2 by the implicit function theorem, and since $\lambda_2 > 0$, we obtain a unique positive ϵ. All the corrections are in W. The aim is to show that the quasi-periodic function W exists and is small as ϵ tends towards 0. By construction we have

$$\lambda_\epsilon U_\epsilon - (1 + \Delta)^2 U_\epsilon - U_\epsilon^3 =: -\epsilon^7 f_\epsilon$$

where f_ϵ is quasi-periodic, of order $\mathcal{O}(1)$ with a finite expansion, and is function of ϵ^2. After substituting (20) into the PDE (11), we obtain an equation of the form

$$\mathcal{F}(\epsilon, W) = 0,$$

with

$$\mathcal{F}(\epsilon, W) =: \mathcal{L}_\epsilon W + \epsilon^3 f_\epsilon + 3\epsilon^4 U_\epsilon W^2 + \epsilon^8 W^3, \tag{21}$$

where

$$\begin{aligned} \mathcal{L}_\epsilon &= (1 + \Delta)^2 - \lambda_\epsilon + 3U_\epsilon^2 =: \mathfrak{L}_\epsilon + \epsilon^6 \mathcal{P}_\epsilon, \tag{22} \\ \mathfrak{L}_\epsilon &= (1 + \Delta)^2 + \epsilon^2 a + \epsilon^4 b, \tag{23} \\ a &= 3u_0^2 - \lambda_2, \quad b = 6u_0 u_1 - \lambda_4, \\ \mathcal{P}_\epsilon &= 6u_0 u_2 + 3(u_1 + \epsilon^2 u_2)^2. \end{aligned}$$

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Remark 7. *The degree of truncation, that is the degree in ϵ in the expansion of U_ϵ, is chosen so that the power of ϵ in front of both W^2 and f_ϵ are greater than 2. This is crucial for the very last step of the proof.*

It is clear that the operator $\mathcal{P}_\epsilon$ is an operator bounded in any $\mathcal{H}_r$, $r \geq 0$ uniformly bounded in ϵ, for $\epsilon \leq \epsilon_0$.

A nice property of the operator $\mathfrak{L}_\epsilon$ is that the averages a_0 and b_0 of a and b are strictly > 0. Indeed, we have for any q

$$a_0 = 3, \quad b_0 = -\lambda_4 > 0.$$

For a_0 this results from (16) and a simple examination of u_0^2 the average of which is $2q$, and for b_0 we observe that the average of $u_0 u_1$ is 0, due to the form of u_1.

Assume that we could prove that $\mathcal{L}_\epsilon^{-1}$ is $\mathcal{O}(\epsilon^{-2})$. Then, provided that we are in $\mathcal{H}_s, s > q/2$ which is a Banach algebra, we should get from (21)

$$W = \mathcal{O}(\epsilon) + \mathcal{O}(\epsilon^2 \|W\|^2)$$

The standard implicit function theorem then would allow to conclude and to get $W = \mathcal{O}(\epsilon)$.

In fact, *it is not expected that the operator $\mathcal{L}_\epsilon$ has a bounded inverse, due to the small divisor problem mentioned in section 1.1.* Notice in particular that it is shown in [IR10] that there exists $c > 0$ such that for any $\mathbf{k} \in \Gamma \backslash \{\mathbf{k}_j; j = 1, 2, ..., 2q\}$

$$\left| \|\mathbf{k}\|^2 - 1 \right| \geq \frac{c}{N_{\mathbf{k}}^{2l_0}}, \tag{24}$$

where $l_0 + 1$ is the order of the algebraic number $\omega = 2 \cos \pi/q$. This estimate is similar to the Siegel's diophantine condition for linearization of vector fields [Arn80].

This lower bound shows that the inverse of $\mathfrak{L}_0$ on the orthogonal complement of its kernel is an unbounded operator in $\mathcal{H}_s$, only bounded from $\mathcal{H}_s$ to $\mathcal{H}_{s-4l_0}$. In other words, 0 *belongs to the continuous spectrum of* $\mathfrak{L}_0$ and the main difficulty to be solved below is to find a bound for the inverse $\mathcal{L}_\epsilon^{-1}$ for small values of ϵ. Notice that $\mathcal{L}_\epsilon$ is selfadjoint in $\mathcal{H}_0$ but not in $\mathcal{H}_s$ for $s > 0$. It is tempting to work on its small (real) eigenvalues to obtain a bound of its inverse. However, we are in infinite dimensions, so the spectrum does not contain only eigenvalues, and an option would be to truncate the space to functions with finite Fourier expansions (with $\mathbf{k}$ such that $N_{\mathbf{k}} \leq N$). Since our method consists in reducing the study to an operator in a smaller space, it is preferable to use the eigenvalues later, on the reduced operator.

2.4 Splitting of the space and first reduction of the problem

Let us split the space $\mathcal{H}_s$ into three mutually othogonal (in any $\mathcal{H}_s$) subspaces. We define

$$E_0 = \left\{ W = \sum_{\mathbf{k} \in \Gamma} W^{(\mathbf{k})} e^{i\mathbf{k} \cdot \mathbf{x}} \in \mathcal{H}_s; \|\mathbf{k}\|^2 - 1|^2 \geq \delta^2, \text{ and } |\mathbf{k} - \mathbf{k}_j| > \delta_1, , j \in \{1, 2q\} \right\},$$

$$E_1 = \left\{ W = \sum_{\mathbf{k} \in \Gamma} W^{(\mathbf{k})} e^{i\mathbf{k} \cdot \mathbf{x}} \in \mathcal{H}_s; \mathbf{k} \in \sigma_1 \right\},$$

$$E_2 = \left\{ W = \sum_{\mathbf{k} \in \Gamma} W^{(\mathbf{k})} e^{i\mathbf{k} \cdot \mathbf{x}} \in \mathcal{H}_s; \exists j \in \{1, ..2q\} \text{ such that } \mathbf{k} \in \sigma_{2,j} \right\},$$
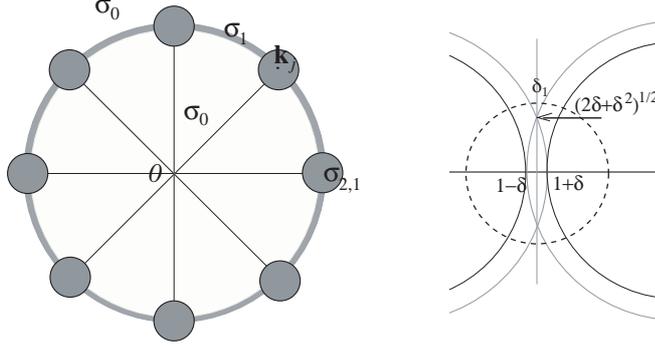
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Figure 2: Division of the Fourier spectrum into σ_0, σ_1 and σ_2.

where (see figure 2)

$$\sigma_1 = \{\mathbf{k} \in \Gamma;\ ||\mathbf{k}|^2 - 1|^2 < \delta^2 \text{ and } |\mathbf{k} - \mathbf{k}_j| > \delta_1 = \sqrt{3\delta}, j \in \{1, 2q\}\},$$
$$\sigma_{2,j} = \{\mathbf{k} \in \Gamma;\ |\mathbf{k} - \mathbf{k}_j| \leq \delta_1\},\ \sigma_2 = \cup_{j=1}^{2q}\sigma_{2,j}.$$

In figure 2 the annulus centered at the unit circle has a thickness 2δ and the little discs should have a radius $\delta_1 = \sqrt{3\delta}$, such that the intersection of the shaded area with the shifted one, centered at the point $(2\mathbf{k}_1, 0)$ is, for δ small enough, reduced to the disc centered at $\mathbf{k}_1$. In the sequel, *we choose $\delta = C\epsilon^{1/2}$ with C large enough* and

$$\delta_1 = \epsilon^{1/4}\sqrt{3C}$$

hence $2\delta + \delta^2 < \delta_1^2$ is verified for $\epsilon^{1/2} < 1/C$ and the intersection $\sigma_1 \cap \{\sigma_1 + 2\mathbf{k}_1\}$ is empty (*hint:* solve $\delta_1^2 = (1 + \delta)^2 - 1$ for intersecting the circle centered in 0, of radius $1 + \delta$ with the line of abscissa 1 parallel to y axis).

This leads to (see figure 3)

$$\sigma_1 \cap \{\sigma_1 + \mathbf{k}_j + \mathbf{k}_l,\ j, l = 1, ..2q,\ \ \mathbf{k}_j + \mathbf{k}_l \neq 0\} = \varnothing.$$

The subspaces E_l are closed as intersections of closed subspaces (kernel of certain coefficients, still continuous functionnals here) and we have the orthogonal decomposition

$$\mathcal{H}_s = E_0 \overset{\perp}{\oplus} E_1 \overset{\perp}{\oplus} E_2.$$

The orthogonal projections associated with this decomposition are denoted by P_0, P_1, P_2. We also notice that the *multiplication operator by a function* having a *finite Fourier expansion* with wave vectors in Γ is a *bounded* linear operator in $\mathcal{H}_r$ for any $r \geq 0$. Indeed a finite Fourier expansion belongs to $\mathcal{H}_s$ for any s, and Lemmas 5 and 6 apply.

2.5 Reduction to the subspace E_2

The aim here is to solve with respect to $U \in \mathcal{H}_s$ the equation

$$\mathcal{L}_\epsilon U = f, \tag{25}$$

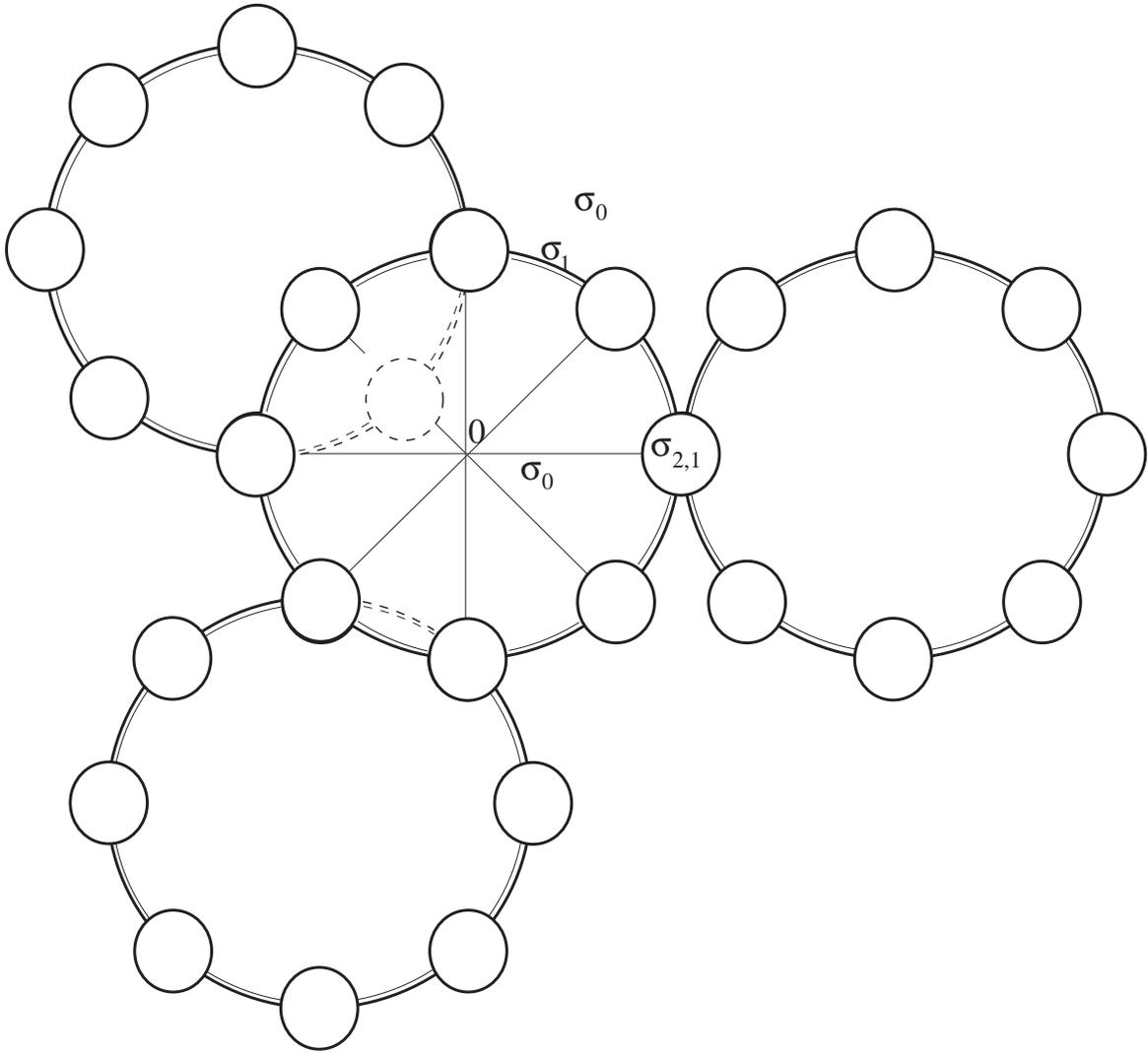where $\mathcal{L}_\epsilon$ is defined in (22) and where $f \in \mathcal{H}_s$ is given.

Figure 3: Empty intersections of σ_1 with σ_1 shifted by $\mathbf{k}_j + \mathbf{k}_l$

Remark 8. *We denote by $\mathcal{L}_0$ the operator $\mathcal{L}_{0,0} = \mathcal{L}_{0,W} = (1 + \Delta)^2 = \mathfrak{L}_0$. For being more explicit in formulae, we denote by $P_j \mathcal{L}_0 P_j$ the restriction of $\mathcal{L}_0$ to the invariant subspace $E_j, j = 0, 1, 2$.*

We decompose this equation on the subspaces E_0, E_1, E_2, which gives, after noticing that the subspaces E_0, E_1, E_2 are invariant under $\mathcal{L}_0$, and that $\mathcal{L}_0|_{E_1}$, $\mathcal{L}_0|_{E_2}$ are bounded operators, and $P_j \mathcal{L}_0 P_0 = 0$ for $j \in \{1, 2\}$,

$$P_0 \mathcal{L}_\epsilon U_0 + P_0 \{(\epsilon^2 a + \epsilon^4 b + \epsilon^6 \mathcal{P}_\epsilon)(U_1 + U_2)\} = f_0 \tag{26}$$

$$P_1 \mathcal{L}_\epsilon U_1 + P_1 \{(\epsilon^2 a + \epsilon^4 b + \epsilon^6 \mathcal{P}_\epsilon)(U_0 + U_2)\} = f_1 \tag{27}$$

$$P_2 \mathcal{L}_\epsilon U_2 + P_2 \{(\epsilon^2 a + \epsilon^4 b + \epsilon^6 \mathcal{P}_\epsilon)(U_0 + U_1)\} = f_2 \tag{28}$$

where $f_j = P_j f, j = 0, 1, 2$.

We have the following

Lemma 9. *Let fix $S \geq 0$ and choose s such that $0 \leq s \leq S$. Then, there exists $\epsilon_0 > 0$, such that choosing C large enough in the definition of δ, $\epsilon \leq \epsilon_0$, the equation $\mathcal{L}_\epsilon U = f$ in $\mathcal{H}_s$ reduces to*

$$\mathcal{L}_\epsilon^{(2)} U_2 = \mathcal{Q}(\epsilon) f$$

where $U_2 = P_2 U$,

$$
\begin{aligned}
\mathcal{L}_\epsilon^{(2)} &= \Lambda_\epsilon + \epsilon^4 R_\epsilon \\
\Lambda_\epsilon U_2 &= P_2(\mathcal{L}_0 + \epsilon^2 a) U_2 \\
R_\epsilon U_2 &= P_2 \{b U_2 - \widetilde{a}(P_0 \mathcal{L}_0 P_0)^{-1} P_0(\widetilde{a} U_2)\} + O(\epsilon^2) U_2,
\end{aligned} \tag{29}
$$

Here $O(\epsilon^2)$ denotes a bounded linear operator in $\mathcal{H}_s$ which norm is bounded by $c\epsilon^2$. The components $U_0 := P_0 U$ and $U_1 := P_1 U$ are functions of U_2 and f and satisfy the following inequalities :

$$
\begin{aligned}
\|U_0\|_s &\leq c\epsilon^2 \|U_2\|_s + \frac{c}{\epsilon} \|(P_0 + P_1) f\|_s \\
\|U_1\|_s &\leq c\epsilon^4 \|U_2\|_s + \frac{c}{\epsilon^2} \|(\epsilon P_0 + P_1) f\|_s
\end{aligned}
$$

where $\widetilde{a} = a - a_0$, for a certain $c > 0$, only depending on s, and R_ϵ and $\mathcal{Q}(\epsilon)$ are bounded linear operators in $\mathcal{H}_s$, depending smoothly on ϵ.

We need the following Lemma

Lemma 10. *For ϵ small enough, we have*

$$\widetilde{a} U_1 \in E_0, \quad P_1(a U_2) = 0, \quad P_1(b U_2) = 0, \tag{30}$$

where $\widetilde{a}$ is the oscillating part of a :

$$\widetilde{a} = a - a_0.$$

Moreover, the Fourier spectra of $P_0(a U_2)$ and $P_0(b U_2)$ are at a distance of order 1 of the unit circle.

By distance of order 1 of the circle, we mean a strictly positive distance as ϵ tends to 0, in the decomposition into subspaces E_j.

Proof. We need to prove that i) for $\mathbf{k} \in \sigma_1$, then $\mathbf{k} + \mathbf{k_m} \notin \sigma_1 \cup \sigma_2$ for $\mathbf{k_m} \neq 0, |\mathbf{m}| = 2$, and ii) for $\mathbf{k} \in \sigma_2$, then $\mathbf{k} + \mathbf{k_m} \notin \sigma_1$ with $|\mathbf{m}| = 2$ or 4.

For showing this, let us first observe that the intersections of the unit circle with all circles of radius 1, centered at $\mathbf{k_m}$, with $\mathbf{k_m} \neq 0$, $|\mathbf{m}| = 2$, are exactly the $2q$ points $\mathbf{k}_r$, $r = 1, ..., 2q$. Let us define the *region* $\mathfrak{a}_0$ (shaded region in figure 2 (a)) defined by

$$\mathfrak{a}_0 = \sigma_1 \cup_{j \in \{1,..,2q\}} \sigma_{2,j}$$

which is the union of the annulus σ_1 and the discs $\cup_{j \in \{1,..,2q\}} \sigma_{2,j}$. Now consider the intersection of $\mathfrak{a}_0$ with the union of shifted analogue annuli $\mathfrak{a}_{\mathbf{k_m}}$

$$\mathfrak{a}_{\mathbf{k_m}} = \{\mathbf{k} + \mathbf{k_m}; \mathbf{k} \in \mathfrak{a}_0, \ \mathbf{k_m} \neq 0, |\mathbf{m}| = 2\}$$

centered at all $\mathbf{k_m}$, such that $|\mathbf{m}| = 2$ (see figure 3). It is clear that for any given q, and ϵ small enough, the little discs of radius δ_1 are such that the intersection $\mathfrak{a}_0 \cap \{\cup_{|\mathbf{m}|=2} \mathfrak{a}_{\mathbf{k_m}}\}$ is exactly the union of the little closed discs centered at each $\mathbf{k}_j$. It results that for $U \in E_1 \oplus E_2$, the product $\widetilde{a}U$ for which the corresponding wave vectors belong to some $\mathfrak{a}_{\mathbf{k_m}}$, has a zero projection on E_1. This proves that

$$P_1(\widetilde{a}U_1) = 0, \ P_1(\widetilde{a}U_2) = 0,$$

which implies

$$P_1(aU_2) = 0.$$

It is also clear that

$$\mathfrak{a}_0 \cap \{\mathbf{k} + \mathbf{k_m}; \mathbf{k} \in \sigma_1, \ \mathbf{k_m} \neq 0, |\mathbf{m}| = 2\} = \varnothing,$$

which moreover implies that

$$P_2(\widetilde{a}U_1) = 0,$$

and (30) is proved for the part concerning a. Now observe that we have $|\mathbf{k}_j + \mathbf{k_m}| \neq 1$ except when $\mathbf{k}_j + \mathbf{k_m} = \mathbf{k}_r$ for some $r \in (1, 2q)$. Since we only consider the finite number of cases $|\mathbf{m}| = 2$ or 4, it is clear that in choosing ϵ small enough (i.e. δ_1 small enough), then for $\mathbf{k} \in \sigma_2$, $\mathbf{k} + \mathbf{k_m} \notin \sigma_1$. It results in particular that

$$P_1(bU_2) = 0.$$

The last assertion of Lemma 10 results from the fact that $|\mathbf{k} + \mathbf{k_m}| \neq 1$ for the Fourier spectrum of terms $\in P_0(aU_2)$ and $P_0(bU_2)$, with a distance to the unit circle equivalent to $||\mathbf{k}_j + \mathbf{k_m}| - 1|$ when it is not zero. ∎

Proof. of Lemma 9 : We know by construction that

$$||(P_0 \mathcal{L}_0 P_0)^{-1}||_s \leq \frac{1}{\epsilon C^2},$$

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Hence, we have

$$(P_0\mathcal{L}_\epsilon P_0)^{-1} = [1 + (P_0\mathcal{L}_0 P_0)^{-1}(\epsilon^2 a + \epsilon^4 b + \epsilon^6 \mathcal{P}_\epsilon)]^{-1}(P_0\mathcal{L}_0 P_0)^{-1}, \qquad (31)$$

and the estimate

$$||\epsilon^2 a + \epsilon^4 b + \epsilon^6 \mathcal{P}_\epsilon||_s \le c(s)\epsilon^2$$

leads for ϵ small enough ($s \le S$) to

$$\begin{aligned}(P_0\mathcal{L}_\epsilon P_0)^{-1} &= [\mathbb{I} - (P_0\mathcal{L}_0 P_0)^{-1}(\epsilon^2 a + \epsilon^4 b + \epsilon^6 \mathcal{P}_\epsilon) + \\ &\quad + \{(P_0\mathcal{L}_0 P_0)^{-1}(\epsilon^2 a + \epsilon^4 b + \epsilon^6 \mathcal{P}_\epsilon)\}^2 + O(\epsilon^3)](P_0\mathcal{L}_0 P_0)^{-1}\end{aligned}$$

with a convergent Neumann power series in the bracket, as soon as $\frac{c(s)\epsilon}{C^2} < 1$, which holds for ϵ small enough ($s \le S$). The first consequence is

$$||(P_0\mathcal{L}_\epsilon P_0)^{-1}||_s \le \frac{1}{\epsilon(C^2 - c(s)\epsilon)}.$$

Notice that

$$U_0 = -(P_0\mathcal{L}_\epsilon P_0)^{-1}P_0\{(\epsilon^2 a + \epsilon^4 b + \epsilon^6 \mathcal{P}_\epsilon)(U_1 + U_2)\} + (P_0\mathcal{L}_\epsilon P_0)^{-1}f_0 \qquad (32)$$

The last property of Lemma 10 and (31) imply that in (32) we have, for $\epsilon \in (0, \epsilon_0)$ and for any s

$$\begin{aligned}||(P_0\mathcal{L}_0 P_0)^{-1}P_0(aU_2)||_s &\le d(s)||U_2||_s, \\ ||(P_0\mathcal{L}_0 P_0)^{-1}P_0(bU_2)||_s &\le d(s)||U_2||_s,\end{aligned}$$

$$||(P_0\mathcal{L}_0 P_0)^{-1}a(P_0\mathcal{L}_0 P_0)^{-1}P_0(aU_2)||_s \le d(s)||U_2||_s.$$

It results that

$$\begin{aligned}-(P_0\mathcal{L}_\epsilon P_0)^{-1}P_0\{(\epsilon^2 a + \epsilon^4 b + \epsilon^6 \mathcal{P}_\epsilon)U_2\} &= -\epsilon^2(P_0\mathcal{L}_0 P_0)^{-1}P_0(aU_2) + \\ &\quad + \mathcal{O}(\epsilon^4||U_2||_s),\end{aligned}$$

hence

$$U_0 = Q_{0,1}(\epsilon)U_1 + Q_{0,2}(\epsilon)U_2 + (P_0\mathcal{L}_\epsilon P_0)^{-1}f_0 \qquad (33)$$

with

$$Q_{0,j}(\epsilon) =: -(P_0\mathcal{L}_\epsilon P_0)^{-1}P_0(\epsilon^2 a + \epsilon^4 b + \epsilon^6 \mathcal{P}_\epsilon)P_j, \quad j = 1, 2,$$

and for $s \le S$, $\epsilon \in (0, \epsilon_0(S))$ the following estimates hold:

$$Q_{0,2}(\epsilon)U_2 = -\epsilon^2(P_0\mathcal{L}_0 P_0)^{-1}P_0(aU_2) + \epsilon^4 Q'_{0,2}(\epsilon)U_2,$$

$$\begin{aligned}||Q_{0,1}(\epsilon)U_1||_s &\le \frac{c(s)\epsilon}{C^2 - c(s)\epsilon}||U_1||_s \le c_1(s)\epsilon||U_1||_s, \qquad (34)\\ ||Q'_{0,2}(\epsilon)U_2||_s &\le c_1(s)||U_2||_s, \\ ||(P_0\mathcal{L}_\epsilon P_0)^{-1}f_0||_s &\le \frac{c_1(s)}{\epsilon}||f_0||_s.\end{aligned}$$

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It results from Lemma 10 that equation (27) leads to

$$P_1(\mathcal{L}_0 + \epsilon^2 a_0 + \epsilon^4 b + \epsilon^6 \mathcal{P}_\epsilon)U_1 + P_1\{(\epsilon^2 a + \epsilon^4 b + \epsilon^6 \mathcal{P}_\epsilon)U_0\} + P_1 \epsilon^6 \mathcal{P}_\epsilon U_2 = f_1 \qquad (35)$$

and since $a_0 = 3 > 0$ and the operator $P_1\mathcal{L}_0 P_1$ is positive, we can invert the operator $P_1(\mathcal{L}_0 + \epsilon^2 a_0)P_1$ with the estimate

$$||\{P_1(\mathcal{L}_0 + \epsilon^2 a_0)P_1\}^{-1}|| \leq \frac{1}{3\epsilon^2}.$$

Now replacing U_0 by its expression (33) into equation (35), we introduce an operator acting on U_1 of the form

$$P_1(\epsilon^4 b + \epsilon^6 \mathcal{P}_\epsilon)P_1 + P_1(\epsilon^2 a + \epsilon^4 b + \epsilon^6 \mathcal{P}_\epsilon)Q_{0,1}(\epsilon)$$

which is bounded by $O(\epsilon^3)$, perturbing $P_1(\mathcal{L}_0 + \epsilon^2 a_0)P_1$ the inverse of which is bounded by $1/3\epsilon^2$. It results that, for ϵ small enough, the operator acting on U_1 has a bounded inverse, with

$$||\{P_1[\mathcal{L}_\epsilon + (\epsilon^2 a + \epsilon^4 b + \epsilon^6 \mathcal{P}_\epsilon)Q_{0,1}(\epsilon)]P_1\}^{-1}||_s \leq \frac{1}{\epsilon^2}.$$

Moreover by Lemma 10 we have $P_1[a(P_0\mathcal{L}_0 P_0)^{-1}P_0(aU_2)] = 0$, hence it results that there are bounded linear operators

$$\mathcal{B}_\epsilon^{(0)} : E_2 \to E_0, \quad \mathcal{B}_\epsilon^{(1)} : E_2 \to E_1,$$

such that

$$U_1 = \epsilon^4 \mathcal{B}^{(1)}(\epsilon)U_2 + \frac{1}{\epsilon}\mathcal{Q}^{(1,0)}(\epsilon)f_0 + \frac{1}{\epsilon^2}\mathcal{Q}^{(1,1)}(\epsilon)f_1, \qquad (36)$$

$$U_0 = \epsilon^2 \mathcal{B}^{(0)}(\epsilon)U_2 + \frac{1}{\epsilon}\mathcal{Q}^{(0,0)}(\epsilon)f_0 + \frac{1}{\epsilon}\mathcal{Q}^{(0,1)}(\epsilon)f_1, \qquad (37)$$

with the estimates

$$\begin{aligned} ||\mathcal{B}^{(j)}(\epsilon)||_s &\leq c_2(s), j = 0, 1, \\ ||\mathcal{Q}^{(i,j)}(\epsilon)||_s &\leq c_2(s), \quad i, j = 0, 1, \end{aligned}$$

uniform in $\epsilon \in (0, \epsilon_0)$ and in W bounded in $\mathcal{H}_s$. Moreover, as $\epsilon \to 0$

$$\begin{aligned} \mathcal{B}^{(0)}(\epsilon) &= \mathcal{B}_0^{(0)} + \mathcal{O}(\epsilon^2), \\ \mathcal{B}_0^{(0)}U_2 &\sim -(P_0\mathcal{L}_0 P_0)^{-1}P_0(\tilde{a}U_2). \end{aligned}$$

Equation (28) now reads

$$P_2\mathcal{L}_\epsilon U_2 + P_2\{(\epsilon^2 a + \epsilon^4 b + \epsilon^6 \mathcal{P}_\epsilon)U_0\} + P_2\{(\epsilon^4 b + \epsilon^6 \mathcal{P}_\epsilon)U_1\} = f_2$$

and replacing U_0 and U_1 by their expressions (37), (36) in function of U_2 leads to

$$\begin{aligned} &P_2\mathcal{L}_\epsilon U_2 + P_2\{(\epsilon^2 a + \epsilon^4 b + \epsilon^6 \mathcal{P}_\epsilon)\epsilon^2 \mathcal{B}^{(0)}(\epsilon)U_2 + \\ &+ P_2\{(\epsilon^4 b + \epsilon^6 \mathcal{P}_\epsilon)\epsilon^4 \mathcal{B}^{(1)}(\epsilon)U_2\} \\ = {} & f_2 - P_2\left((\epsilon^2 a + \epsilon^4 b + \epsilon^6 \mathcal{P}_\epsilon)[\frac{1}{\epsilon}\mathcal{Q}^{(0,0)}(\epsilon)f_0 + \frac{1}{\epsilon}\mathcal{Q}^{(0,1)}(\epsilon)f_1]\right) + \\ & - P_2\left((\epsilon^4 b + \epsilon^6 \mathcal{P}_\epsilon)[\frac{1}{\epsilon}\mathcal{Q}^{(1,0)}(\epsilon)f_0 + \frac{1}{\epsilon^2}\mathcal{Q}^{(1,1)}(\epsilon)f_1]\right), \end{aligned}$$

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this gives (see the definition of Λ_ϵ in Lemma 9)

$$\Lambda_\epsilon U_2 + \epsilon^4 R_\epsilon U_2 = \mathcal{Q}(\epsilon)f$$

with the announced properties for bounded operators R_ϵ and $\mathcal{Q}(\epsilon)$ in $\mathcal{H}_s$. Lemma 9 is proved. ∎

2.6 Structure of the reduced operator Λ_ϵ

2.6.1 General structure - Invariant subspaces

We study in this section the structure of the operator Λ_ϵ defined by (29). We first observe that we deal with functions U which are invariant under the rotation $R_{\frac{\pi}{q}}$ of the plane. So, let us define a new subspace of $\mathcal{H}_s$ for such functions:

$$E_2^{(S)} = \{U \in E_2; U(R_{\frac{\pi}{q}}\mathbf{x}) = U(\mathbf{x})\}.$$

This implies immediately that

$$U^{(\mathbf{k})} = U^{(R_{\frac{\pi}{q}}\mathbf{k})}, \tag{38}$$

and for any $U \in E_2^{(S)}$ we have the following decomposition

$$U = \sum_{j=1,...,2q} U_{2,j}$$

where

$$U_{2,j}(\mathbf{x}) = \sum_{\mathbf{k}\in\sigma_{2,j}} U^{(\mathbf{k})}e^{i\mathbf{k}\cdot\mathbf{x}} = U_{2,1}(R_{\frac{(1-j)\pi}{q}}\mathbf{x}). \tag{39}$$

It results that any $U \in E_2^{(S)}$ may be written as

$$U(\mathbf{x}) = \sum_{j=1,...,2q} U_{2,1}(R_{\frac{(1-j)\pi}{q}}\mathbf{x}).$$

Let us notice that in the little disc $\sigma_{2,1}$ we have

$$\sigma_{2,1} \ni \mathbf{k} = \mathbf{k}_1 + \mathbf{k}', |\mathbf{k}'| \le \delta_1 = \epsilon^{1/4}\sqrt{3C},$$

and let decompose the discs $\sigma_{2,l}$ into $2q$ equal sectors $\mathbf{k}_l + \Sigma_m, m = 1, ..., 2q$ such that

$$\Sigma_m = \left\{\mathbf{k}' \in \Gamma; |\mathbf{k}'| \le \delta_1, \arg \mathbf{k}' \in [\frac{(m-1)\pi}{q} - \frac{\pi}{2q}, \frac{(m-1)\pi}{q} + \frac{\pi}{2q})\right\}. \tag{40}$$

For any $\mathbf{k}' \in \Sigma_l$, we define the set of $2q$ spectral points $\sigma_{\mathbf{k}'}^{(l)}$ by

$$\sigma_{\mathbf{k}'}^{(l)} = \left\{\mathbf{k} = \mathbf{k}_j + \mathbf{k}' \in \sigma_{2,j}; \; j = 1, ..., 2q\right\}.$$

The subspace of $\mathcal{H}_s$ associated with $\sigma^{(l)}_{\mathbf{k}'}$ is denoted by $E^{(l)}_{2,\mathbf{k}'}$, so that any $U \in E^{(S)}_2$ may be written as

$$
\begin{aligned}
U(\mathbf{x}) &= \sum_{j=1,...,2q} \sum_{\mathbf{k}_1+\mathbf{k}'\in\sigma_{2,1}} U^{(\mathbf{k}_1+\mathbf{k}')} e^{i(\mathbf{k}_j+R_{\frac{(j-1)\pi}{q}}\mathbf{k}')\cdot\mathbf{x}} \\
&= \sum_{l=1,...,2q} \sum_{\mathbf{k}'\in\Sigma_l} \sum_{j=1,...,2q} U^{(\mathbf{k}_1+R_{\frac{(1-j)\pi}{q}}\mathbf{k}')} e^{i(\mathbf{k}_j+\mathbf{k}')\cdot\mathbf{x}},
\end{aligned}
$$

hence

$$
U = \sum_{l=1}^{2q} \sum_{\mathbf{k}'\in\Sigma_l} U^{(l,\mathbf{k}')}, \quad U^{(l,\mathbf{k}')} \in E^{(l)}_{2,\mathbf{k}'},
$$

$$
U^{(l,\mathbf{k}')}(\mathbf{x}) = \sum_{j=1,...,2q} U^{(\mathbf{k}_1+R_{\frac{(1-j)\pi}{q}}\mathbf{k}')} e^{i(\mathbf{k}_j+\mathbf{k}')\cdot\mathbf{x}} = \sum_{j=1,...,2q} U^{(\mathbf{k}_j+\mathbf{k}')} e^{i(\mathbf{k}_j+\mathbf{k}')\cdot\mathbf{x}}
$$

and any $U \in E^{(S)}_2$ is *completely determined by the set of* $2q-$ *dimensional* $U^{(1,\mathbf{k}')} \in E^{(1)}_{2,\mathbf{k}'}$, $\mathbf{k}' \in \Sigma_1$, identified with the set of components

$$
\{ U^{(\mathbf{k}_j+\mathbf{k}')} = U^{(\mathbf{k}_1+R_{\frac{(1-j)\pi}{q}}\mathbf{k}')}, \quad j = 1,...,2q \}.
$$

Indeed we have

$$
U^{(l,\mathbf{k}')}(\mathbf{x}) = U^{(l+1,R_{\frac{\pi}{q}}\mathbf{k}')}(R_{\frac{\pi}{q}}\mathbf{x}),
$$

hence

$$
U^{(l,\mathbf{k}')}(\mathbf{x}) = U^{(1,R_{(1-l)\frac{\pi}{q}}\mathbf{k}')}(R_{(1-l)\frac{\pi}{q}}\mathbf{x}), \quad \mathbf{k}' \in \Sigma_l,
$$

$$
U^{(l,R_{\frac{(l-1)\pi}{q}}\mathbf{k}')}(\mathbf{x}) = \sum_{j=1,...,2q} U^{(\mathbf{k}_1+R_{\frac{(l-j)\pi}{q}}\mathbf{k}')} e^{i(\mathbf{k}_j+R_{\frac{(l-1)\pi}{q}}\mathbf{k}')\cdot\mathbf{x}}, \quad \mathbf{k}' \in \Sigma_1,
$$

where

$$
U^{(1,\mathbf{k}')}(\mathbf{x}) = \sum_{j=1,...,2q} U^{(\mathbf{k}_1+R_{\frac{(1-j)\pi}{q}}\mathbf{k}')} e^{i(\mathbf{k}_j+\mathbf{k}')\cdot\mathbf{x}}, \quad \mathbf{k}' \in \Sigma_1, \tag{41}
$$

and we observe that the coordinates of $U^{(l,R_{\frac{(l-1)\pi}{q}}\mathbf{k}')}, \mathbf{k}' \in \Sigma_1$, correspond to those *shifted* of $U^{(1,\mathbf{k}')}$. Moreover we have

$$
U(\mathbf{x}) = \sum_{l=1}^{2q} \sum_{\mathbf{k}'\in\Sigma_1} U^{(1,\mathbf{k}')}(R_{(1-l)\frac{\pi}{q}}\mathbf{x}). \tag{42}
$$

From now on, we denote by $E_{2,\mathbf{k}'}$ the previously defined 2q-dimensional subspace $E^{(1)}_{2,\mathbf{k}'}$.

Looking at the form of the operator Λ_ϵ we see that the wave vector $\mathbf{k}$ of U is shifted by $\mathbf{k_m}, |\mathbf{m}| = 2$ at order ϵ^2, and $|\mathbf{m}| = 4$ at order ϵ^4. Now, we observe that for a fixed finite $|\mathbf{m}|$, if the combination

$$
\mathbf{k_m} - (\mathbf{k}_1 - \mathbf{k}_j)
$$

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is not 0, then it has a minimal length of order 1 as ϵ tends to 0. It results that for $\mathbf{k} = \mathbf{k}_1 + \mathbf{k}' \in \sigma_{2,1}$, and $\mathbf{l} = \mathbf{k}_j + \mathbf{l}' \in \sigma_{2,j}$ with

$$\mathbf{k_m} - (\mathbf{k} - \mathbf{l}) \neq 0$$

then, for ϵ small enough

$$\mathbf{k_m} - (\mathbf{k} - \mathbf{l}) = \mathcal{O}(1).$$

It results that the only possibility for going from $\mathbf{l} \in \sigma_{2,j}$ to $\sigma_{2,1}$ is to add $\mathbf{k_m} = \mathbf{k}_1 + \mathbf{k}_{j+q} = \mathbf{k}_1 - \mathbf{k}_j$. It results that the system

$$\mathbf{k_m} + \mathbf{l} = \mathbf{k}, \ \mathbf{l} = \mathbf{k}_j + \mathbf{l}', \ \mathbf{k} = \mathbf{k}_j + \mathbf{k}'$$

has the only solution

$$\mathbf{l}' = \mathbf{k}'. \tag{43}$$

It should be clear that $\mathbf{k_m}$ comes from terms with many possible combinations, not only trivial ones as for $q = 4$. For example for $q = 6$, the terms occuring in the coefficients giving $\mathbf{k_m}$ are even more frequent at order ϵ^4 because of the existing special combinations $\mathbf{k}_j + R_{\pi/3}\mathbf{k}_j + R_{2\pi/3}\mathbf{k}_j = 0$. However, *in all cases* we can write, for $\mathbf{k} = \mathbf{k}_1 + \mathbf{k}' \in \sigma_{2,1}$

$$(\Lambda_\epsilon U)^{(\mathbf{k}_1 + \mathbf{k}')} = \sum_{j=1}^{Q} \gamma_j(\mathbf{k}, \epsilon) U^{(\mathbf{k}_j + \mathbf{k}')}. \tag{44}$$

Remark 11. *The argument ϵ in $\gamma_j(\mathbf{k}, \epsilon)$ only refers to the perturbation of $P_2\mathcal{L}_0 P_2$ in Λ_ϵ (see (29)), and not on the fact that P_2 also depends on ϵ via the radii of the little discs composing the set σ_2 which are $O(\epsilon^{1/4})$.*

Remark 12. *Due to the form of orders ϵ^2 and ϵ^4 in Λ_ϵ and $\mathcal{L}_\epsilon^{(2)}$, and because of (43), we notice that the dependency in $\mathbf{k}$ of the coefficients $\gamma_j(\mathbf{k}, \epsilon)$ only occurs at orders ϵ^0 and ϵ^4. Indeed, the dependency in $\mathbf{k}$ comes from operators $\mathcal{L}_0$ at order 0 and $(P_0\mathcal{L}_0 P_0)^{-1}$ in the term $P_2\{\widetilde{a}(P_0\mathcal{L}_0 P_0)^{-1} P_0(\widetilde{a}U)\}$ at order ϵ^4.*

The property that $\Lambda_\epsilon U$ is invariant under the rotation $R_{\frac{\pi}{q}}$ and the identity

$$R_{\frac{-\pi}{q}}\widetilde{\mathbf{k}} + \mathbf{k}_j - \mathbf{k}_1 = R_{\frac{-\pi}{q}}(\widetilde{\mathbf{k}} + \mathbf{k}_{j+1} - \mathbf{k}_2)$$

lead to

$$(\Lambda_\epsilon U)^{(\widetilde{\mathbf{k}})} = \sum_{j=1}^{2q} \gamma_j(R_{\frac{-\pi}{q}}\widetilde{\mathbf{k}}, \epsilon) U^{(\widetilde{\mathbf{k}} + \mathbf{k}_{j+1} - \mathbf{k}_2)}, \ \widetilde{\mathbf{k}} = R_{\frac{\pi}{q}}\mathbf{k} \in \sigma_{2,2}.$$

Choosing $\sigma_{2,2} \ni \widetilde{\mathbf{k}} = \mathbf{k}' + \mathbf{k}_2$, $|\mathbf{k}'| \leq \delta_1$, we then have

$$(\Lambda_\epsilon U)^{(\mathbf{k}' + \mathbf{k}_2)} = \sum_{j=1}^{2q} \gamma_j(R_{\frac{-\pi}{q}}\mathbf{k}' + \mathbf{k}_1, \epsilon) U^{(\mathbf{k}' + \mathbf{k}_{j+1})}. \tag{45}$$

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In the same way, after identifying $j + 2q$ with j, we obtain for $r = 1, ..., 2q$

$$
\begin{aligned}
(\Lambda_\epsilon U)^{(\mathbf{k}' + \mathbf{k}_r)} &= \sum_{j=1}^{2q} \gamma_j \left(R_{\frac{\pi(1-r)}{q}} \mathbf{k}' + \mathbf{k}_1, \epsilon \right) U^{(\mathbf{k}' + \mathbf{k}_{j+r-1})}, \\
&= \sum_{j=1}^{2q} \gamma_{j+1-r} \left(R_{\frac{\pi(1-r)}{q}} \mathbf{k}' + \mathbf{k}_1, \epsilon \right) U^{(\mathbf{k}' + \mathbf{k}_j)}.
\end{aligned}
$$

The important result is that, for a fixed $\mathbf{k} = \mathbf{k}_1 + \mathbf{k}', \mathbf{k}' \in \Sigma_1$, the subspace $E_{2, \mathbf{k}'}$ is invariant under the operator Λ_ϵ then denoted $\Lambda_\epsilon^{(\mathbf{k}')}$. Hence the $2q \times 2q$ *matrices of* $\Lambda_\epsilon^{(\mathbf{k}')}$ *are uncoupled for different* $\mathbf{k}' \in \Sigma_1$. We notice that if γ_j were independent of $\mathbf{k}$, the lines of the matrix of $\Lambda_\epsilon^{(\mathbf{k}')}$ would be deduced each from the previous one by a simple right shift.

The next useful property of Λ_ϵ is its *self-adjointness* in E_2 with the Hilbert structure of $\mathcal{H}_0$. This property is immediate from the definition (29) with the scalar product of the space $\mathcal{H}_s$ for $s = 0$. It should be noticed that the full linear operator $\mathcal{L}_{\epsilon, W}^{(2)}$ acting in E_2 *is not selfadjoint in general*. Now, isolating the coordinates $U^{(\mathbf{k}' + \mathbf{k}_j)}$, $j = 1, ..., 2q$ for $\mathbf{k}' \in \Sigma_1$, we still have, for any fixed $\mathbf{k}'$, a $2q \times 2q$ self-adjoint matrix $\Lambda_\epsilon^{(\mathbf{k}')}$ due to the previous self-adjointness of the operator Λ_ϵ in $\mathcal{H}_0$. It results that we have

$$
\gamma_{j+1-r} \left(R_{\frac{\pi(1-r)}{q}} \mathbf{k}' + \mathbf{k}_1, \epsilon \right) = \gamma_{r+1-j} \left(R_{\frac{\pi(1-j)}{q}} \mathbf{k}' + \mathbf{k}_1, \epsilon \right). \tag{46}
$$

We sum up these results in the following

Lemma 13. *The subspace $E_2^{(S)}$ of $\mathcal{H}_s$ consisting of functions invariant under rotations by π/q may be decomposed into the following Hilbert sum*

$$
E_2^{(S)} = \overset{\perp}{\underset{\mathbf{k}' \in \Sigma_1}{\oplus}} E_{2, \mathbf{k}'}
$$

where we identify the wave vector $\mathbf{k}$ with $R_{\frac{\pi}{q}} \mathbf{k}$ i.e., $\mathbf{k}_1 + \mathbf{k}' \in \sigma_{2,1}$ with $\mathbf{k}_j + R_{\frac{\pi(j-1)}{q}} \mathbf{k}' \in \sigma_{2,j}$ (see (42)).

The $2q-$ dimensional subspace $E_{2, \mathbf{k}'}$ is invariant under the operator Λ_ϵ. Defining the coefficients $\gamma_j(\mathbf{k}, \epsilon)$ by

$$
(\Lambda_\epsilon U)^{(\mathbf{k}' + \mathbf{k}_1)} = \sum_{j=1}^{2q} \gamma_j(\mathbf{k}, \epsilon) U^{(\mathbf{k}' + \mathbf{k}_j)}, \ \ \mathbf{k} \in \sigma_{2,1}
$$

the $2q \times 2q$ matrix of the restriction $\Lambda_\epsilon^{(\mathbf{k}')}$ of Λ_ϵ to $E_{2, \mathbf{k}'}$ is symmetric and satisfies

$$
\gamma_{j+1-r} \left(R_{\frac{\pi(1-r)}{q}} \mathbf{k}' + \mathbf{k}_1, \epsilon \right) = \gamma_{r+1-j} \left(R_{\frac{\pi(1-j)}{q}} \mathbf{k}' + \mathbf{k}_1, \epsilon \right)
$$

for any $\mathbf{k}' \in \Sigma_1$.

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Let us define $\Lambda_0^{(\mathbf{k}')}$ which is a *diagonal matrix* with

$$\gamma_1(\mathbf{k}, 0) = (|\mathbf{k}|^2 - 1)^2, \gamma_2(\mathbf{k}, 0) = 0, ... \gamma_{2q}(\mathbf{k}, 0) = 0.$$

For $\mathbf{k} = \mathbf{k}' + \mathbf{k}_j \in \sigma_{2,j}$, we define

$$\beta_j(\mathbf{k}') = (|\mathbf{k}' + \mathbf{k}_j|^2 - 1)^2, \quad j = 1, ..., 2q. \tag{47}$$

Hence for $j = 1, ..., 2q$

$$
\begin{aligned}
\gamma_1\left(R_{\frac{\pi(1-j)}{q}}\mathbf{k}' + \mathbf{k}_1, 0\right) &= \left(\left|R_{\frac{\pi(1-j)}{q}}\mathbf{k}' + \mathbf{k}_1\right|^2 - 1\right)^2 = \beta_j(\mathbf{k}') \\
&= (2\mathbf{k}_j \cdot \mathbf{k}' + |\mathbf{k}'|^2)^2,
\end{aligned}
$$

and $\Lambda_0^{(\mathbf{k}')}$ reads

$$
\Lambda_0^{(\mathbf{k}')} = \begin{pmatrix}
\beta_1(\mathbf{k}') & 0 & . & . & 0 & . & 0 & 0 \\
0 & \beta_2(\mathbf{k}') & . & . & 0 & . & 0 & 0 \\
. & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . \\
0 & 0 & . & . & \beta_j(\mathbf{k}') & . & . & 0 \\
. & . & . & . & . & . & . & . \\
0 & 0 & . & . & 0 & . & \beta_{2q-1}(\mathbf{k}') & 0 \\
0 & 0 & . & . & 0 & . & 0 & \beta_{2q}(\mathbf{k}')
\end{pmatrix}.
$$

Then, according to the definition of $\Lambda_\epsilon^{(\mathbf{k}')}$, we can write

$$\Lambda_\epsilon^{(\mathbf{k}')} = \Lambda_0^{(\mathbf{k}')} + \epsilon^2 \Lambda_1.$$

According to (46), in the case when the coefficients $\gamma_j(\mathbf{k}, \epsilon)$ are independent of $\mathbf{k}$, (which corresponds here to the order ϵ^2), this leads to a first line for the $2q \times 2q$ matrix, of the form

$$\gamma_1, \gamma_2, ... \gamma_q, \gamma_{q+1}, \gamma_q, \gamma_{q-1}, .. \gamma_3, \gamma_2 \tag{48}$$

and next lines are deduced by a right shift, making a symmetric matrix. For example in the case $q = 4$, we obtain for Λ_1 a matrix of the form (easily generalizable for any q)

$$
\begin{pmatrix}
\gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 & \gamma_5 & \gamma_4 & \gamma_3 & \gamma_2 \\
\gamma_2 & \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 & \gamma_5 & \gamma_4 & \gamma_3 \\
\gamma_3 & \gamma_2 & \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 & \gamma_5 & \gamma_4 \\
\gamma_4 & \gamma_3 & \gamma_2 & \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 & \gamma_5 \\
\gamma_5 & \gamma_4 & \gamma_3 & \gamma_2 & \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 \\
\gamma_4 & \gamma_5 & \gamma_4 & \gamma_3 & \gamma_2 & \gamma_1 & \gamma_2 & \gamma_3 \\
\gamma_3 & \gamma_4 & \gamma_5 & \gamma_4 & \gamma_3 & \gamma_2 & \gamma_1 & \gamma_2 \\
\gamma_2 & \gamma_3 & \gamma_4 & \gamma_5 & \gamma_4 & \gamma_3 & \gamma_2 & \gamma_1
\end{pmatrix}. \tag{49}
$$

where $\gamma_1, ... \gamma_{q+1}$ are independent of $\mathbf{k}'$.

2.6.2 Computation of coefficients γ_j in Λ_1

Let us compute the operator

$$E_2^{(S)} \ni U \mapsto \epsilon^2 P_2(aU),$$
$$a = 3u_0^2 - 3(2q - 1).$$

where u_0 is given by (14). We have for $U \in E_2^{(S)}$ (see(39))

$$P_{2,1}(u_0^2 U) = 2qU_{2,1}(\mathbf{x}) + U_{2,q+1}(\mathbf{x})e^{2i\mathbf{k}_1 \cdot \mathbf{x}} + \sum_{j=2,..,q,q+2,...,2q} 2U_{2,j}(\mathbf{x})e^{i(\mathbf{k}_1 - \mathbf{k}_j) \cdot \mathbf{x}},$$

where we denote by $P_{2,1}$ the orthogonal projection corresponding to the part $\sigma_{2,1}$ of the spectrum. Since we have

$$U_{2,j}(\mathbf{x})e^{i(\mathbf{k}_1 - \mathbf{k}_j) \cdot \mathbf{x}} = \sum_{\mathbf{k} \in \sigma_{2,j}} U^{(\mathbf{k})}e^{i(\mathbf{k} + \mathbf{k}_1 - \mathbf{k}_j) \cdot \mathbf{x}} = \sum_{\mathbf{k} \in \sigma_{2,1}} U^{(\mathbf{k} + \mathbf{k}_j - \mathbf{k}_1)}e^{i\mathbf{k} \cdot \mathbf{x}}$$

we obtain

$$P_{2,1}(aU) = \sum_{\mathbf{k} \in \sigma_{2,1}} 3e^{i\mathbf{k} \cdot \mathbf{x}} \left\{ U^{(\mathbf{k})} + U^{(\mathbf{k} - 2\mathbf{k}_1)} + \sum_{j=2,..,q,q+2,...,2q} 2U^{(\mathbf{k} + \mathbf{k}_j - \mathbf{k}_1)} \right\}. \quad (50)$$

As expected, it appears that the linear operator

$$U \mapsto P_2(\epsilon^2 a)U, \ U \in E_2^{(S)} \quad (51)$$

leaves invariant the subspaces $E_{2,\mathbf{k}'}$ and in this subspace it takes the form of a matrix with 4 identical blocks for the set of $2q$ coordinates $U^{(1,\mathbf{k}')}$, and coefficients γ_j are independent of $\mathbf{k}'$ and have the form (48) with:

$$\gamma_1 = \gamma_{1+q} = 3, \quad \gamma_2 = \gamma_3 = ...\gamma_q = 6.$$

Hence, $\Lambda_1^{(\mathbf{k}')}$ **is independent of** $\mathbf{k}'$ **and we write** $\Lambda_1^{(\mathbf{k}')} = \Lambda_1$.
For example, in the case $q = 4$, we have the following corresponding matrix for

$$U \mapsto (P_2(a\cdot)^{(\mathbf{k}')})U, \ U \in E_{2,\mathbf{k}'}$$

$$\Lambda_1 = 3 \begin{pmatrix} 1 & 2 & 2 & 2 & 1 & 2 & 2 & 2 \\ 2 & 1 & 2 & 2 & 2 & 1 & 2 & 2 \\ 2 & 2 & 1 & 2 & 2 & 2 & 1 & 2 \\ 2 & 2 & 2 & 1 & 2 & 2 & 2 & 1 \\ 1 & 2 & 2 & 2 & 1 & 2 & 2 & 2 \\ 2 & 1 & 2 & 2 & 2 & 1 & 2 & 2 \\ 2 & 2 & 1 & 2 & 2 & 2 & 1 & 2 \\ 2 & 2 & 2 & 1 & 2 & 2 & 2 & 1 \end{pmatrix}.$$

for each fixed $\mathbf{k}' \in \Sigma_1$.

2.7 Eigenvalues of Λ_ϵ

From (29) we have

$$\Lambda_\epsilon^{(\mathbf{k}')} = \Lambda_0^{(\mathbf{k}')} + \epsilon^2 \Lambda_1.$$

We show below the following

Lemma 14. *For any given $q \geq 4$, and $\mathbf{k} \in \sigma_2$ the eigenvalues μ_j of $\Lambda_\epsilon^{(\mathbf{k}')}$ take the form*

$$
\begin{aligned}
\mu_j &= (2\mathbf{k}_j \cdot \mathbf{k}' + |\mathbf{k}'|^2)^2 + 3\epsilon^2 + O(\epsilon^4), \\
&= [|\mathbf{k}' + \mathbf{k}_j|^2 - 1]^2 + 3\epsilon^2 + O(\epsilon^4), \quad j = 1, .., 2q.
\end{aligned}
$$

Proof. The eigenvalues $\mu \in \mathbb{R}$ of $\Lambda_\epsilon^{(\mathbf{k}')}$ satisfy for a certain $\zeta \in \mathbb{R}^{2q}$

$$\{\Lambda_0^{(\mathbf{k}')} + \epsilon^2 \Lambda_1\}\zeta = \mu\zeta. \tag{52}$$

Since we deal with selfadjoint operators, any eigenvalue takes the form (see [Kat95] in the $2q$-dimensional subspace $E_{2,\mathbf{k}'}$.)

$$\mu_j = \mu_{j,0}(\mathbf{k}') + \epsilon^2 \mu_{j,1}(\mathbf{k}') + O(\epsilon^4), \quad j = 1, ..., 2q$$

with

$$\mu_{j,0}(\mathbf{k}') = (2\mathbf{k}_j \cdot \mathbf{k}' + |\mathbf{k}'|^2)^2 = \beta_j(\mathbf{k}')$$

by definition (47). Eigenvectors take the form

$$\zeta_j = \zeta_{j,0} + \epsilon^2 \zeta_{j,1} + O(\epsilon^4), j = 1, ..., 2q,$$

with

$$\zeta_{j,0} = (0, .., 0, 1, 0.., 0)^t, \ 1 \text{ taking the } j\text{th place.}$$

A simple identification at order ϵ^2 leads to

$$(\Lambda_0^{(\mathbf{k}')} - \mu_{j,0})\zeta_{j,1} + (\Lambda_1 - \mu_{j,1})\zeta_{j,0} = 0, \tag{53}$$

Taking the scalar product of (53) with $\zeta_{j,0}$ gives, taking into account the form of Λ_1,

$$\mu_{j,1} = \frac{\langle \Lambda_1 \zeta_{j,0}, \zeta_{j,0}\rangle}{\langle \zeta_{j,0}, \zeta_{j,0}\rangle} = 3, \quad j = 1, ..., 2q,$$

which is independent of $\mathbf{k}'$, and which gives the result of Lemma 14. ∎

2.8 Inverse of $\mathcal{L}_\epsilon$ in $\mathcal{H}_s$

We already have the following estimate in $\mathcal{H}_0$:

Lemma 15. *For any given $q \geq 4$, and for ϵ small enough, the linear operator Λ_ϵ is invertible in $\mathcal{H}_0$ with*

$$\|\Lambda_\epsilon^{-1}\|_0 \leq \frac{1}{2\epsilon^2}.$$

The proof of Lemma 15 follows directly from Lemma 14 for $\epsilon \leq \epsilon_0$, since $\mathbf{k}'$ is bounded, and all eigenvalues for $\mathbf{k}' \in \Sigma_1$ are positive and larger than $2\epsilon^2$.

For extending the estimate to $\mathcal{H}_s$, we need next property

Lemma 16. *For any $K > 0$, $|x - y| \leq K$, x and $y > 0$, and any $p \geq 0$ there exists $d(p, K) > 0$ such that*

$$|(1 + x)^p - (1 + y)^p| \leq d(p, K)(1 + x)^{p-1}.$$

The proof of this Lemma is in Appendix A.

Then we prove the following

Lemma 17. *For any given $q \geq 4$, and for ϵ small enough, the linear operator Λ_ϵ is invertible in $\mathcal{H}_s$ for $s \geq 0$, with $c_s > 0$ such that*

$$\|\Lambda_\epsilon^{-1}\|_s \leq \frac{c_s}{\epsilon^2}.$$

Proof. Let us assume that $f \in \mathcal{H}_s$, and define $f_s \in \mathcal{H}_0$ by its Fourier coefficients

$$f_s^{(\mathbf{k})} = (1 + N_{\mathbf{k}}^2)^{s/2} f^{(\mathbf{k})}.$$

We then have $\|f_s\|_0 = \|f\|_s$. Then $\Lambda_\epsilon U = f$ leads to $(\Lambda_\epsilon U)_s \in \mathcal{H}_0$, $\|(\Lambda_\epsilon U)_s\| = \|f\|_s$.

By definition

$$(\Lambda_\epsilon U)^{(\mathbf{k})} = (1 - |\mathbf{k}|^2)^2 U^{(\mathbf{k})} + \epsilon^2 \Sigma_{\mathbf{l} \in \sigma_2} a^{(\mathbf{k}-\mathbf{l})} U^{(\mathbf{l})},$$

where $\mathbf{k} \in \sigma_2$. Now

$$(\Lambda_\epsilon U)_s^{(\mathbf{k})} - (\Lambda_\epsilon U_s)^{(\mathbf{k})} = \epsilon^2 \Sigma_{\mathbf{l} \in \sigma_2} [(1 + N_{\mathbf{k}}^2)^{s/2} - (1 + N_{\mathbf{l}}^2)^{s/2}] a^{(\mathbf{k}-\mathbf{l})} U^{(\mathbf{l})},$$

and since $|N_{\mathbf{k}} - N_{\mathbf{l}}| \leq 2$ from the form of a, we have from Lemma 16

$$|(1 + N_{\mathbf{k}}^2)^{s/2} - (1 + N_{\mathbf{l}}^2)^{s/2}| \leq d(s/2, 2)(1 + N_{\mathbf{k}}^2)^{s/2-1}.$$

Now define $\widetilde{U}$ for any $U \in \mathcal{H}_s$ by

$$\widetilde{U}^{(\mathbf{k})} = |\widetilde{U}^{(\mathbf{k})}|.$$

Then $\|\widetilde{U}\|_s = \|U\|_s$ and since for $0 < s \leq 2$, $(1 + N_{\mathbf{k}}^2)^{s/2-1} \leq 1$,

$$|[(\Lambda_\epsilon U)_s - (\Lambda_\epsilon U_s)]^{(\mathbf{k})}| \leq \epsilon^2 d(s/2, 2))(\widetilde{a}\widetilde{U})^{(\mathbf{k})}$$

(where $\tilde{a}$ differs from the one defined at Lemma 9), hence for $0 < s \leq 2$

$$\|(\Lambda_\epsilon U)_s - (\Lambda_\epsilon U_s)\|_0 \leq d(s/2, 2)\epsilon^2 \|\widetilde{a}\widetilde{U}\|_0 \leq d_s \epsilon^2 \|U\|_0.$$

Hence we obtain

$$\|\Lambda_\epsilon U_s\|_0 \leq \|(\Lambda_\epsilon U)_s\|_0 + d_s/2\|f\|_0 = \|f\|_s + d_s/2\|f\|_0,$$

and finally, for $0 \leq s \leq 2$

$$\|U\|_s = \|U_s\|_0 \leq \frac{1}{2\epsilon^2} \|\Lambda_\epsilon U_s\|_0 \leq \frac{c_s}{\epsilon^2} \|f\|_s.$$

Let us prove by induction on $s \geq 0$ that $||\Lambda_\epsilon^{-1}||_s \leq c_s \epsilon^{-2}$. This holds for $0 \leq s \leq 2$. Assume that it holds for $s - 2$, then

$$|[(\Lambda_\epsilon U)_s - (\Lambda_\epsilon U_s)]^{(\mathbf{k})}| \leq \epsilon^2 d(s/2, 2))(1 + N_{\mathbf{k}}^2)^{s/2-1}(\widetilde{a}\widetilde{U})^{(\mathbf{k})}.$$

Hence, we have

$$||(\Lambda_\epsilon U)_s - (\Lambda_\epsilon U_s)||_0 \leq d(s/2, 2)\epsilon^2||\widetilde{a}\widetilde{U}||_{s-2} \leq d_s \epsilon^2||U||_{s-2}.$$

We assumed that $||\Lambda_\epsilon^{-1}||_{s-2} \leq \frac{c_{s-2}}{\epsilon^2}$, hence we obtain

$$||\Lambda_\epsilon U_s||_0 \leq ||(\Lambda_\epsilon U)_s||_0 + d_s c_{s-2}||f||_{s-2} = ||f||_s + d_s c_{s-2}||f||_{s-2},$$

hence

$$||U||_s = ||U_s||_0 \leq \frac{1}{2\epsilon^2}||\Lambda_\epsilon U_s||_0 \leq \frac{c_s}{\epsilon^2}||f||_s.$$

This ends the proof of Lemma 17. ∎

Then we finally have

Lemma 18. *For any $q \geq 4$, and $s \geq 0$, there exists $\epsilon_0 > 0$, such that for $0 < \epsilon \leq \epsilon_0$ the linear operator $\mathcal{L}_\epsilon$ has a bounded inverse in $\mathcal{H}_s$, with*

$$||\mathcal{L}_\epsilon^{-1}||_s \leq \frac{c(s)}{\epsilon^2},$$

where $c(s)$ is a positive constant only depending on s.

Proof. From Lemma 17 we have

$$(\mathcal{L}_\epsilon^{(2)})^{-1} = (1 + \epsilon^4 \Lambda_\epsilon^{-1} R_\epsilon)^{-1} \Lambda_\epsilon^{-1},$$

and $||\epsilon^4 \Lambda_\epsilon^{-1} R_\epsilon||_s \leq c_s \epsilon^2||R_\epsilon||_s \leq c_s' \epsilon^2$. For ϵ small enough we then have

$$||(\mathcal{L}_\epsilon^{(2)})^{-1}|| \leq \frac{2c_s}{\epsilon^2}.$$

Then, from Lemma 9 we deduce immediately that there exists a constant $c(s)$ such that

$$||U||_s \leq \frac{c(s)}{\epsilon^2}||f||_s,$$

which proves the Lemma. ∎

3 Existence of the solution

Below we prove our main result

Theorem 19. *For any $q \geq 4$ and for any $s > q/2$, there exists $\lambda_0 > 0$ such that for $0 < \lambda < \lambda_0$, there exists a quasipattern solution of the Swift-Hohenberg steady equation (11) in $\mathcal{H}_s$, bifurcating from 0 and invariant under rotations of angle π/q. Its asymptotic expansion at the origin is given by the formal expansion computed in [IR10].*

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Proof. We want to solve (21) with respect to W in $\mathcal{H}_s, s > q/2$. Taking into account of Lemma 18, this equation takes the following form for $\epsilon \neq 0$:

$$W = -\epsilon^3 \mathcal{L}_\epsilon^{-1}[f_\epsilon + 3\epsilon U_\epsilon W^2 + \epsilon^5 W^3], \tag{54}$$

which we write as

$$W = \mathcal{G}(\epsilon, W)$$

where $\mathcal{G}$ is well defined in $]0, \epsilon_0[\times\mathcal{H}_s$, depending smoothly on its arguments for $0 < \epsilon < \epsilon_0$ and $W \in \mathcal{H}_s$. In fact for $\epsilon \neq 0$ fixed, $\mathcal{G}$ is analytic in W, and observing that $\epsilon^3 \mathcal{L}_\epsilon^{-1} = O(\epsilon)$, we see that $\mathcal{G}$ is continuous in ϵ on $]0, \epsilon_0[$.

The map $\mathcal{G}$ is Lipschitz in W in a fixed ball of $\mathcal{H}_s$, with a small Lipschitz constant for $\epsilon \in]0, \epsilon_0[$. Indeed, we have

$$\|\mathcal{G}(\epsilon, W) - \mathcal{G}(\epsilon, W')\|_s \leq \|\epsilon^3 \mathcal{L}_\epsilon^{-1}[3\epsilon U_\epsilon(W^2 - W'^2) + \epsilon^5(W^3 - W'^3)]\|_s,$$

$\epsilon^3 \|\mathcal{L}_\epsilon^{-1}\|_s \leq c_s \epsilon$ and $\|U_\epsilon\|_s \leq c'_s \epsilon$. Moreover, we have

$$\mathcal{G}(\epsilon, 0) = -\epsilon^3 \mathcal{L}_\epsilon^{-1}(f_\epsilon) = 0(\epsilon).$$

Than according Dieudonné's version of the implicit function theorem [D60][(10.1.1)], there exists a unique mapping $W(\epsilon)$ into a ball in $\mathcal{H}_s$ of size $O(\epsilon)$, such that

$$W(\epsilon) = \mathcal{G}(\epsilon, W(\epsilon))$$

for all $\epsilon \in]0, \epsilon_0[$ and W is continuous there. Finally we have a solution $U = U_\epsilon + \epsilon^4 W(\epsilon)$ of (11) of the form (20). $\blacksquare$

A   Proof of Lemma 6

We follow and modify the argument of Iooss-Rucklidge [IR10][appendix C] used to prove that $\mathcal{H}_s$ is an algebra [IR10][lemma 4.2]. Let

$$u = \sum_{\mathbf{k} \in \Gamma} u^{(\mathbf{k})}(\nu) e^{i\mathbf{k}.\mathbf{x}}, \quad v = \sum_{\mathbf{k} \in \Gamma} v^{(\mathbf{k})}(\nu) e^{i\mathbf{k}.\mathbf{x}}$$

be elements of $\mathcal{H}_s \cap \mathcal{H}_{s'}$. We have

$$2^{-2s+1} \|uv\|_s^2 \leq \underbrace{\sum_{\mathbf{K}} \left\| \sum_{\mathbf{k}+\mathbf{k}'=\mathbf{K}} u^{(\mathbf{k})} v^{(\mathbf{k}')} \right\|_{p,K}^2 (1 + N_\mathbf{k}^2)^s}_{S_1} + \underbrace{\sum_{\mathbf{K}} \left\| \sum_{\mathbf{k}+\mathbf{k}'=\mathbf{K}} u^{(\mathbf{k})} v^{(\mathbf{k}')} \right\|_{p,K}^2 (1 + N_{\mathbf{k}'}^2)^s}_{S_2}.$$

Moreover, we have

$$\frac{1}{2} S_1 \leq \underbrace{\sum_{\mathbf{K}} \left| \sum_{\substack{\mathbf{k}+\mathbf{k}'=\mathbf{K} \\ N_\mathbf{k} \leq 3N_{\mathbf{k}'}}} u^{(\mathbf{k})} v^{(\mathbf{k}')} \right|^2 (1 + N_\mathbf{k}^2)^s}_{S_1'} + \underbrace{\sum_{\mathbf{K}} \left| \sum_{\substack{\mathbf{k}+\mathbf{k}'=\mathbf{K} \\ N_\mathbf{k} > 3N_{\mathbf{k}'}}} u^{(\mathbf{k})} v^{(\mathbf{k}')} \right|^2 (1 + N_\mathbf{k}^2)^s}_{S_1''}.$$

Using the fact that $1 \leq \left(\frac{16(1+N_{\mathbf{k}'}^2)}{(1+N_{\mathbf{K}}^2)} \right)^{s'}$ and Cauchy-Schwarz inequality, we obtain $(s' > q/2)$

$$S_1' \leq (K^*)^2 \|u\|_s^2 \|v\|_{s'}^2 \sum_{\mathbf{K}} \left(\frac{16}{(1+N_{\mathbf{K}}^2)} \right)^{s'} \leq C \|u\|_s^2 \|v\|_{s'}^2.$$

To obtain a similar bound for S_1'', we use the Iooss-Rucklidge dyadic decomposition of S_1'' : $\Delta_p u := \sum_{2^p \leq N_{\mathbf{k}} < 2^{p+1}} u^{(\mathbf{k})} e^{i\mathbf{k}.\mathbf{x}}$, $\Delta_{-1} u := u^{(\mathbf{0})}$ and $S_k u := \sum_{p=-1}^k \Delta_p u$. Then, $u = \sum_{p \geq -1} \Delta_p u \in \mathcal{H}_s$ if and only if $\sum_{p \geq -1} 2^{2ps} \|\Delta_p u\|_0^2 < +\infty$. According to the computation of [IR10][p. 387], we have

$$S_1'' = \left\| \sum_{\mathbf{K}} \left(\sum_{\substack{\mathbf{k}+\mathbf{k}'=\mathbf{K} \\ N_{\mathbf{k}}>3N_{\mathbf{k}'}}} u^{(\mathbf{k})} v^{(\mathbf{k}')} \right) e^{i\mathbf{K}.\mathbf{x}} \right\|_s \leq C \sum_{j=-1}^{+\infty} 2^{2js} \left\| \left(\sum_{p=j-1}^{j+1} \Delta_j (S_{p-1} v \Delta_p u) \right) \right\|_0^2 \quad (55)$$

Since $s' > q/2$, by Cauchy-Schwarz, we have $\sum |v^{(\mathbf{k}')}| \leq c \|v\|_{s'}$. So, following the computations [IR10][p.388], we obtain $\|S_{p-1} v \Delta_p u\|_0^2 \leq C \|\Delta_p u\|_0^2 \|v\|_{s'}^2$. From this and following the same computation as in [IR10][p. 388], we obtain

$$S_1'' \leq C'' \|v\|_{s'}^2 \sum_{p=-1}^{\infty} 2^{2ps} \|\Delta_p u\|_0^2 \leq C_1 \|v\|_{s'}^2 \|u\|_s^2$$

To get an estimate for S_2, we just need to interchange the role of u and v and the result is proved.

B   Proof of Lemma 16

We assume $x, y > 0$, and $p > 0$ and

$$|x - y| \leq K.$$

Then, we prove that

$$|(1+x)^p - (1+y)^p| \leq d(p)(1+x)^{p-1},$$

with

$$
\begin{aligned}
d(p) &= pK(1+K) \text{ for } p > 1, \\
&= pK(1+K)^{1-p} \text{ for } p \leq 1.
\end{aligned}
$$

Proof. For some t between x and y, we have

$$(1+x)^p - (1+y)^p = p(x-y)(1+t)^{p-1}.$$

If $p \geq 1$ we use $(1+t)/(1+x) = 1 + (t-x)/(1+x) \leq 1 + K$ if $x < t < y$ and ≤ 1 if $y < t < x$. This proves the lemma if $p \geq 1$.

Similarly if $p \leq 1$ we use $(1+x)/(1+t) \leq 1$ if $x < t < y$ and $= 1 + (x-t)/(1+t) \leq 1+K$ if $y < t < x$ and so $(1+t)^{p-1} \leq const.(1+x)^{p-1}$. ∎

References

[AG12]   M. Argentina and G.Iooss. Quasipatterns in a parametrically forced horizontal fluid film. *PhysicaD: Nonlinear Phenomena*, 241, 16:1306–1321, 2012.

[Arn80]   V.I. Arnold. *Chapitres supplémentaires de la théorie des équations différentielles ordinaires.* Mir, 1980.

[BBP10]   M. Berti, P. Bolle, and M. Procesi. An abstract Nash-Moser theorem with parameters and applications to PDEs. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 27(1):377–399, 2010.

[BCM92]   P. Alstrom B. Christiansen and M.T.Levinsen. Ordered capillary-wave states: Quasi-cristals, hexagons,and radial waves. *Phys. rev. Lett.*, 68:2157–2160, 1992.

[Ber07]   Massimiliano Berti. *Nonlinear oscillations of Hamiltonian PDEs.* Progress in Nonlinear Differential Equations and their Applications, 74. Birkhäuser Boston Inc., Boston, MA, 2007.

[Bou95]   J. Bourgain. Construction of periodic solutions of nonlinear wave equations in higher dimension. *Geom. Funct. Anal.*, 5(4):629–639, 1995.

[Cra00]   Walter Craig. *Problèmes de petits diviseurs dans les équations aux dérivées partielles*, volume 9 of *Panoramas et Synthèses [Panoramas and Syntheses].* Société Mathématique de France, Paris, 2000.

[D60]   J. Dieudonné. Foundations of modern analysis. Pure and Applied Mathematics, Vol. X Academic Press,1960.

[EF94]   W. S. Edwards and S. Fauve. Patterns and quasi-patterns in the Faraday experiment. *J. Fluid Mech.*, 278:123–148, 1994.

[IR10]   G. Iooss and A. M. Rucklidge. On the existence of quasipattern solutions of the Swift-Hohenberg equation. *J. Nonlinear Sci.*, 20(3):361–394, 2010.

[Kat95]   Tosio Kato. *Perturbation theory for linear operators.* Classics in Mathematics. Springer-Verlag, Berlin, 1995. Reprint of the 1980 edition.

[RR03]   A. M. Rucklidge and W. J. Rucklidge. Convergence properties of the 8, 10 and 12 mode representations of quasipatterns. *Phys. D*, 178(1-2):62–82, 2003.

[RS07]   A. M. Rucklidge and M. Silber. Quasipatterns in parametrically forced systems. *Phys. Rev. E (3)*, 75(5):055203, 4, 2007.