On the Existence of Quasipattern Solutions
of the Swift–Hohenberg Equation

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Abstract

Quasipatterns (two-dimensional patterns that are quasiperiodic in any spatial direction) remain one of the outstanding problems of pattern formation. As with problems involving quasiperiodicity, there is a small divisor problem. In this paper, we consider 8-fold, 10-fold, 12-fold, and higher order quasipattern solutions of the Swift–Hohenberg equation. We prove that a formal solution, given by a divergent series, may be used to build a smooth quasiperiodic function which is an approximate solution of the pattern-forming PDE up to an exponentially small error.

Keywords: bifurcations, quasipattern, small divisors, Gevrey series

AMS: 35B32, 35C20, 40G10, 52C23

1   Introduction

Quasipatterns remain one of the outstanding problems of pattern formation. These are two-dimensional patterns that have no translation symmetries and are quasiperiodic in any spatial direction (see figure 1). In spite of the lack of translation symmetry, the spatial Fourier transforms of quasipatterns have discrete rotational order (most often, 8, 10 or 12-fold). In contrast, regular patterns have translation symmetries (they are periodic in space) and so cannot have 8, 10 or 12-fold rotation symmetry. Quasipatterns were first discovered in nonlinear pattern-forming systems in the Faraday wave experiment [10,14], in which a layer of fluid is subjected to vertical oscillation. Since their discovery, they have also been found in nonlinear optical systems [19], shaken convection [28,32] and in liquid crystals [25], as well as being investigated in detail in large aspect ratio Faraday wave experiments [1, 4, 5, 24].
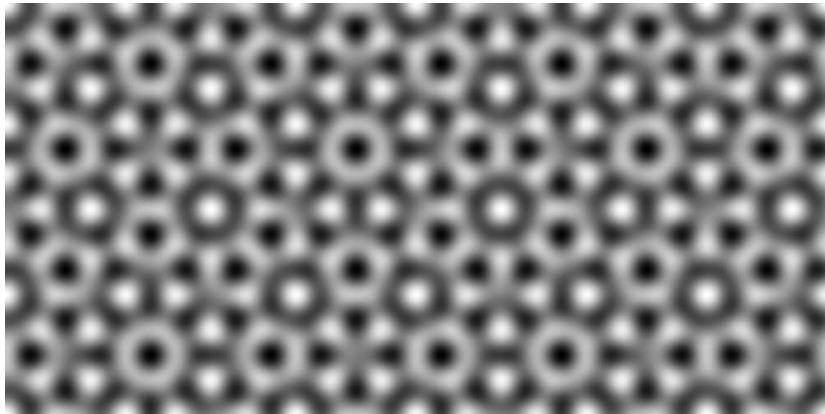
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Figure 1: Example 8-fold quasipattern. This is an approximate solution of the Swift–Hohenberg equation (1) with $\mu = 0.1$, computed by using Newton iteration to find an equilibrium solution of the PDE truncated to wavenumbers satisfying $|\mathbf{k}| \leq \sqrt{5}$ and to the quasilattice Γ_{27}.

In many of these experiments, the domain is large compared to the size of the pattern, and the boundaries appear to have little effect. Furthermore, the pattern is usually formed in two directions (x and y), while the third direction (z) plays little role. Mathematical models of the experiments are therefore often posed with two unbounded directions, and the basic symmetry of the problem is $E(2)$, the Euclidean group of rotations, translations and reflections of the (x, y) plane.

The mathematical basis for understanding the formation of regular patterns is well founded in *equivariant bifurcation theory* [16]. With regular (spatially periodic) patterns, the pattern-forming problem (usually a PDE) is posed in a periodic spatial domain instead of the infinite plane. Spatially periodic patterns have Fourier expansions with wavevectors that live on a lattice. There is a parameter μ in the PDE, and at the point of onset of the pattern-forming instability ($\mu = 0$), the primary modes have zero growth rate and all other modes on the lattice have negative growth rates that are bounded away from zero. In this case, the infinite-dimensional PDE can be reduced rigorously to a finite-dimensional set of equations for the amplitudes of the primary modes [8,9, 17,20,31], and existence of regular patterns as solutions of the pattern-forming PDE can be proved. The coefficients of leading order terms in these amplitude equations can be calculated and the values of these coefficients determine how the amplitude of the pattern depends on the parameter μ, and which of the regular patterns that fit into the periodic domain are stable. The solutions of the PDE are expressed as power series in $\sqrt{\mu}$, which can be computed, and which has a non-zero radius of convergence.

In contrast, quasipatterns do not fit into any spatially periodic domain and

have Fourier expansions with wavevectors that live on a *quasilattice* (defined below). At the onset of pattern formation, the primary modes have zero growth rate but there are other modes on the quasilattice that have growth rates arbitrarily close to zero, and techniques that are used for regular patterns cannot be applied. (These small growth rates appear as *small divisors*, as seen below.) If weakly nonlinear theory is applied in this case without regard to its validity, this results in a divergent power series [29], and this approach does not lead to a convincing argument for the existence of quasipattern solutions of the pattern-forming problem.

This paper is primarily concerned with proving the *existence* of quasipatterns as steady solutions of the simplest pattern-forming PDE, the Swift–Hohenberg equation:

$$\frac{\partial U}{\partial t} = \mu U - (1 + \Delta)^2 U - U^3 \tag{1}$$

where $U(x, y, t)$ is real and μ is a parameter. We are not concerned with the *stability* of these quasipatterns: in fact, they are almost certainly unstable in the Swift–Hohenberg equation. Stability of a pattern depends on the coefficients in the amplitude equations (as computed using weakly nonlinear theory). In the Faraday wave experiment, and in more general parametrically forced pattern forming problems, resonant mode interactions have been identified as the primary mechanism for the stabilisation of quasipatterns and other complex patterns (see [30] and references therein). These mode interactions are not present in the Swift–Hohenberg equation, though their presence will not significantly alter our existence results.

In many situations involving a combination of nonlinearity and quasiperiodicity, small divisors can be handled using *hard implicit function theorems* [13], of which the KAM theorem is an example. Unfortunately, there is as yet no successful existence proof for quasipatterns using this approach, although these ideas have been applied successfully to a range of small-divisor problems arising in other types of PDEs [12, 21, 22]. There are also alternative approaches to describing quasicrystals based on Penrose tilings and on projections of high-dimensional regular lattices onto low-dimensional spaces [23].

We take a different approach in this paper: we show how the divergent power series that is generated by the naive application of weakly nonlinear theory can be used to generate a smooth quasiperiodic function that (a) shares the same asymptotic expansion as the naive divergent series, and (b) satisfies the PDE (1) with an exponentially small error as μ tends to 0. This approach is based on summation techniques for divergent power series: see [2,7,27] for other examples. In order to make the paper self-contained, we put in Appendices some proofs of useful results, even though they are "known".

In section 2, we define the quasilattice and derive Diophantine bounds for the small divisors that will arise in the nonlinear problem, for Q-fold quasilattices: Lemma 2.1 extends the results of [29] covering the cases $Q = 8$, 10, 12 to any even $Q \geq 8$. We then compute in section 3 (following [29]) the power series in $\sqrt{\mu}$ for a formal quasipattern solution U of the Swift–Hohenberg equation, where μ is the bifurcation parameter in the PDE.

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In section 4, we define an appropriate function space $\mathcal{H}_s$: each term in the formal power series U is in this space. In section 5, we prove (Theorem 5.1) bounds on the norm of each term in the formal power series solution of the PDE. In the Q-fold case, the norm of the $\mu^{n+\frac{1}{2}}$ term in the power series for the quasipattern is bounded by a constant times $K^n(n!)^{2\alpha}$, where K is a constant and $\alpha/2+1$ is the order of the algebraic number $\omega = 2\cos(2\pi/Q)$, which is also half of *Euler's Totient function* $\varphi(Q)$ ($\alpha = 2$ for $Q = 8$, 10 and 12, $\alpha = 4$ for $Q = 14$ and 18, $\alpha = 6$ for $Q = 16$, 20, 24, 30, ...). This result was announced in [29] for $Q \leq 12$, and is extended here to $Q \geq 14$. With a bound that grows in this way with n, the power series is *Gevrey - 2α*, taking values in a space of quasiperiodic functions.

In section 6, for convenience, we consider the cases $Q = 8$, 10 and 12. We introduce a small parameter ζ related to the bifurcation parameter μ by $\zeta = \sqrt[4]{\mu}$, so that the norm of the ζ^{4n+2} term in the power series for U is also bounded by a constant times $K^n(n!)^4 < K^n(4n!)$. We use the *Borel transform* $\widehat{U}$ of the formal solution U: the ζ^{4n+2} term in the power series for $\widehat{U}$ is the ζ^{4n+2} term in the power series for U divided by $(4n + 2)!$. With this definition, $\widehat{U}$ is an analytic function of ζ in the disk $|\zeta| < K^{-1/4}$, and for each ζ in this disk, $\widehat{U}$ is a quasiperiodic function of (x, y) in the space $\mathcal{H}_s$. Of course the new function $\widehat{U}$ does not satisfy the original PDE, but we prove that it satisfies a transformed PDE (Theorem 6.2).

The next stage would be to invert the Borel transform: however, the usual inverse Borel transform is a line integral (related to the Laplace transform) taking ζ from 0 to ∞, and $\widehat{U}$ is only an analytic function of ζ for ζ in a disk. If the definition of $\widehat{U}$ could be extended to a line in the complex ζ plane, the inverse Borel transform would provide a quasiperiodic solution of the PDE – this remains an open problem.

Since the full inverse Borel transform cannot be used, in section 7, we use a truncated integral to define $\bar{U}(\nu)$. This involves integrating ζ along a line segment inside the disk where $\widehat{U}$ is analytic, weighted by an exponential that decays rapidly as $\nu \to 0$. We show that $\bar{U}(\nu)$ and $U(\mu)$ have the same power series expansion when we set $\nu = \sqrt[4]{\mu}$, but unlike U, $\bar{U}(\nu)$ is a C^∞ function of ν in a neighbourhood of 0, taking values in $\mathcal{H}_s$. In other words, $\bar{U}(\mu^{1/4})$ is a quasiperiodic function of (x, y) for small enough μ. This function is not an exact solution of the Swift–Hohenberg PDE, but we show in Theorem 7.2 that the residual when $\bar{U}(\mu^{1/4})$ is substituted into the PDE is exponentially small as $\mu \to 0$.

In conclusion, we have shown that, for any even $Q \geq 8$, the divergent power series $U(\mu)$ generated by the naive application of weakly nonlinear theory can be used to find a smooth quasiperiodic function $\bar{U}(\mu^{1/2\alpha})$ that shares the same asymptotic expansion as U, and that satisfies the PDE with an exponentially small error.

This technique does not prove the existence of a quasiperiodic solution of the PDE. However, this is a first step towards an existence proof for quasiperiodic solutions of PDEs like (1). In particular, we may hope to use $\bar{U}$ as a

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starting point for the Newton iteration process that would form part of an existence proof using the Nash–Moser theorem. As an aside, ordinary numerical Newton iteration succeeds in finding an approximate solution of the truncated PDE for values of μ where the formal power series has already diverged, as in figure 3.

We believe that an analogous result might be proved for example in the Rayleigh–Bénard convection problem, since the dispersion equation possesses the same property as in the present model: at the critical value of the parameter there is a circle of critical wavevectors in the plane. The method may also extend to the case of the Faraday wave experiment by considering fixed points of a stroboscopic map.

In the present work we consider quasilattices generated by regularly spaced wavevectors on the unit circle, and solutions invariant under $2\pi/Q$ rotations. It might be worth studying the case of solutions having less symmetry on the same quasilattice, or quasilattices (still dense in the plane) generated by wavevectors that are irregularly spaced.

2   Small divisors: Quasilattices and Diophantine bounds

Let $Q \in \mathbb{N}$ ($Q \geq 8$) be the order of a quasipattern and define wavevectors

$$\mathbf{k}_j = \left(\cos\left(2\pi \frac{j-1}{Q} \right), \sin\left(2\pi \frac{j-1}{Q} \right) \right), \qquad j = 1, 2, \ldots, Q$$

(see figure 2a). We define the *quasilattice* $\Gamma \subset \mathbb{R}^2$ to be the set of points spanned by integer combinations $\mathbf{k_m}$ of the form

$$\mathbf{k_m} = \sum_{j=1}^{Q} m_j \mathbf{k}_j, \qquad \text{where} \qquad \mathbf{m} = (m_1, m_2, \ldots, m_Q) \in \mathbb{N}^Q. \qquad (2)$$

The set Γ is dense in $\mathbb{R}^2$.

We are interested in real functions $U(\mathbf{x})$ that are linear combinations of Fourier modes $e^{i\mathbf{k} \cdot \mathbf{x}}$, with $\mathbf{x} \in \mathbb{R}^2$ and $\mathbf{k} \in \Gamma$. If $U(\mathbf{x})$ is to be a real function, we need Q to be even, with $\mathbf{k}_j$ and $-\mathbf{k}_j$ in Γ, hence the quasilattice Γ is symmetric with respect to the origin.

Define $|\mathbf{m}| = \sum_j m_j$, then, for a given wavevector $\mathbf{k} \in \Gamma$, we define the order $N_{\mathbf{k}}$ of $\mathbf{k}$ by

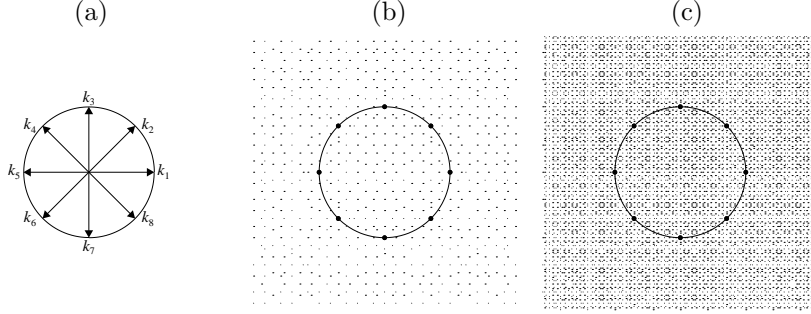$$N_{\mathbf{k}} = \min\{|\mathbf{m}|; \mathbf{k} = \mathbf{k_m}\}.$$

Figure 2: Example quasilattice with $Q = 8$, after [29]. (a) The 8 wavevectors with $|\mathbf{k}| = 1$ that form the basis of the quasilattice. (b,c) The truncated quasilattices Γ_9 and Γ_{27}. The small dots mark the positions of combinations of up to 9 or 27 of the 8 basis vectors on the unit circle. Note how the density of points increases with $N_{\mathbf{k}}$.

The reason for this is that, for a given $\mathbf{k}$, there is an infinite set of $\mathbf{m}$'s satisfying $\mathbf{k} = \mathbf{k_m}$. For example, we could increase m_j and $m_{j+Q/2}$ by 1: this increases $|\mathbf{m}|$ by 2 but does not change $\mathbf{k_m}$.

Whenever solutions are computed numerically, it is necessary to use only a finite number of Fourier modes, so we define the *truncated quasilattice* Γ_N to be:

$$\Gamma_N = \{\mathbf{k} \in \Gamma : N_{\mathbf{k}} \leq N\}. \tag{3}$$

Figure 2(b,c) shows the truncated quasilattices Γ_9 and Γ_{27} in the case $Q = 8$.

In the calculations that follow, we will require Diophantine bounds on the magnitude of the small divisors in terms of $N_{\mathbf{k}}$. We see below that the small divisors are $\left||\mathbf{k}|^2 - 1\right|$, for $\mathbf{k} \in \Gamma$. To compute the required bound, we start with

$$|\mathbf{k_m}|^2 = \sum_{1 \leq j_1 \leq j_2 \leq Q} 2m_{j_1} m_{j_2} \cos \frac{2\pi}{Q}(j_1 - j_2),$$

hence

$$|\mathbf{k_m}|^2 - 1 = q_0 + \omega q_1 + \cdots + \omega^l q_l \tag{4}$$

where $q_0 + 1$ and q_j, $j = 1, \ldots, l$ are integer-valued *quadratic forms* of $\mathbf{m}$, and ω is an *algebraic number* defined by

$$\omega = 2 \cos \frac{2\pi}{Q}.$$

This number is solution of a polynomial of degree $l + 1 \leq Q/2$ with integer

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coefficients. For example, we have in the case $Q = 8$:

$$|\mathbf{k_m}|^2 = \sum_{j=1}^{4} m_j'^2 + \sqrt{2} \left(m_1' m_2' + m_2' m_3' + m_3' m_4' - m_4' m_1' \right), \qquad (5)$$

$$N_{\mathbf{k}} = \sum_{j=1}^{4} |m_j'| \qquad (6)$$

where $m_j' = m_j - m_{j+Q/2}$. More generally we have

$$N_{\mathbf{k}} \leq \sum_{j=1}^{Q/2} |m_j'|.$$

The above inequality can occur strictly (for example) in the case $Q = 12$, because only 4 of the 12 vectors $\mathbf{k}_j$ are rationally independent in this case.

In the cases $Q = 8$, 10 and 12, the irrational numbers $\omega = 2\cos(2\pi/Q)$ are $\sqrt{2}$, $\frac{1+\sqrt{5}}{2}$ and $\sqrt{3}$: these are quadratic algebraic numbers ($l + 1 = 2$), while for $Q = 14$, ω is cubic ($l + 1 = 3$). For an algebraic number ω of order $l + 1$, the quantity $|q_0 + \omega q_1 + \cdots + \omega^l q_l|$ may be as small as we want for good choices of large integers q_j.

In [29], it was proved that in the cases $Q = 8$, 10 and 12, there is a constant $c > 0$ such that

$$\left| |\mathbf{k}|^2 - 1 \right| \geq \frac{c}{N_{\mathbf{k}}^2}, \qquad \text{for any } \mathbf{k} \in \Gamma \text{ with } |\mathbf{k}| \neq 1. \qquad (7)$$

The proof relies on the fact that for quadratic algebraic numbers, there exists $C > 0$ such that

$$|p - \omega q| \geq \frac{C}{q}$$

holds for any $(p, q) \in \mathbb{Z}^2$, $q \neq 0$ [18]. Now using the fact that q is quadratic in $\mathbf{m}$ (see (5)) we have

$$q \leq Q N_{\mathbf{k}}^2 \qquad (8)$$

from which (7) can be deduced.

We extend the Diophantine bound (7) to any even $Q \geq 8$, and prove that there exists $c > 0$ depending only on Q, such that for any $\mathbf{k} \in \Gamma$, with $|\mathbf{k}| \neq 1$, there exists $\alpha \geq 2$ such that

$$\left| |\mathbf{k}|^2 - 1 \right| \geq \frac{c}{N_{\mathbf{k}}^{\alpha}}. \qquad (9)$$

To show this, we use the following known result (see [11]) proved in Appendix A:

Lemma 2.1 *Let ω be an algebraic number of order $l + 1$, that is, a solution of $P(\omega) = 0$ where P is a polynomial of degree $l + 1$ with integer coefficients, that is irreducible on $\mathbb{Q}$. Then, there exists a constant C and an integer α with*

$2 \leq \alpha \leq 2l$ *such that for any* $\mathbf{q} = (q_0, q_1, \ldots, q_l) \in \mathbb{Z}^{l+1} \backslash \{0\}$, *the following estimate*

$$|q_0 + q_1\omega + q_2\omega^2 + \cdots + q_l\omega^l| \geq \frac{C}{|\mathbf{q}|^{\alpha/2}} \tag{10}$$

holds, where $|\mathbf{q}| = \sum_{0 \leq j \leq l} |q_j|$.

As an aside, the polynomials P are related to cyclotomic polynomials, and the order $l + 1$ of the algebraic number ω is $\varphi(Q)/2$, where $\varphi(Q)$ is Euler's Totient function [3], the number of positive integers $j < Q$ such that j and Q are relatively prime. For example, $\varphi(14) = 6$ since the 6 numbers 1, 3, 5, 9, 11 and 13 have no factors in common with 14, and so $l + 1 = 3$ in the case $Q = 14$.

The expression $2\cos\frac{2p\pi}{Q}$ as a polynomial in ω, for $1 \leq p \leq Q - 1$, as

$$2\cos\frac{2p\pi}{Q} = \omega^p - p\omega^{p-2} + \frac{p(p-3)}{2}\omega^{p-4}.....$$

has integer coefficients which only depend on Q (easy proof by induction). Hence the estimate (8) is replaced by

$$|q| \leq c(Q)N_{\mathbf{k}}^2$$

where $c(Q)$ depends only on Q. Then estimate (9) is satisfied by taking

$$c = \frac{C}{[c(Q)]^{\alpha/2}},$$

It should be observed that the estimate $\alpha \leq 2l$ might be not optimal, however for $l + 1 = 2$ we have indeed $\alpha = 2$ and for $Q = 14$, this gives $\alpha \leq 4$, and computations of $\left||\mathbf{k}|^2 - 1\right|$ in the case $Q = 14$ up to $N_{\mathbf{k}} = 1000$ suggest that indeed $\alpha = 4$ in this case [29].

3 Formal power series computation

Let us consider the *steady Swift–Hohenberg equation*

$$(1 + \Delta)^2 U - \mu U + U^3 = 0 \tag{11}$$

where we look for a Q-fold quasiperiodic function U of $\mathbf{x} \in \mathbb{R}^2$, defined by Fourier coefficients $U_{\mathbf{k}}$ on a quasilattice Γ as defined above. We write formally

$$U(\mathbf{x}) = \sum_{\mathbf{k} \in \Gamma} U_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{x}},$$

the meaning of this sum being given in section 4. We seek a solution of (11), bifurcating from the origin when $\mu = 0$, that is *invariant under rotations by* $2\pi/Q$. First we look for a formal solution in the form of a power series of an amplitude. More precisely we look for the series

$$U(\mathbf{x}, \mu) = \sqrt{\frac{\mu}{\beta}} \sum_{n \geq 0} \mu^n U^{(n)}(\mathbf{x}) \tag{12}$$

as a formal solution of (11), where all coefficients $U^{(n)}$ are invariant under rotations by $2\pi/Q$ of the plane. The coefficient β will be given by fixing $U^{(0)}_{\mathbf{k}_1}$.

At order $\mathcal{O}(\sqrt{|\mu|})$ in (11) we have

$$0 = (1 + \Delta)^2 U^{(0)} \tag{13}$$

and we choose the solution

$$U^{(0)} = \sum_{j=1}^{Q} e^{i\mathbf{k}_j \cdot \mathbf{x}}, \tag{14}$$

which is invariant under rotations by $2\pi/Q$ and defined up to a factor which we take equal to 1. Let us rewrite (11) in the form

$$\mathcal{L}_0 U = \mu U - U^3 \tag{15}$$

where

$$\mathcal{L}_0 = (1 + \Delta)^2.$$

At order $\mathcal{O}(|\mu|^{3/2})$ we have

$$\mathcal{L}_0 U^{(1)} = U^{(0)} - \beta^{-1}(U^{(0)})^3. \tag{16}$$

We need to impose a solvability condition, namely that the coefficients of $e^{i\mathbf{k}_j \cdot \mathbf{x}}$, for $j = 1, \ldots, Q$ on the left hand side of this equation must be zero. Because of the invariance under rotations by $2\pi/Q$, it is sufficient to cancel the coefficient of $e^{i\mathbf{k}_1 \cdot \mathbf{x}}$. This yields

$$\beta = 3(Q - 1) > 0, \tag{17}$$

and $U^{(1)}$ is known up to an element $\beta^{(1)}U^{(0)}$ in $\ker \mathcal{L}_0$, which is determined at the next step:

$$U^{(1)} = \widetilde{U}^{(1)} + \beta^{(1)}U^{(0)}, \qquad \widetilde{U}^{(1)} = \sum_{\mathbf{k} \in \Gamma, N_{\mathbf{k}}=3} \alpha_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{x}}, \tag{18}$$

$$\alpha_{3\mathbf{k}_j} = -1/64, \qquad \alpha_{2\mathbf{k}_j + \mathbf{k}_l} = -\frac{3}{(1 - |2\mathbf{k}_j + \mathbf{k}_l|^2)^2}, \qquad \mathbf{k}_j + \mathbf{k}_l \neq 0,$$

$$\alpha_{\mathbf{k}_j + \mathbf{k}_l + \mathbf{k}_r} = -\frac{6}{(1 - |\mathbf{k}_j + \mathbf{k}_l + \mathbf{k}_r|^2)^2}, \qquad j \neq l \neq r \neq j,$$

$$\mathbf{k}_j + \mathbf{k}_l \neq 0, \qquad \mathbf{k}_j + \mathbf{k}_r \neq 0, \qquad \mathbf{k}_r + \mathbf{k}_l \neq 0,$$

where $\widetilde{U}^{(1)}$ has no component on $e^{i\mathbf{k}_j \cdot \mathbf{x}}$.

Order $|\mu|^{n+1/2}$ in (15) leads for $n \geq 2$ to

$$\mathcal{L}_0 U^{(n)} = U^{(n-1)} - \beta^{-1} \sum_{\substack{k+l+r=n-1, \\ k,l,r \geq 0}} U^{(k)}U^{(l)}U^{(r)}. \tag{19}$$
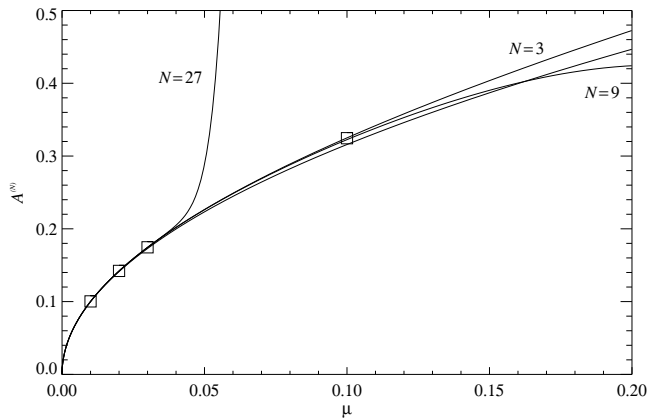
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Figure 3: Amplitude $A^{(N)}$ of the quasipattern, as a function of μ and of N, with $Q = 8$, $N = 1, 3, 9$ and 27, and scaled so that $A^{(1)} = \sqrt{\mu}$. Increasing the order of the truncation leads to divergence for smaller values of μ. The squares represent amplitudes computed by solving the PDE by Newton iteration, truncated to the quasilattice Γ_{27} ($N_{\mathbf{k}} \leq 27$) and restricted to wavevectors with $|\mathbf{k}| \leq \sqrt{5}$. Note that for $\mu = 0.1$, the Newton iteration succeeds in finding an equilibrium solution of the PDE, while the formal power series has diverged. The spatial form of the solution with $\mu = 0.1$ is shown in figure 1.

For $n = 2$, the solvability condition on the right hand side gives $\beta^{(1)}$, and $U^{(2)}$ is then determined up to $\beta^{(2)}U^{(0)}$. Indeed we obtain on the right hand side

$$
\begin{aligned}
U^{(1)} - 3\beta^{-1}U^{(1)}U^{(0)2} &= \widetilde{U}^{(1)} + \beta^{(1)}U^{(0)} - 3\beta^{-1}\beta^{(1)}U^{(0)3} - 3\beta^{-1}\widetilde{U}^{(1)}U^{(0)2} \\
&= -2\beta^{(1)} + \widetilde{U}^{(1)} - 3\beta^{-1}\widetilde{U}^{(1)}U^{(0)2} - 3\beta^{-1}\mathcal{L}_0\widetilde{U}^{(1)}, \quad (20)
\end{aligned}
$$

where we used the fact that the component of $U^{(0)3}$ on $e^{i\mathbf{k}_1 \cdot \mathbf{x}}$ is β (see (16)). Hence $2\beta^{(1)}$ is the coefficient of $e^{i\mathbf{k}_1 \cdot \mathbf{x}}$ of $-3\beta^{-1}\widetilde{U}^{(1)}U^{(0)2}$, and since all coefficients of $\widetilde{U}^{(1)}$ are negative, we find $\beta^{(1)} > 0$. We obtain in the same way the coefficients $\beta^{(n-1)}U^{(0)}$ of $U^{(n-1)}$ in using the solvability condition on the right hand side of (19).

It is then clear that we can continue to compute this expansion as far as we wish, where at each step we use the formal inverse of $\mathcal{L}_0$ on the complement of the kernel, which is one-dimensional because of the invariance under rotations by $2\pi/Q$. However, applying $\mathcal{L}_0^{-1}$ to $e^{i\mathbf{k} \cdot \mathbf{x}}$ introduces a factor

$$
\frac{1}{(1 - |\mathbf{k}|^2)^2},
$$

which may be very large for combinations $\mathbf{k} = \mathbf{k_m}$ with large $\mathbf{m}$, since points $\mathbf{k_m}$ of the quasilattice Γ sit as close as we want to the unit circle. This is a *small*

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divisor problem and computations indicate that the series (12) seems to diverge numerically [29]. We illustrate this in figure 3, plotting the amplitude $A^{(N)}$ against μ, where

$$A^{(N)} = ||\mathbf{P}_0 \sqrt{\frac{\mu}{\beta}} \sum_{n=0}^{(N-1)/2} \mu^n U^{(n)}||_s = \sqrt{\frac{\mu}{\beta}} \left(\sum_{n=0}^{(N-1)/2} \mu^n \beta^{(n)} \right) ||U^{(0)}||_s,$$

and the norm $|| \cdot ||_s$ and the projection operator $\mathbf{P}_0$ are defined below: $A^{(N)}$ is essentially the magnitude of the coefficient of $e^{i\mathbf{k}_1 \cdot \mathbf{x}}$ as a function of μ and of N, the maximum order of wavevectors included in the truncated power series.

However, we prove in section 5 that in all cases we can control the divergence of the coefficients of the series (12), and obtain a Gevrey estimate $||U^{(n)}||_s \leq \gamma K^n (n!)^{2\alpha}$, where the norm $|| \cdot ||_s$ is defined below.

4 Function spaces

We characterise the functions of interest by their Fourier coefficients on the quasilattice Γ generated by the Q unit vectors $\mathbf{k}_j$:

$$U(\mathbf{x}) = \sum_{\mathbf{k} \in \Gamma} U_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{x}}$$

Recall that for each $\mathbf{k} \in \Gamma$, there exists a vector $\mathbf{m} \in \mathbb{N}^Q$ such that $\mathbf{k} = \mathbf{k_m} = \sum_{j=1}^{Q} m_j \mathbf{k}_j$ and we can choose $\mathbf{m}$ such that

$$|\mathbf{m}| = N_{\mathbf{k}} = \min\{|\mathbf{m}| : \mathbf{k} = \mathbf{k_m}\}.$$

We have the following properties, proved in Appendix B:

Lemma 4.1 *(i) We define $m'_j = m_j - m_{j+Q/2}$, then*

$$N_{\mathbf{k}} = \min \left\{ \sum_{j=1}^{Q/2} |m'_j| : \sum_{j=1}^{Q/2} m'_j \mathbf{k}_j = \mathbf{k},\ m'_j \in \mathbb{Z},\ j = 1, \ldots, Q/2 \right\}. \tag{21}$$

(ii) We have the following inequalities:

$$N_{\mathbf{k}+\mathbf{k}'} \leq N_{\mathbf{k}} + N_{\mathbf{k}'}, \quad N_{-\mathbf{k}} = N_{\mathbf{k}}, \tag{22}$$

$$|\mathbf{k}| \leq N_{\mathbf{k}}. \tag{23}$$

(iii) We have the following estimate of the numbers of vectors $\mathbf{k}$ having a given $N_{\mathbf{k}}$:

$$\operatorname{card}\{\mathbf{k} : N_{\mathbf{k}} = N\} \leq c_1(Q) N^{Q/2-1} \tag{24}$$

where $c_1(Q)$ only depends on Q.

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Define now the space of functions

$$\mathcal{H}_s = \left\{ U = \sum_{\mathbf{k} \in \Gamma} U_\mathbf{k} e^{i\mathbf{k} \cdot \mathbf{x}} : ||U||_s^2 = \sum_{\mathbf{k} \in \Gamma} (1 + N_\mathbf{k}^2)^s |U_\mathbf{k}|^2 < \infty \right\}, \qquad (25)$$

which becomes a Hilbert space with the scalar product

$$\langle W, V \rangle_s = \sum_{\mathbf{k} \in \Gamma} (1 + N_\mathbf{k}^2)^s W_\mathbf{k} \overline{V}_\mathbf{k}. \qquad (26)$$

In the sequel we need the following lemma, proved in Appendix C:

Lemma 4.2 *The space $\mathcal{H}_s$ is a Banach algebra for $s > Q/4$. In particular there exists $c_s > 0$ such that*

$$||UV||_s \le c_s ||U||_s ||V||_s. \qquad (27)$$

For $\ell \ge 0$ and $s > \ell + Q/4$, $\mathcal{H}_s$ is continuously embedded into $\mathcal{C}^\ell$.

From now on, all inner products are s unless otherwise stated, so that we can remove the s subscripts throughout in scalar products.

We will also use the orthogonal projection on $\ker \mathcal{L}_0$: for any $U \in \mathcal{H}_s$, let

$$\mathbf{P}_0 U = \sum_{j=1,\dots,Q} U_{\mathbf{k}_j} e^{i\mathbf{k}_j \cdot \mathbf{x}},$$

and we denote by Q_0 the orthogonal projection:

$$\mathbf{Q}_0 = \mathbb{I} - \mathbf{P}_0,$$

which consists in suppressing the Fourier components of $e^{i\mathbf{k}_j \cdot \mathbf{x}}$, $j = 1, \dots, Q$. The norm of the linear operator $\mathbf{Q}_0$ is 1 in all spaces $\mathcal{H}_s$.

5   Gevrey estimates

In this section we prove rigorously a Gevrey estimate of the coefficients $U^{(n)}$ in (12). The estimate for $Q = 8$, 10 and 12 ($\alpha = 2$) was announced in [29]. Recall that a formal power series $\sum_{n=0}^\infty u_n \zeta^n$ is *Gevrey - k* [15], where k is a positive integer, if there are constants $\delta > 0$ and $K > 0$ such that

$$|u_n| \le \delta K^n (n!)^k \qquad \forall n \ge 0. \qquad (28)$$

Theorem 5.1 *For any even $Q \ge 8$, assume that $s > Q/4$. Then there exist positive numbers $K(Q, c, s)$ and $\delta(Q, s)$ such that there exists a unique $U(\mu)$ as a power series in $\mu^{1/2}$, all coefficients belonging to $\mathcal{H}_s$, that is a formal solution*

of (11), and that satisfies

$$U = \sqrt{\frac{\mu}{\beta}} \sum_{n \geq 0} \mu^n U^{(n)}, \tag{29}$$

$$U^{(n)} = \beta^{(n)} U^{(0)} + \widetilde{U}^{(n)}, \qquad \langle \widetilde{U}^{(n)}, e^{i\mathbf{k}_j \cdot \mathbf{x}} \rangle_s = 0, \quad j = 1, \dots, Q,$$

$$\|\widetilde{U}^{(n)}\|_s \leq \delta \frac{(Q-1)}{2^{s/2} c_s^2 Q} K^n (n!)^{2\alpha}, \quad n \geq 1,$$

$$|\beta^{(n)}| \leq \delta K^n (n!)^{2\alpha}, \quad n \geq 1.$$

where α is the integer defined in Lemma 2.1. From the above inequalities, it follows that

$$\|U^{(n)}\|_s \leq \gamma K^n (n!)^{2\alpha}, \quad n \geq 0,$$

where γ is related to δ, Q and s only.

Remark 5.2 *The above Theorem claims that the series U in powers of $\sqrt{\mu}$ is Gevrey - α taking its values in $\mathcal{H}_s$.*

Proof We choose $s > Q/4$ since Lemma 4.2 insures that $\mathcal{H}_s$ is then a Banach algebra. We notice that

$$\|e^{i\mathbf{k}_j \cdot \mathbf{x}}\|_s = 2^{s/2},$$

and

$$\|U^{(0)}\|_s = 2^{s/2} \sqrt{Q}. \tag{30}$$

We also have $\beta^{(0)} = 1$ and $\widetilde{U}^{(0)} = 0$. Now we notice from (9) that for $|\mathbf{k}| \neq 1$ we have

$$\left| |\mathbf{k}|^2 - 1 \right|^{-2} \leq \frac{N_{\mathbf{k}}^{2\alpha}}{c^2},$$

which controls the unboundedness of the pseudo-inverse $\widetilde{\mathcal{L}}_0^{-1}$ (inverse of $\mathcal{L}_0$ restricted to the orthogonal complement of its kernel). Indeed $\widetilde{\mathcal{L}}_0^{-1}$ is bounded from $\mathcal{H}_s$ to $\mathcal{H}_{s-2\alpha}$. The basic observation here is that the coefficient $U^{(n)}$ of μ^n has a finite Fourier expansion in $e^{i\mathbf{k} \cdot \mathbf{x}}$, with $\mathbf{k} = \sum_{j=1}^{Q} m_j \mathbf{k}_j$, $\sum m_j \leq 2n+1$, hence $N_{\mathbf{k}} \leq 2n+1$. Since for $\widetilde{U}^{(1)}$ we have $|\mathbf{m}| = 3$ in all $\mathbf{k}_{\mathbf{m}}$'s, equation (16) leads to

$$\|\widetilde{U}^{(1)}\|_s \leq \frac{3^{2\alpha} c_s^2 2^{3s/2} Q^{3/2}}{c^2 3(Q-1)}. \tag{31}$$

We set

$$U^{(n)} = \beta^{(n)} U^{(0)} + \widetilde{U}^{(n)}, \qquad \widetilde{U}^{(n)} = \mathbf{Q}_0 U^{(n)}, \tag{32}$$

and replacing this decomposition in (19) we obtain, by taking the scalar product with $e^{i\mathbf{k}_1 \cdot \mathbf{x}}$

$$\beta^{(n-1)} 2^s - \frac{1}{\beta} \left\langle 3 U^{(n-1)} U^{(0)2}, e^{i\mathbf{k}_1 \cdot \mathbf{x}} \right\rangle - \frac{1}{\beta} \left\langle \sum_{\substack{k+l+r=n-1, \\ 0 \leq k,l,r \leq n-2}} U^{(k)} U^{(l)} U^{(r)}, e^{i\mathbf{k}_1 \cdot \mathbf{x}} \right\rangle = 0,$$

where we have used $\left\langle U^{(0)}, e^{i\mathbf{k}_1\cdot\mathbf{x}}\right\rangle = ||e^{i\mathbf{k}_1\cdot\mathbf{x}}||_s^2 = 2^s$. Next, we use

$$
\begin{aligned}
\left\langle 3U^{(n-1)}U^{(0)2}, e^{i\mathbf{k}_1\cdot\mathbf{x}}\right\rangle &= \beta^{(n-1)}\left\langle 3U^{(0)3}, e^{i\mathbf{k}_1\cdot\mathbf{x}}\right\rangle + \left\langle 3\widetilde{U}^{(n-1)}U^{(0)2}, e^{i\mathbf{k}_1\cdot\mathbf{x}}\right\rangle \\
&= 3\beta\beta^{(n-1)}2^s + \left\langle 3\widetilde{U}^{(n-1)}U^{(0)2}, e^{i\mathbf{k}_1\cdot\mathbf{x}}\right\rangle,
\end{aligned}
$$

and we are led to solve with respect to $\beta^{(n-1)}, \widetilde{U}^{(n)}$ the following system for $n \geq 2$

$$
\mathcal{L}_0\widetilde{U}^{(n)} = \widetilde{U}^{(n-1)} - \beta^{-1}\mathbf{Q}_0 \sum_{\substack{k+l+r=n-1, \\ k,l,r\geq 0}} U^{(k)}U^{(l)}U^{(r)}, \tag{33}
$$

$$
\beta^{(n-1)} = \frac{-1}{2^{1+s}\beta}\left\langle 3\widetilde{U}^{(n-1)}U^{(0)2} + \sum_{\substack{k+l+r=n-1, \\ 0\leq k,l,r\leq n-2}} U^{(k)}U^{(l)}U^{(r)}, e^{i\mathbf{k}_1\cdot\mathbf{x}}\right\rangle. \tag{34}
$$

Now we make the following recurrence assumption: there exist positive constants γ_1, δ and K, depending on Q, s and α, such that

$$
\begin{aligned}
||\widetilde{U}^{(p)}||_s &\leq \gamma_1 K^p(p!)^{2\alpha}, \qquad p = 0, 1, \ldots, n-1, \\
|\beta^{(p)}| &\leq \delta K^p(p!)^{2\alpha}, \qquad p = 1, \ldots, n-2.
\end{aligned} \tag{35}
$$

These estimates hold for $\widetilde{U}^{(0)} = 0$ and for $\widetilde{U}^{(1)}$ provided that γ_1 and K satisfy

$$
\frac{3^{2\alpha}c_s^2 2^{3s/2}Q^{3/2}}{c^2 3(Q-1)} \leq \gamma_1 K. \tag{36}
$$

Putting these together results in

$$
||U^{(p)}||_s = ||\beta^{(p)}U^{(0)} + \widetilde{U}^{(p)}||_s \leq \left(2^{s/2}\delta\sqrt{Q} + \gamma_1\right)K^p(p!)^{2\alpha},
$$

or

$$
||U^{(p)}||_s \leq \gamma K^p(p!)^{2\alpha}, \qquad \text{with} \qquad \gamma = 2^{s/2}\delta\sqrt{Q} + \gamma_1. \tag{37}
$$

The resolution of (33) and (34) provides $\beta^{(n-1)}$ and $\widetilde{U}^{(n)}$, starting with $n = 2$. A useful lemma is the following, proved in Appendix D.

Lemma 5.3 *The following estimates hold true for $\alpha \geq 2$:*

$$
\Pi_{3,n} = \sum_{\substack{k+l+r=n \\ k,l,r\geq 0}} (k!l!r!)^{2\alpha} \leq 4(n!)^{2\alpha}, \quad n \geq 1
$$

$$
\Pi_{3,n}' = \sum_{\substack{k+l+r=n \\ 0\leq k,l,r\leq n-1}} (k!l!r!)^{2\alpha} \leq 10((n-1)!)^{2\alpha}, \quad n \geq 2.
$$

Thanks to Lemma 5.3 and the estimate for $||U^{(p)}||_s$ in (37), we observe that

$$
\left\|\sum_{\substack{k+l+r=n-1 \\ 0\leq k,l,r\leq n-2}} U^{(k)}U^{(l)}U^{(r)}\right\|_s \leq 10c_s^2\gamma^3 K^{n-1}((n-2)!)^{2\alpha}.
$$

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From this it follows that

$$|\beta^{(n-1)}| \leq \frac{c_s^2}{2^{1+s/2}\beta} K^{n-1}((n-1)!)^{2\alpha} \left\{ 3\gamma_1 2^s Q + 10\gamma^3 \right\},$$

and the recurrence assumption is realized if

$$\frac{c_s^2}{3(Q-1)2^{1+s/2}} \left\{ 3\gamma_1 2^s Q + 10\gamma^3 \right\} \leq \delta \tag{38}$$

holds. Now we have, still by using Lemma 5.3

$$
\begin{aligned}
\|\widetilde{U}^{(n)}\|_s &\leq \frac{(2n+1)^{2\alpha}K^{n-1}((n-1)!)^{2\alpha}}{c^2} \left\{ \gamma_1 + \frac{4c_s^2}{\beta}\gamma^3 \right\} \tag{39}\\
&\leq K^n(n!)^{2\alpha}(2+\frac{1}{n})^{2\alpha}\frac{1}{Kc^2} \left\{ \gamma_1 + \frac{4c_s^2}{\beta}\gamma^3 \right\}.
\end{aligned}
$$

The factor $(2n+1)^{2\alpha}/c^2$ here comes from the pseudo-inverse of $\mathcal{L}_0$ acting on functions containing modes of order up to $2n+1$. The recurrence assumption is realized if

$$\frac{3^{2\alpha}}{c^2} \left\{ \gamma_1 + \frac{4c_s^2}{\beta}\gamma^3 \right\} \leq \gamma_1 K. \tag{40}$$

We now must choose γ_1, δ and K in such a way as to satisfy the three conditions (36), (38) and (40). Indeed, we may choose γ_1 and δ small enough, and K large enough for having

$$
\begin{aligned}
\gamma_1 &= \delta\frac{(Q-1)}{2^{s/2}c_s^2 Q}, \\
\delta^2 &\leq \frac{3(Q-1)2^{s/2-1}}{5c_s^2} \left(2^{s/2}\sqrt{Q} + \frac{Q-1}{2^{s/2}c_s^2 Q} \right)^{-3}, \\
K &= \max\left\{ \frac{3^{2\alpha}}{c^2} \left(1 + \frac{2^{s+1}c_s^2 Q}{5(Q-1)} \right), \frac{1}{\delta}\frac{3^{2\alpha-1}2^{2s}c_s^4 Q^{5/2}}{c^2(Q-1)^2} \right\}.
\end{aligned}
$$

We conclude that the bounds on $\|\widetilde{U}^{(n)}\|_s$ and $|\beta^{(n)}|$ in Theorem 5.1 hold, and that (37), which holds for $0 \leq p \leq n-1$, also holds for $p = n$, and so

$$\|U^{(n)}\|_s \leq \gamma K^n(n!)^{2\alpha}, \quad n \geq 1.$$

This ends the proof of Theorem 5.1.

6 Borel transform of the formal solution

In this and subsequent sections, we consider the case with $\alpha = 2$ ($Q = 8$, 10 and 12) and set

$$\sqrt{\mu} = \zeta^2.$$

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Remark 6.1 *In the cases with $\alpha > 2$, we can always assume that α is an integer (large enough) and set $\zeta = \mu^{1/2\alpha}$.*

The formal expansion (29) becomes, after incorporating $\beta^{-1/2}$ into $U^{(n)}$,

$$U = \zeta^2 \sum_{n \geq 0} \zeta^{4n} U^{(n)}, \tag{41}$$

and we have the estimate

$$||U^{(n)}||_s \leq \gamma K^n (n!)^4 \leq \gamma K^n (4n!).$$

Thus the formal power series (41) is a Gevrey 1 series in ζ.

Let us now consider the new function $\zeta \mapsto \widehat{U}(\zeta)$, taking its values in $\mathcal{H}_s$, defined by

$$\widehat{U}(\zeta) = \sum_{n \geq 0} \frac{\zeta^{4n+2}}{(4n+2)!} U^{(n)}.$$

Indeed, by construction, this function is analytic in the disc $|\zeta| < K_1^{-1} = K^{-1/4}$, with values in the Hilbert space $\mathcal{H}_s$ and invariant under rotations of angle $2\pi/Q$. The mapping $U \mapsto \widehat{U}$, where we divide the coefficient of ζ^n by $n!$, is the *Borel transform* [6] applied to the series U. Since U satisfies a Gevrey 1 estimate, the Borel transform $\widehat{U}$ is analytic in a disc.

We now need to show that this function $\widehat{U}(\zeta)$ is solution of a certain partial differential equation. Let us recall a simple property of Gevrey 1 series. Consider two scalar Gevrey 1 series u and v

$$u = \sum_{n \geq 1} u_n \zeta^n, \qquad v = \sum_{n \geq 1} v_n \zeta^n,$$
$$|u_n| \leq c_1 K_1^n n!, \qquad |v_n| \leq c_2 K_1^n n!,$$

then we have

$$(uv)_n = \sum_{1 \leq k \leq n-1} u_k v_{n-k},$$
$$|(uv)_n| \leq c_1 c_2 K_1^n n!,$$

as this results from Appendix D, by using the following inequality for $n \geq 3$

$$\frac{1}{(n-1)!} \sum_{1 \leq k \leq n-1} k!(n-k)! \leq 1 + 2\left(\frac{1}{2} + \cdots + \frac{1}{n-1}\right) \leq n,$$

which shows that in our case we can multiply two Gevrey 1 series with coefficients belonging to $\mathcal{H}_s$ (the factor $c_1 c_2$ is then multiplied by c_s) and obtain a new Gevrey 1 series with coefficients in $\mathcal{H}_s$. It is then classical that we can write

$$\widehat{U^3} = \widehat{U} *_G \widehat{U} *_G \widehat{U} \tag{42}$$

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where the *convolution product*, written as $*_G$, is well defined by

$$(\hat{u} *_G \hat{v})(\zeta) = \sum_{n \geq 1} \sum_{1 \leq k \leq n-1} \frac{u_k v_{n-k}}{n!} \zeta^n,$$

and satisfies

$$(\hat{u} *_G \hat{v}) = \widehat{(uv)}.$$

This convolution product is easily extended for two functions $f(\zeta)$ and $g(\zeta)$, analytic in the disc $|\zeta| < K_1^{-1}$, and with no zero order term, by

$$(f * g)(\zeta) = \sum_{n \geq 1} \sum_{1 \leq k \leq n-1} f_k g_{n-k} \frac{k!(n-k)!}{n!} \zeta^n. \tag{43}$$

It is clear that for $f = \hat{u}$, and $g = \hat{v}$ we have

$$f * g = (\hat{u} *_G \hat{v}) = \widehat{(uv)}.$$

Since we have (42), it is clear from (19) that we have

$$((1 \widehat{+ \Delta})^2 U)(\mathbf{x}, \zeta) = (1 + \Delta)^2 \widehat{U}(\mathbf{x}, \zeta).$$

Now let us define a bounded linear operator $\mathcal{K}$ as follows: for any function $\zeta \mapsto V(\zeta)$ analytic in the disc $|\zeta| < K_1^{-1}$, taking values in $\mathcal{H}_s$, canceling for $\zeta = 0$, and satisfying

$$V(\zeta) = \sum_{n \geq 1} V_n \zeta^n, \quad ||V_n||_s \leq c K_1^n,$$

we define

$$(\mathcal{K}V)(\zeta) = \sum_{n \geq 1} \frac{n!}{(n+4)!} \zeta^{n+4} V_n.$$

It is then clear for $V = \widehat{U}$ that

$$(\mathcal{K}\widehat{U})(\zeta) = \sum_{n \geq 0} \frac{\zeta^{4n+6}}{(4n+6)!} U^{(n)} = \widehat{(\zeta^4 U)},$$

and we see that

$$\partial_\zeta^4 (\mathcal{K}\widehat{U}) = \widehat{U}.$$

We now claim the following:

Theorem 6.2 *The Borel transform $\widehat{U}(\mathbf{x}, \zeta)$ of the Gevrey solution found in Theorem 5.1 for $\alpha = 2$ is the unique solution, analytic in the disc $|\zeta| < K^{-1/4}$, cancelling for $\zeta = 0$, and taking values in $\mathcal{H}_s$ invariant under rotations of angle $2\pi/Q$, of the equation*

$$(1 + \Delta)^2 V - \mathcal{K}V + V * V * V = 0. \tag{44}$$

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Proof. We still assume $\alpha = 2$ in what follows. The changes needed for larger α's are left to the reader. Let us look for a solution V in the form

$$V = \sum_{n \geq 1} \zeta^n V_n,$$

where $V_n \in \mathcal{H}_s$ is invariant under rotations of angle $2\pi/Q$. Then defining a formal series

$$U = \sum_{n \geq 1} \zeta^n U_n, \qquad U_n = n! V_n,$$

it is clear that U satisfies formally

$$(1 + \Delta)^2 U - \zeta^4 U + U^3 = 0,$$

and by identifying powers of ζ:

$$
\begin{aligned}
\mathcal{L}_0 U_1 &= 0, \\
\mathcal{L}_0 U_2 &= 0, \\
\mathcal{L}_0 U_3 + U_1^3 &= 0,
\end{aligned}
$$

which leads to $U_1 = 0$ because of the last equation where the solvability condition cannot be satisfied. Then we have

$$U_1 = 0, \qquad \mathcal{L}_0 U_j = 0, j = 2, 3, 4, 5,$$

and

$$\mathcal{L}_0 U_6 - U_2 + (U_2)^3 = 0.$$

We observe that U_2 and U_6 satisfy the equations verified by $\beta^{-1/2} U^{(0)}$ and $\beta^{-1/2} U^{(1)}$ (see (16)). This is indeed the only solution invariant under rotations of $2\pi/Q$. Hence

$$
\begin{aligned}
U_2 &= \beta^{-1/2} U^{(0)}, \\
U_6 &= \beta^{-1/2} U^{(1)}.
\end{aligned}
$$

Now at order ζ^7 we get

$$\mathcal{L}_0 U_7 - U_3 + 3 U_2^2 U_3 = 0$$

and since $U_3 = C U^{(0)}$, where C is a constant, the solvability condition gives

$$C = \frac{3C}{\beta} \langle U^{(0)3}, e^{i \mathbf{k}_1 \cdot \mathbf{x}} \rangle_s = 3C$$

hence $C = 0$ and $U_3 = 0$. It is the same for $U_4 = U_5 = 0$, and we obtain $\mathcal{L}_0 U_7 = \mathcal{L}_0 U_8 = \mathcal{L}_0 U_9 = 0$. Then the computation of higher orders is exactly as the one for the computation of $U^{(n)}$, since the cubic term cancels if the sum of the 3 indices p in U_p is not 2 mod 4. Coming back to the definition of $U_n = n! V_n$, it is then clear that Theorem 6.2 is proved. ■

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7 Truncated Laplace transform

Let us take $K' > K_1$ and define a linear mapping $U \mapsto \bar{U}$ in the set of Gevrey 1 series taking values in $\mathcal{H}_s$

$$\bar{U}(\nu) = \frac{1}{\nu} \int_0^{\frac{1}{K'}} e^{-\frac{\zeta}{\nu}} \widehat{U}(\zeta) \, d\zeta, \tag{45}$$

where $\widehat{U}(\zeta)$ is the Borel transform of U as defined above, which is analytic in the disc $|\zeta| < 1/K_1$. The function $\nu \mapsto \bar{U}(\nu)$ is a truncated Laplace transform of the Borel transform of U.

Remark 7.1 *If $\widehat{U}(\zeta)$ could be shown to be analytic on a line in the complex ζ plane extending to ∞, instead of just in a disk, then the Laplace transform in (45) would be the inverse Borel transform, and would provide a quasiperiodic solution of (11) in $\mathcal{H}_s$.*

It is clear that $\bar{U}(\nu)$ is a $\mathcal{C}^\infty$ function of ν in a neighborhood of 0, taking its values in $\mathcal{H}_s$, as this results from

$$\bar{U}(\nu) = \int_0^{\frac{1}{K'\nu}} e^{-z} \widehat{U}(\nu z) \, dz$$

and from the dominated convergence theorem. Moreover $\bar{U}(\nu)$ and $U(\mu)$ have the same asymptotic expansion in powers on ν, when we set $\mu = \nu^{1/4}$, as this results from

$$\frac{1}{\nu} \int_0^{\frac{1}{K'}} e^{-\frac{\zeta}{\nu}} \frac{\zeta^n}{n!} \, d\zeta = \nu^n - e^{-\frac{1}{K'\nu}} \left(\frac{\nu^n}{1} + \frac{\nu^{n-1}}{K'1!} + \cdots + \frac{\nu}{K'^{n-1}(n-1)!} + \frac{1}{K'^n n!} \right). \tag{46}$$

It is also clear that in a little disc near the origin

$$\widehat{\bar{U}} = \widehat{U},$$

but this does not imply that $\bar{U} = U$ since U has no meaning as a function of ν, and an asymptotic expansion does not define a unique function. The real question is whether or not $\bar{U}$ is solution of (11) in $\mathcal{H}_s$.

By construction, we know that the Gevrey 1 expansion of

$$V(\mu^{1/4}) =: (1 + \Delta)^2 \bar{U}(\mu^{1/4}) - \mu \bar{U}(\mu^{1/4}) + \bar{U}(\mu^{1/4})^3$$

in powers of $\mu^{1/4}$ is identically 0, but we don't know whether this function (smooth in $\mu^{1/4}$), which is in $\mathcal{H}_{s-4}$, is indeed 0. In fact we have the following

Theorem 7.2 *For any even $Q \geq 8$, take $s > Q/4$. Then, α being defined by Lemma 2.1, the quasiperiodic function $\bar{U}(\mu^{1/2\alpha}) \in \mathcal{H}_s$, with $s > Q/4$, defined from the series found in Theorem 5.1, is solution of the Swift–Hohenberg PDE up to an exponentially small term bounded by $C(K') e^{-\frac{1}{K'\mu^{1/2\alpha}}}$ in $\mathcal{H}_{s-4}$, for any $K' > K^{1/2\alpha}$.*

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Proof. The result of the Theorem follows directly from two elementary lemmas E.1 and E.2 on Gevrey 1 series shown in Appendix E, and which may be understood in the function space $\mathcal{H}_s$ instead of $\mathbb{C}$. Indeed, for $\alpha = 2$ this gives an estimate of the difference beween $V(\mu^{1/4})$ and the truncated Laplace transform of the equation (44) (which is then 0), taking into account of

$$(1 + \Delta)^2 \bar{U}(\mu^{1/4}) = \frac{1}{\mu^{1/4}} \int_0^{\frac{1}{K'}} e^{-\frac{\zeta}{\nu}} (1 + \Delta)^2 \widehat{U}(\zeta) d\zeta.$$

Using Remark 6.1, the extension to larger α's is left to the reader. ■

A Proof of Lemma 2.1

We give below an elementary proof of Lemma 2.1.

The polynomial P being irreducible on $\mathbb{Q}$ of degree $l + 1$ and the polynomial Q defined by

$$Q(x) = \sum_{0 \le j \le l} q_j x^j,$$

being of degree l, then by the Bezout Theorem there exist two polynomials $A(x)$ of degree $l - 1$ and $B(x)$ of degree l, with coefficients in $\mathbb{Q}$ such that

$$A(x)P(x) + B(x)Q(x) = 1. \tag{47}$$

Defining coefficients p_j, $0 \le j \le l + 1$, a_j, $0 \le j \le l - 1$ and b_j, $0 \le j \le l$ of polynomials P, A and B, the identity (47) becomes a linear system of $2l + 1$ equations, of the form

$$\mathbf{M}X = \xi_0, \tag{48}$$

where the unknown is X with

$$X = \begin{pmatrix} a_{j-1} \\ a_{j-2} \\ . \\ a_0 \\ b_l \\ b_{l-1} \\ . \\ b_0 \end{pmatrix}, \xi_0 = \begin{pmatrix} 0 \\ 0 \\ . \\ . \\ . \\ . \\ 0 \\ 1 \end{pmatrix},$$

$$\mathbf{M} = \begin{pmatrix} p_{l+1} & 0 & . & 0 & q_l & 0 & . & . & 0 \\ p_l & p_{l+1} & . & . & q_{l-1} & q_l & 0 & . & . \\ . & . & . & 0 & q_{l-2} & q_{l-1} & q_l & . & . \\ . & . & . & p_{l+1} & . & . & . & . & 0 \\ p_1 & . & . & p_l & q_0 & . & . & . & q_l \\ p_0 & . & . & p_{l-1} & 0 & q_0 & . & . & q_{l-1} \\ 0 & p_0 & . & p_{l-2} & 0 & 0 & . & . & . \\ . & . & . & . & . & . & . & . & . \\ 0 & . & 0 & p_0 & 0 & . & . & 0 & q_0 \end{pmatrix}.$$

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The $(2l+1) \times (2l+1)$ matrix $\mathbf{M}$ has integer coefficients and is *invertible* (otherwise it would contradict the Bezout Theorem). Hence its determinant is integer valued and is an homogeneous polynomial of degree $l+1$ in $q = (q_0, \ldots, q_l)$. We may invert the system (48) by Cramer's formulas and we observe that the coefficients b_j are rational numbers, with a common denominator of degree $l+1$ in q and with a numerator of degree l only (we replace in the determinant one column containing the q_j's by ξ_0). It results that the polynomial $B(x)$ is the ratio of a *polynomial with integer coefficients B_0 of degree l in q*, with an integer d, homogeneous polynomial of q of degree $l+1$ and which is different from 0 $(\det \mathbf{M} \neq 0)$. Now taking $x = \omega$ in (47) leads to

$$|Q(\omega)| = \frac{d}{|B_0(\omega)|},$$

and since $d \geq 1$ and the coefficients of B_0 are bounded by $C'|q|^l$, this completes the proof of Lemma 2.1.

B Proof of Lemma 4.1

By construction we have (21), so assertion (i) is clear. This implies directly assertion (iii), since in the $Q/2-$ dimensional space of $\{m'_j, j = 1, \ldots, Q/2\}$ the set $\sum_{j=1}^{Q/2} |m'_j| = N$ is a union of $2^{Q/2}$ simplexes of area of order $O(N^{Q/2-1})$. To prove (22) we observe that

$$
\begin{aligned}
N_{\mathbf{k}+\mathbf{l}} &= \min\{|m+n|; \mathbf{k}+\mathbf{l} = \sum_{j=1}^{Q}(m_j+n_j)\mathbf{k}_j\} \\
&\leq \min\{|m|; \mathbf{k} = \sum_{j=1}^{Q} m_j\mathbf{k}_j\} + \min\{|n|; \mathbf{l} = \sum_{j=1}^{Q} n_j\mathbf{k}_j\} \\
&\leq N_{\mathbf{k}} + N_{\mathbf{l}},
\end{aligned}
$$

where

$$N_{\mathbf{k}} = \min_{\mathbf{k}=\mathbf{k_m}} \sum_{j=1}^{Q} m_j\mathbf{k}_j, \quad N_{\mathbf{l}} = \min_{\mathbf{l}=\mathbf{l_n}} \sum_{j=1}^{Q} n_j\mathbf{k}_j.$$

We notice that $N_{\mathbf{0}} = 0$, and $N_{-\mathbf{k}} = N_{\mathbf{k}}$ (each m'_j for $\mathbf{k}$ is just the opposite for $-\mathbf{k}$); we deduce that inequality (22) may be strict, since

$$0 = N_{\mathbf{0}} = N_{\mathbf{k}-\mathbf{k}} < N_{-\mathbf{k}} + N_{\mathbf{k}} = 2N_{\mathbf{k}}.$$

The last inequality (23) is easily deduced from

$$\mathbf{k} = \sum_{j=1}^{Q} m_j\mathbf{k}_j$$

where $\{m_j\}$ gives precisely the "norm" $N_{\mathbf{k}}$; which implies (since $|\mathbf{k}_j| = 1$)

$$|\mathbf{k}| \le \sum_{j=1}^{Q} |m_j| = N_{\mathbf{k}},$$

and the Lemma is proved.

C   Proof of Lemma 4.2

Let $u \in \mathcal{H}_s$, then by Cauchy–Schwarz inequality in $l^2(\Gamma)$ (Γ is countable) we have

$$
\begin{aligned}
\left| \sum_{\mathbf{k} \in \Gamma} u_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{x}} \right|^2
&\le \left(\sum_{\mathbf{k} \in \Gamma} (1 + N_{\mathbf{k}}^2)^s |u_{\mathbf{k}}|^2 \right) \sum_{\mathbf{k} \in \Gamma} \frac{1}{(1 + N_{\mathbf{k}}^2)^s} \\
&\le \|u\|_{\mathcal{H}_s}^2 \sum_{\mathbf{k} \in \Gamma} \frac{1}{(1 + N_{\mathbf{k}}^2)^s}.
\end{aligned}
$$

Now by (24) we have the following estimate

$$\sum_{\mathbf{k} \in \Gamma} \frac{1}{(1 + N_{\mathbf{k}}^2)^s} \le c_1(Q) \sum_{n \in \mathbb{N}} \frac{n^{Q/2 - 1}}{(1 + n^2)^s}$$

which is bounded when $s > Q/4$. Hence for $s > Q/4$ the series $\sum_{\mathbf{k} \in \Gamma} u_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{x}}$ converges absolutely and represents a continuous quasiperiodic function, the norm (uniform norm) of which being bounded as soon as the norm in $\mathcal{H}_s$ is bounded. We may proceed in the same way for the derivatives in using (23), and show that the series

$$\sum_{\mathbf{k} \in \Gamma} |\mathbf{k}|^l u_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{x}}$$

is absolutely convergent for $s > Q/4 + l$. This ends the proof of the last assertion of the Lemma. Let us now prove the first assertion which is necessary for our nonlinear problem.

First step: We first use the following inequality due to (22)

$$(1 + N_{\mathbf{k}+\mathbf{k}'}^2)^{s/2} \le 2^{s-1} \left\{ (1 + N_{\mathbf{k}}^2)^{s/2} + (1 + N_{\mathbf{k}'}^2)^{s/2} \right\}$$

valid for any $s \ge 1$, because of (22) and a simple convexity argument (this inequality is in fact valid for $s > 0$). Then the following decomposition holds

$$\sum_{\mathbf{K}} \left| \sum_{\mathbf{k}+\mathbf{k}'=\mathbf{K}} u_{\mathbf{k}} v_{\mathbf{k}'} \right|^2 (1 + N_{\mathbf{K}}^2)^s \le 2^{2s-1}(S_1 + S_2)$$

with

$$S_1 \;=\; \sum_{\mathbf{K}} \left| \sum_{\mathbf{k}+\mathbf{k}'=\mathbf{K}} u_{\mathbf{k}} v_{\mathbf{k}'} \right|^2 (1 + N_{\mathbf{k}}^2)^s$$

$$S_2 \;=\; \sum_{\mathbf{K}} \left| \sum_{\mathbf{k}+\mathbf{k}'=\mathbf{K}} u_{\mathbf{k}} v_{\mathbf{k}'} \right|^2 (1 + N_{\mathbf{k}'}^2)^s.$$

For symmetry reasons in the space $(\mathbf{k}, \mathbf{k}')$, it is then sufficient to estimate S_1. Let us split the bracket in the sum S_1 into two terms: a sum S_1' containing $(\mathbf{k}, \mathbf{k}')$ such that

$$N_{\mathbf{k}} \leq 3 N_{\mathbf{k}'},$$

and a sum S_1'' containing $(\mathbf{k}, \mathbf{k}')$ such that $N_{\mathbf{k}} > 3 N_{\mathbf{k}'}$. Hence we have now

$$S_1 \leq 2(S_1' + S_1'')$$

with

$$S_1' \;=\; \sum_{\mathbf{K}} \left| \sum_{\substack{\mathbf{k}+\mathbf{k}'=\mathbf{K}, \\ N_{\mathbf{k}} \leq 3 N_{\mathbf{k}'}}} u_{\mathbf{k}} v_{\mathbf{k}'} \right|^2 (1 + N_{\mathbf{k}}^2)^s,$$

$$S_1'' \;=\; \sum_{\mathbf{K}} \left| \sum_{\substack{\mathbf{k}+\mathbf{k}'=\mathbf{K}, \\ N_{\mathbf{k}} > 3 N_{\mathbf{k}'}}} u_{\mathbf{k}} v_{\mathbf{k}'} \right|^2 (1 + N_{\mathbf{k}}^2)^s.$$

To estimate S_1' we use (22) which gives $N_{\mathbf{K}} \leq 4 N_{\mathbf{k}'}$, hence

$$\frac{1}{1 + N_{\mathbf{k}'}^2} \leq \frac{16}{1 + N_{\mathbf{K}}^2},$$

and, in using again Cauchy–Schwarz

$$\sum_{\substack{\mathbf{k}+\mathbf{k}'=\mathbf{K}, \\ N_{\mathbf{k}} \leq 3 N_{\mathbf{k}'}}} |u_{\mathbf{k}} v_{\mathbf{k}'}| (1 + N_{\mathbf{k}}^2)^{s/2} \;\leq\; \sum_{\substack{\mathbf{k}+\mathbf{k}'=\mathbf{K}, \\ N_{\mathbf{k}} \leq 3 N_{\mathbf{k}'}}} 4^s |u_{\mathbf{k}} v_{\mathbf{k}'}| \frac{(1 + N_{\mathbf{k}}^2)^{s/2}(1 + N_{\mathbf{k}'}^2)^{s/2}}{(1 + N_{\mathbf{K}}^2)^{s/2}}$$

$$\leq \frac{4^s}{(1 + N_{\mathbf{K}}^2)^{s/2}} ||u||_{\mathcal{H}_s} ||v||_{\mathcal{H}_s}.$$

It results that

$$S_1' \leq ||u||_{\mathcal{H}_s}^2 ||v||_{\mathcal{H}_s}^2 \sum_{\mathbf{K}} \frac{4^{2s}}{(1 + N_{\mathbf{K}}^2)^s}$$

which, for $s > Q/4$ leads to

$$S_1' \leq C ||u||_{\mathcal{H}_s}^2 ||v||_{\mathcal{H}_s}^2.$$

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Second step: We now find a bound for S_1'', which is more technical, since we split this sum into packets of increasing lengths.

Let us define

$$\Delta_p u = \sum_{2^p \leq N_{\mathbf{k}} < 2^{p+1}} u_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}, \quad \Delta_{-1} u = u_{\mathbf{0}}.$$

It is clear that for $s > Q/4$ (the series is absolutely convergent)

$$u = \sum_{p=-1}^{\infty} \Delta_p u.$$

Moreover, it is clear from the definition that the norm of $u \in \mathcal{H}_s$ is equivalent to

$$\left(\sum_{p=-1}^{\infty} 2^{2ps} ||\Delta_p u||_0^2 \right)^{1/2}.$$

To estimate the sum S_1'', we notice that in the product uv the terms $\Delta_p u \Delta_q v$ only take into account the wavevectors $\mathbf{k}$ and $\mathbf{k}'$ such that

$$2^p \leq N_{\mathbf{k}} < 2^{p+1}, \quad 2^q \leq N_{\mathbf{k}'} < 2^{q+1}, \quad N_{\mathbf{k}} > 3 N_{\mathbf{k}'}.$$

This implies

$$N_{\mathbf{k}'} < 2^p, \qquad 2^{q+1} < N_{\mathbf{k}},$$

hence in S_1''

$$\Delta_p u \Delta_q v = 0, \text{ for } p \leq q.$$

Now, we use (for the sum in S_1'')

$$\frac{2}{3} N_{\mathbf{k}} \leq N_{\mathbf{K}}$$

$$S_1'' \leq \left(\frac{2}{3}\right)^{2s} \sum_{\mathbf{K}} \left| \sum_{\substack{\mathbf{k}+\mathbf{k}'=\mathbf{K}, \\ N_{\mathbf{k}} > 3 N_{\mathbf{k}'}}} u_{\mathbf{k}} v_{\mathbf{k}'} \right|^2 (1 + N_{\mathbf{K}}^2)^s$$

and the right hand side is the square of the norm of the product uv computed on terms such that $N_{\mathbf{k}} > 3 N_{\mathbf{k}'}$, $\mathbf{k} + \mathbf{k}' = \mathbf{K}$. We now use the equivalent norm defined above with the decomposition in packets, hence

$$S_1'' \leq C \sum_{j=-1}^{\infty} 2^{2js} \left\| \Delta_j \left(\sum_{p \geq 0} \left(\sum_{q=-1}^{p-1} \Delta_q v \right) \Delta_p u \right) \right\|_0^2.$$

Let us define $S_{p-1} v = \sum_{q=-1}^{p-1} \Delta_q v$, then we have

$$\Delta_j \left(\sum_p S_{p-1} v \Delta_p u \right) = \sum_{p=j-1}^{j+1} \Delta_j (S_{p-1} v \Delta_p u) 2^{ps} 2^{-ps}$$

hence by Cauchy–Schwarz

$$2^{2js}\left\|\Delta_j\left(\sum_p S_{p-1}v\Delta_p u\right)\right\|_0^2 \le \left(\sum_{p=j-1}^{j+1} 2^{2(j-p)s}\right)\sum_{p=j-1}^{j+1} 2^{2ps}\|\Delta_j(S_{p-1}v\Delta_p u)\|_0^2$$

Now

$$\|S_{p-1}v\Delta_p u\|_0^2 = \sum_{\mathbf{K}} |\sum_{\mathbf{k}+\mathbf{k}'=\mathbf{K},\,0\le N_{\mathbf{k}'}<2^p\le N_{\mathbf{k}}<2^{p+1}} u_{\mathbf{k}}v_{\mathbf{k}'}|^2$$

and a classical computation (convolution $l^1 * l^2$) using Cauchy–Schwarz gives

$$\begin{aligned}
\sum_{\mathbf{K}}|\sum_{\mathbf{k}+\mathbf{k}'=\mathbf{K}} u_{\mathbf{k}}v_{\mathbf{k}'}|^2 &\le \sum_{\mathbf{K}}\left\{(\sum_{\mathbf{k}+\mathbf{k}'=\mathbf{K}} |v_{\mathbf{k}'}||u_{\mathbf{k}}|^2)(\sum_{\mathbf{k}'} |v_{\mathbf{k}'}|)\right\} \\
&\le (\sum_{\mathbf{k}'} |v_{\mathbf{k}'}|)(\sum_{\mathbf{k}}\sum_{\mathbf{K}} |v_{\mathbf{K}-\mathbf{k}}||u_{\mathbf{k}}|^2) \\
&\le \left((\sum_{\mathbf{k}'} |v_{\mathbf{k}'}|)\right)^2 \sum_{\mathbf{k}} |u_{\mathbf{k}}|^2)
\end{aligned}$$

which leads to

$$\|S_{p-1}v\Delta_p u\|_0^2 \le \|\Delta_p u\|_0^2 \left((\sum_{\mathbf{k}'} |v_{\mathbf{k}'}|)\right)^2$$

and since the series $\sum |v_{\mathbf{k}'}| \le c\|v\|_{\mathcal{H}_s}$ for $s > Q/4$, as shown at the beginning of the proof of Lemma 4.2, we have

$$\|S_{p-1}v\Delta_p u\|_0^2 \le C\|\Delta_p u\|_0^2\|v\|_{\mathcal{H}_s}^2.$$

Finally, we obtain

$$\begin{aligned}
\sum_{p=j-1}^{j+1} 2^{2ps}\|\Delta_j(S_{p-1}v\Delta_p u)\|_0^2 &\le \sum_{p=j-1}^{j+1} 2^{2ps}\|S_{p-1}v\Delta_p u\|_0^2 \\
&\le C'\|v\|_{\mathcal{H}_s}^2 \sum_{p=j-1}^{j+1} 2^{2ps}\|\Delta_p u\|_0^2,
\end{aligned}$$

and

$$2^{2js}\left\|\Delta_j\left(\sum_p S_{p-1}v\Delta_p u\right)\right\|_0^2 \le C''\|v\|_{\mathcal{H}_s}^2 \sum_{p=j-1}^{j+1} 2^{2ps}\|\Delta_p u\|_0^2,$$

hence

$$\begin{aligned}
S_1'' &\le 3C''\|v\|_{\mathcal{H}_s}^2 \sum_{p=-1}^{\infty} 2^{2ps}\|\Delta_p u\|_0^2 \\
&\le C_1\|u\|_{\mathcal{H}_s}^2\|v\|_{\mathcal{H}_s}^2
\end{aligned}$$

and Lemma 4.2 is proved.

D   Proof of Lemma 5.3

Let us define the two sums

$$\Pi_{2,n} = \sum_{k=0}^{n}(k!(n-k)!)^{2\alpha}$$

$$\Pi'_{2,n} = \sum_{k=1}^{n-1}(k!(n-k)!)^{2\alpha}$$

we have already

$$\Pi_{2,0} = 1, \quad \Pi_{2,1} = 2, \qquad \Pi_{2,2} = (2+\frac{1}{2^{2\alpha}})(2!)^{2\alpha},$$

$$\Pi'_{2,2} = 1, \quad \Pi'_{2,3} = 2(2!)^{2\alpha},$$

which shows that $\Pi_{2,n} \leq (2+\frac{1}{16})(n!)^4$ for $n = 0, 1, 2$, and $\alpha \geq 2$. Now we have for $n \geq 2$

$$\frac{\Pi_{2,n+1}}{((n+1)!)^{2\alpha}} - \frac{\Pi_{2,n}}{(n!)^{2\alpha}} = \sum_{k=2}^{n-2}\left(\frac{k!(n-k)!}{n!}\right)^{2\alpha}\left\{\left(\frac{n+1-k}{n+1}\right)^{2\alpha}-1\right\} +$$

$$+\frac{2}{(n+1)^{2\alpha}} - \frac{2}{n^{2\alpha}} + \frac{2^{2\alpha}}{(n(n+1))^{2\alpha}},$$

and since $n^{2\alpha} - (n+1)^{2\alpha} + 2^{2\alpha-1} < 0$ for $n \geq 1$ the above right hand side terms are negative. It results that for $n \geq 2$

$$\Pi_{2,n+1} \leq \left(\frac{(n+1)!}{n!}\right)^{2q}\Pi_{n,2},$$

hence

$$\Pi_{2,n} \leq (2+\frac{1}{16})(n!)^{2\alpha}, \qquad n \geq 0. \tag{49}$$

In the same way

$$\frac{\Pi'_{2,n+1}}{(n!)^{2\alpha}} - \frac{\Pi'_{2,n}}{((n-1)!)^{2\alpha}} = \sum_{k=2}^{n-2}\left(\frac{k!(n-k)!}{n!}\right)^{2\alpha}\left\{\left(\frac{n+1-k}{n+1}\right)^{2\alpha}-1\right\} + \frac{2^{2\alpha}}{n^{2\alpha}},$$

hence for $n \geq 2$

$$\frac{\Pi'_{2,n+1}}{(n!)^{2\alpha}} \leq \frac{\Pi'_{2,n}}{((n-1)!)^{2\alpha}} + \frac{2^{2\alpha}}{n^{2\alpha}},$$

and

$$\frac{\Pi'_{2,n}}{((n-1!)^{2\alpha}} \leq 2^{2\alpha}\left(\frac{1}{(n-1)^{2\alpha}}+\cdots+\frac{1}{2^{2\alpha}}\right) + \Pi'_{2,2}$$

$$\leq 2 + 2^{2\alpha}\left(\frac{1}{(n-1)^{2\alpha}}+\cdots+\frac{1}{3^{2\alpha}}\right)$$

$$\leq 2 + \frac{2}{2\alpha-1} \leq 3.$$

Finally
$$\Pi'_{2,n} \le 3((n-1!)^{2\alpha} \text{ for } n \ge 2. \tag{50}$$

Consider now $\Pi_{3,n}$ defined by
$$\Pi_{3,n} = \sum_{\substack{k+l+r=n \\ k,l,r \ge 0}} (k!l!r!)^{2\alpha}.$$

We already have
$$\Pi_{3,0} = 1, \quad \Pi_{3,1} = 3, \quad \Pi_{3,2} = (3 + \frac{3}{2^{2\alpha}})(2!)^{2\alpha} \le 4(2!)^{2\alpha},$$

In splitting the sum we obtain easily for $n \ge 3$

$$
\begin{aligned}
\Pi_{3,n} &= \Pi_{2,n} + (n!)^{2\alpha} + \sum_{r=1}^{n-1} (r!)^{2\alpha} \Pi_{2,n-r} \\
&\le (3 + \frac{1}{16})(n!)^{2\alpha} + (2 + \frac{1}{16})\Pi'_{2,n} \\
&\le (n!)^{2\alpha}(3 + \frac{1}{16} + 3(2 + \frac{1}{16})\frac{1}{n^{2\alpha}}) \\
&\le (3 + \frac{3}{16} + \frac{9}{3^4})(n!)^{2\alpha} \le 4(n!)^{2\alpha}.
\end{aligned}
$$

Hence
$$\Pi_{3,n} \le 4(n!)^{2\alpha} \tag{51}$$

holds for any $n \ge 0$. Consider now $\Pi'_{3,n}$ defined for $n \ge 2$ by
$$\Pi'_{3,n} = \sum_{\substack{k+l+r=n \\ 0 \le k,l,r \le n-1}} (k!l!r!)^{2\alpha}.$$

We already have
$$\Pi'_{3,2} = 1,$$

and for $n \ge 3$, we obtain in the same way

$$
\begin{aligned}
\Pi'_{3,n} &= \Pi'_{2,n} + \sum_{r=1}^{n-1} (r!)^{2\alpha} \Pi_{2,n-r} \\
&\le \left(3 + 3(2 + \frac{1}{16})\right)(n-1!)^{2\alpha} \\
&\le 10(n-1!)^{2\alpha}. \tag{52}
\end{aligned}
$$

Hence, with estimates (51) and (52), Lemma 5.3 is proved.

E Lemmas on Gevrey 1 series

Below we give elementary proofs of two useful lemmas. The interested reader will find more general results in [27] and [26].

In the following we denote by $\mathcal{L}_{K'}$ the linear operator defined for analytic functions v on the disc $\{|z| < 1/K_1\}$ by

$$(\mathcal{L}_{K'}v)(\nu) = \frac{1}{\nu} \int_0^{1/K'} e^{-\frac{z}{\nu}} v(z)dz, \quad K' > K_1.$$

We also use the notations

$$\|v\|_{0,K'} = \sup_{z \in (0,1/K')} |v(z)|, \quad \|v\|_{1,K'} = \sup_{z \in (0,1/K')} |v'(z)|,$$

and when $v(0) = 0$, we notice that (integrating by parts for the second estimate)

$$|(\mathcal{L}_{K'}v)(\nu)| \leq \|v\|_{0,K'}, \tag{53}$$

$$\left|(\mathcal{L}_{K'}v)(\nu) - \int_0^{1/K'} e^{-\frac{z}{\nu}} v'(z)dz\right| \leq e^{-\frac{1}{K'\nu}}\|v\|_{0,K'}.$$

Then we have the following Lemmas giving estimates of the commutator of $\mathcal{L}_{K'} \circ \mathcal{B}$ (where $\mathcal{B}$ is the Borel transform) with the multiplication by ν^4 and with the mapping $u \mapsto u^3$ in the space of Gevrey series.

Lemma E.1 *Assume that $u(\nu)$ is a Gevrey 1 series, with $u_0 = 0$, then for $\nu < 1/K'$*

$$\left|\left(\mathcal{L}_{K'}(\widehat{u^3})\right)(\nu) - (\mathcal{L}_{K'}\widehat{u})^3(\nu)\right| \leq \frac{e^{-\frac{1}{K'\nu}}}{(K'\nu)^3}\|\widehat{u}\|_{0,K'}(\|\widehat{u}\|_{0,K'} + \nu\|\widehat{u}\|_{1,K'})^2.$$

For any given Gevrey 1 series u, with $u_0 = 0$, there is $C(K') > 0$ such that for $\nu < \nu_0(K')$ we have the estimate

$$\left|\left(\mathcal{L}_{K'}(\widehat{u^3})\right)(\nu) - (\mathcal{L}_{K'}\widehat{u})^3(\nu)\right| \leq C(K')e^{-\frac{1}{K'\nu}}, \quad K' > K_1.$$

Lemma E.2 *Assume that $u(\nu)$ is a Gevrey 1 series, with $u_0 = 0$, then for $\nu < 1/K'$ there exists $C(K')$ such that*

$$\left|(\mathcal{L}_{K'}\mathcal{K}\widehat{u})(\nu) - \nu^4 \mathcal{L}_{K'}\widehat{u}\right| \leq C(K')\|\widehat{u}\|_{0,K'}e^{-\frac{1}{K'\nu}}.$$

Proof of Lemma E.1. From the identity

$$\int_0^z \left(\int_0^{z_1} \frac{z_1^{k-1} z_2^{m-1}(z - z_1 - z_2)^l}{(k-1)!(m-1)!l!}dz_2\right) dz_1 = \frac{z^{k+m+l}}{(k+m+l)!},$$

from the definition (43) of the convolution product, and from the analyticity of $\widehat{u}$ in the disc $\{|z| < 1/K_1\}$, we have

$$\left(\mathcal{L}_{K'}(\widehat{u} * \widehat{u} * \widehat{u})\right)(\nu) \;=\; \left(\mathcal{L}_{K'}(\widehat{u^3})\right)(\nu) =$$
$$= \;\frac{1}{\nu}\int_0^{1/K'} e^{-\frac{z}{\nu}}\left(\int_0^z\left(\int_0^{z_1}\widehat{u}'(z_1)\widehat{u}'(z_2)\widehat{u}(z-z_1-z_2)dz_2\right)dz_1\right)dz.$$

By Fubini's theorem and a simple change of variables, we obtain

$$\left(\mathcal{L}_{K'}(\widehat{u^3})\right)(\nu) = \frac{1}{\nu}\int_{D_{K'}} e^{-\frac{z_1+z_2+z_3}{\nu}}\widehat{u}'(z_1)\widehat{u}'(z_2)\widehat{u}(z_3)dz_1dz_2dz_3 \qquad (54)$$

where $D_{K'} = \{z_1, z_2, z_3 > 0; z_1 + z_2 + z_3 < 1/K'\}$. Now, we have

$$(\mathcal{L}_{K'}\widehat{u})^3(\nu) = \frac{1}{\nu^3}\int_{(0,1/K')^3} e^{-\frac{z_1+z_2+z_3}{\nu}}\widehat{u}(z_1)\widehat{u}(z_2)\widehat{u}(z_3)dz_1dz_2dz_3,$$

and from (53) we obtain

$$\left|(\mathcal{L}_{K'}\widehat{u})^3(\nu) - \frac{1}{\nu}\int_{(0,1/K')^3} e^{-\frac{z_1+z_2+z_3}{\nu}}\widehat{u}'(z_1)\widehat{u}'(z_2)\widehat{u}(z_3)dz_1dz_2dz_3\right| \leq$$

$$\leq e^{-\frac{1}{K'\nu}}\|\widehat{u}\|_{0,K'}^2(\|\widehat{u}\|_{0,K'} + \nu\|\widehat{u}\|_{1,K'}). \qquad (55)$$

Now, we observe that $(0,1/K')^3\backslash D_{K'}$ is such that $z_1 + z_2 + z_3 > 1/K'$, hence

$$\left|\frac{1}{\nu}\int_{(0,1/K')^3\backslash D_{K'}} e^{-\frac{z_1+z_2+z_3}{\nu}}\widehat{u}'(z_1)\widehat{u}'(z_2)\widehat{u}(z_3)dz_1dz_2dz_3\right| \leq \frac{e^{-\frac{1}{K'\nu}}}{\nu^3 K'^3}\|\widehat{u}\|_{0,K'}(\nu\|\widehat{u}\|_{1,K'})^2.$$
$$(56)$$

Collecting (54), (55) and (56) the first result of Lemma E.1 is proved. Notice that by choosing $K'' > K'$, then for ν small enough $\frac{e^{-\frac{1}{K'\nu}}}{\nu^3 K'^3} \leq e^{-\frac{1}{K''\nu}}$. Since K' is chosen arbitrarily larger than K_1, we can assert that u being given, there is $C(K')$ such that

$$\left|\left(\mathcal{L}_{K'}(\widehat{u^3})\right)(\nu) - (\mathcal{L}_{K'}\widehat{u})^3(\nu)\right| \leq C(K')e^{-\frac{1}{K'\nu}}, \quad K' > K_1.$$

$\blacksquare$

Proof of Lemma E.2. By integrating by parts, we obtain

$$(\mathcal{L}_{K'}\mathcal{K}\widehat{u})(\nu) \;=\; -e^{-\frac{1}{K'\nu}}\left[(\mathcal{K}\widehat{u}) + \nu(\mathcal{K}\widehat{u})' + \nu^2(\mathcal{K}\widehat{u})'' + \nu^3(\mathcal{K}\widehat{u})'''\right]|_{1/K'} +$$
$$+\nu^4(\mathcal{L}_{K'}\widehat{u})(\nu).$$

Hence

$$\left|(\mathcal{L}_{K'}\mathcal{K}\widehat{u})(\nu) - \nu^4(\mathcal{L}_{K'}\widehat{u})(\nu)\right| \leq e^{-\frac{1}{K'\nu}}\|\widehat{u}\|_{0,K'}\left\{\frac{\nu^3}{K'} + \frac{\nu^2}{2K'^2} + \frac{\nu}{6K'^3} + \frac{1}{24K'^4}\right\}$$

which proves Lemma E.2. $\blacksquare$

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