Ramanujan summation of divergent series

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Abstract

In Chapter VI of his second Notebook Ramanujan introduce the Euler-MacLaurin formula to define the "constant " of a series. When the series is divergent he uses this "constant" like a sum of the series. We give a rigorous definition of Ramanujan summation and some properties and applications of it. These properties of the summation seems very unusual so in the last chapter we give a general algebraic view on summation of series that unify Ramanujan summation with the classical summations procedures.

Contents

Introduction

It is possible to give a meaning to the sum

$$\sum_{n=1}^{+\infty} n = 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + ...$$

which appears in the study of Casimir effect, by the analytic continuation of the Riemann zeta function defined for $\mathrm{Re}(s) > 1$ by $\zeta : s \mapsto \sum_{n\geq 1}^{+\infty} \frac{1}{n^s}$. This function has an analytic continuation to $\mathbb{C}\backslash\{1\}$ and thus we can set for example $\sum_{n=1}^{+\infty} n = \zeta(-1)$. But this strategy does not work with the series

$$\sum_{n=1}^{+\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + ...$$

since the zeta function has a pole at $s = 1$.

These series are examples of divergent series. The classical definition of convergence of a series $\sum_{n\geq 1} a_n$ by

$$\sum_{n=1}^{+\infty} a_n = \lim_{n\to+\infty} \sum_{k=1}^{n} a_k \quad \text{(when this limit is finite)}$$

was introduced by Cauchy in order to avoid frequent mistakes in working with series.

But non convergent series that is "divergent series" appear elsewhere in analysis. Thus some other methods of summation of series have been introduced by several mathematicians such that Cesaro, Euler, Abel, Borel and others, in particular Ramanujan. These methods of summation assign to a series $\sum_{n\geq 1} a_n$ a number S obtained by taking the limit of some means of the partial sums $S_n = \sum_{k=1}^{n} a_k$.

In the last chapter of his book "Divergent series" Hardy introduced the method employed by Ramanujan. This method is based on the Euler-MacLaurin summation formula which is an asymptotic expansion when $n \to +\infty$ of the partial sum $S_n = \sum_{k=1}^{n} f(k)$ where f is a sufficiently regular function. For Ramanujan we can always find, in this expansion, a natural constant (not depending on n) which he call "the constant of the series" and treats like a sort of sum of the series.

The precise definition of this summation is not given xplicitely by Ramanujan who uses intuitive and formal calculations. We define this summation in limiting us with the series $\sum_{n\geq 1} f(n)$ when f is a function not too increasing.

The coherence of our definition constrains us to deal, in the case of convergence of the series $\sum_{n\geq 1} f(n)$, with an integral term which is the difference between the Ramanujan-sum and the classical Cauchy-sum of the series. There is also the apparition of an integral when we compare the sums of the series $\sum_{n\geq 1} f(n)$ and $\sum_{n\geq 1} f(n+1)$.

If we account for these properties we can give a precise meaning to some formal manipulations and obtain rigorous results in applying Ramanujan summation to some divergent series.

These properties of the summation seems very unusual so in the last chapter we give a general algebraic view on summation of series that unify Ramanujan summation with the classical summations procedures.

In appendix we give the classical Euler-MacLaurin and Euler-Boole formula.

Evidently we can not claim that our version of the Ramanujan summation is exactly the summation procedure that Ramanujan had in his powerful mind, so we give an exact copy of the Chapter VI of the second notebook in which Ramanujan introduce the "constant of a series".

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Chapter 1

Ramanujan summation

1.1 The Ramanujan constant of a series

Let f a smooth function defined for real $x > 0$. In the beginning of Chapter VI of his Notebook 2, Ramanujan introduce the sum

$$f(1) + f(2) + f(3) + f(4) + ... + f(x) = \phi(x),$$

which is solution of

$$\phi(x) - \phi(x-1) = f(x) \quad \text{with} \quad \phi(0) = 0$$

Let the numbers $\mathcal{B}_r$ are defined when $r = 2, 4, 6, ...$ by (second notebook chapter V, entry 9)

$$\frac{x}{e^x - 1} = 1 - \frac{x}{2} + \sum_{k \geq 1} \frac{(-1)^{k-1}\mathcal{B}_{2k}}{(2k)!} x^{2k}$$

then Ramanujan writes the Euler-McLaurin series

$$\phi(x) = c + \int f(x)dx + \frac{1}{2}f(x) + \frac{\mathcal{B}_2}{\lfloor 2} f'(x) - \frac{\mathcal{B}_4}{\lfloor 4} f'''(x) + \frac{\mathcal{B}_6}{\lfloor 6} f^V(x) + \frac{\mathcal{B}_8}{\lfloor 8} f^{VII}(x) + ...$$

and he said about the constant c: *The algebraic constant of a series is the constant obtained by completing the remaining part in the above theorem. We can substitute this constant which is like the centre of gravity of a body instead of its divergent infinite series.*

In Ramanujan notation $\mathcal{B}_{2n} = (-1)^{n-1}B_{2n}$, where the B_n are the usual Bernoulli numbers given by $B_n = B_n(0)$ defined by

$$\frac{ze^{zx}}{e^z - 1} = \sum_{n \geq 0} \frac{B_n(x)}{n!}$$

Thus we can write the above Euler-McLaurin series in the form

$$\varphi(x) = c + \int f(x)dx + \frac{1}{2}f(x) + \sum_{k \geq 2} \frac{B_{2k}}{(2k)!} \partial^{2k-1} f(x)$$

Unfortunately in this last formula the series $\sum_{k \geq 2} \frac{B_k}{k!} \partial^{k-1} f(x)$ can be a *divergent* series and in the integral of f *the low limit of integration is not precisely defined.* Thus we must replace this series by the finite sum and give a precise meaning to the integral. Then we get the **Euler-MacLaurin formula** (cf. appendix):

$$f(1) + ... + f(n) \quad = \quad C_m + \int_1^n f(x)dx + \frac{f(n)}{2} + \sum_{k=1}^{m} \frac{B_{2k}}{(2k)!} \partial^{2k-1} f(n) - \int_n^{+\infty} \frac{b_{2m+1}(x)}{(2m+1)!} \partial^{2m+1} f(x)dx$$

where

$$C_m = \frac{f(1)}{2} - \sum_{k=1}^{m} \frac{B_{2k}}{(2k)!} \partial^{2k-1} f(1) + \int_1^{+\infty} \frac{b_{2m+1}(x)}{(2m+1)!} \partial^{2m+1} f(x)dx$$

and the b_n are the periodic Bernoulli functions defined by $b_n(x) = B_n(x - [x])$, In this formula it is assumed that the function f is an infinitely differentiable function and that the integral $\int_1^{+\infty} b_{2m+1}(x)\partial^{2m+1} f(x)dx$ is convergent. If it is convergent for all $m \geq M$ then by integration by parts we verify that the constant C_m does not depend on m if $m \geq M$ thus we set $C_m = C$.

We use the notation

$$C(f) = \sum_{n\geq 1}^{\mathcal{R}} f(n) = \frac{f(1)}{2} - \sum_{k=1}^{m} \frac{B_{2k}}{(2k)!}\partial^{2k-1} f(1) + \int_1^{+\infty} \frac{b_{2m+1}(x)}{(2m+1)!}\partial^{2m+1} f(x)dx$$

and call the constant C(f) the *Ramanujan sum of the series.*

Example

If f is a constant function then $\partial f = 0$ thus $\sum_{n\geq 1}^{\mathcal{R}} f(n) = \frac{f(1)}{2}$. Thus

$$\boxed{\sum_{n\geq 1}^{\mathcal{R}} 1 = \frac{1}{2}}$$

.

If $f(x) = x$ then $\partial^3 f = 0$ thus

$$\boxed{\sum_{n\geq 1}^{\mathcal{R}} n = \frac{1}{2} - \frac{B_2}{2} = \frac{5}{12}}$$

The case of convergence

Assume that the integrals $\int_1^{+\infty} b_n(x)\partial^n f(x)dx$ are convergent for $n \geq 1$.

Then the Euler-MacLaurin formula is valid for $m = 0$ and we get

$$f(1) + ... + f(n) = C(f) + \int_1^n f(x)dx + \frac{f(n)}{2} - \int_n^{+\infty} b_1(x)\partial f(x)dx$$

Since $\int_n^{+\infty} b_1(x)\partial f(x)dx \to 0$ when $n \to +\infty$ we get

$$\sum_{n\geq 1}^{\mathcal{R}} f(n) = C(f) = \lim_{n\to+\infty} \left(f(1) + ... + f(n) - \int_1^n f(x)dx - \frac{f(n)}{2}\right)$$

Note that *if the series* $\sum f(n)$ *and the integral* $\int_1^{+\infty} f(x)dx$ *are convergent* we get

$$\sum_{n\geq 1}^{\mathcal{R}} f(n) = \sum_{n\geq 1}^{+\infty} f(n) - \int_1^{+\infty} f(x)dx \qquad (1.1)$$

Remark: Since $b_1(x) = x - [x] - 1/2$ we have also the integral formula

$$\sum_{n\geq 1}^{\mathcal{R}} f(n) = \frac{f(1)}{2} + \int_1^{+\infty} (x - [x] - 1/2)f(x)dx$$

Examples

1) If $f(x) = \frac{1}{x^z}$ with $Re(z) > 1$ then $|\partial f(x)| = |z|\frac{1}{x^{Re(z)+1}}$ thus $\int_1^{+\infty} |\partial f(x)|dx < +\infty$ and

$$\sum_{n\geq 1}^{\mathcal{R}} \frac{1}{n^z} = \sum_{n\geq 1}^{+\infty} \frac{1}{n^z} - \int_1^{+\infty} \frac{1}{x^z}dx = \sum_{n\geq 1}^{+\infty} \frac{1}{n^z} - \frac{1}{z-1}$$

thus for $Re(z) > 1$ we have

$$\boxed{\sum_{n\geq 1}^{\mathcal{R}} \frac{1}{n^z} = \zeta(z) - \frac{1}{z-1}}$$

2) If $f(x) = \frac{1}{x}$ then

$$\sum_{n\geq 1}^{\mathcal{R}} \frac{1}{n} = \lim_{n\to+\infty} (\sum_{k=1}^{n} \frac{1}{k} - Log(n) - \frac{1}{2n}) = \lim_{n\to+\infty} (\sum_{k=1}^{n} \frac{1}{k} - Log(n))$$

Thus

$$\boxed{\sum_{n\geq 1}^{\mathcal{R}} \frac{1}{n} = \gamma}$$

where γ is the Euler constant.
3) If $f(x) = Log(x)$ then by Euler-MacLaurin

$$\sum_{n\geq 1}^{\mathcal{R}} Log(n) = \lim_{n\to+\infty} (\sum_{k=1}^{n} Log(k) - (nLog(n) - n + 1 + \frac{1}{2}Log(n))$$

Thus the Stirling formula we have

$$\lim_{n\to+\infty} \sum_{k=1}^{n} Log(k) - (nLog(n) - n + \frac{1}{2}Log(n)) = Log(\sqrt{2\pi})$$

is equivalent to

$$\boxed{\sum_{n\geq 1}^{\mathcal{R}} Log(n) = Log(\sqrt{2\pi}) - 1}$$

4) We have by Euler-MacLaurin

$$\sum_{n\geq 1}^{\mathcal{R}} nLog(n) = \lim_{n\to+\infty} (\sum_{k=1}^{n} kLog(k) - Log(n)(\frac{n^2}{2} + \frac{n}{2} + \frac{1}{12}) + \frac{n^2}{4}) - \frac{1}{3}$$

Thus

$$\boxed{\sum_{n\geq 1}^{\mathcal{R}} nLog(n) = Log(A) - \frac{1}{3}}$$

where A is the Glaisher-Kinkelin constant (cf. Srivastava and Choi p.39).

Remark
Note that if the derivatives $\partial^m f(x)$ are sufficiently decreasing at infinity for $m \geq M$ so that $\int_1^{+\infty} \frac{b_m(x)}{m!}\partial^m f(x)dx$ is convergent, then for all $a > 0$ we can write

$$f(1) + ... + f(n) =$$

$$\int_1^a f(x)dx + \frac{f(1)}{2} - \sum_{k=1}^{m} \frac{B_{2k}}{(2k)!}\partial^{2k-1} f(1) + \int_1^{+\infty} \frac{b_{2m+1}(x)}{(2m+1)!}\partial^{2m+1} f(x)dx$$

$$+ \int_a^n f(x)dx + \frac{f(n)}{2} + \sum_{k=1}^{m} \frac{B_{2k}}{(2k)!}\partial^{2k-1} f(n) - \int_n^{+\infty} \frac{b_{2m+1}(x)}{(2m+1)!}\partial^{2m+1} f(x)dx$$

thus we see that the constant $C(f) = \sum_{n\geq 1}^{\mathcal{R}} f(n)$ is replaced by

$$C_a(f) = \int_1^a f(x)dx + \sum_{n\geq 1}^{\mathcal{R}} f(n)$$

It seems that Ramanujan *let the possibility for the constant on the series that the choice of a is on the series in consideration*:

* If the series $\sum f(n)$ and the integral $\int_1^{+\infty} f(x)dx$ are convergent then with $a = +\infty$ we get

$$C_\infty(f) = \int_1^{+\infty} f(x)dx + \sum_{n\geq 1}^{\mathcal{R}} f(n)$$

and with our preceding formula (1.1) we get

$$C_\infty(f) = \sum_{n=1}^{+\infty} f(n)$$

This is compatible with an affirmation of Ramanujan in Chapter 6 p.62 of Notebook2:
"If f (1) + f (2) + ... + f (x) be a convergent series then its constant is the sum of the series."

* If for example $f(x) = x$ then $C_\infty(f)$ is not defined but

$$C_0(f) = \int_1^0 xdx + \sum_{n\geq 1}^{\mathcal{R}} n = -\frac{B_2}{2}$$

To get simple properties of Ramanujan summation we fix the parameter a in the integral, we make the choice $a = 1$ in order to have

$$\sum_{n\geq 1}^{\mathcal{R}} \frac{1}{n} = \gamma$$

With the use of Euler-Maclaurin formula we have the definition of the constant of a series by

$$\sum_{n\geq 1}^{\mathcal{R}} f(n) = \lim_{n\to+\infty} (f(1) + ... + f(n) - [\int_1^n f(x)dx + \frac{f(n)}{2} + \sum_{k=1}^{m} \frac{B_{2k}}{(2k)!}\partial^{2k-1} f(n)] \tag{1.2}$$

this needs convergence of the integral $\int_1^{+\infty} b_{2m+1}(x)\partial^{2m+1} f(x)dx$.

This hypothesis is not always satisfied, for example by a simple series like $\sum e^n$. Thus we need to avoid the systematic use of Euler-McLaurin formula and define in another way the Ramanujan summation.

Remark

A nice example of this Ramanujan's flexibility in the choice of a is the following derivation of the functional equation for the zeta function by Ramanujan.

For $r = 2, 4, 6, ...$ we have the classical Euler formula

$$\sum_{n=1}^{+\infty} \frac{1}{n^r} = \frac{1}{2}\frac{(2\pi)^r}{r!}\mathcal{B}_r$$

thus (second notebook chapter V, entry 25) Ramanujan *define $\mathcal{B}_r$ for $r > 1$ by*

$$\frac{1}{2}\frac{(2\pi)^r}{r!}\mathcal{B}_r = \sum_{n=1}^{+\infty} \frac{1}{n^r} = \zeta(r)$$

Let r a positive integer the Euler-MacLaurin formula gives for $f(x) = x^r$

$$\phi(x) = \int_0^x t^r dt + C_0(r) + \frac{x^r}{2} + \sum_{k\geq 1} \frac{(-1)^{k-1}\mathcal{B}_{2k}}{(2k)!} r(r-1)...(r-(2k-2))x^{r-(2k-1)}$$

where the series in the right side is a finite sum.

If $r = 2m+1$ the value at $x = 0$ of the right side is

$$\phi(0) = C_0(2m+1) + \frac{(-1)^{k-1}\mathcal{B}_{2k}}{(2k)!} r(r-1)...(r-2k+2) \text{ with } k = m+1$$

thus it is

$$\phi(0) = C_0(2m+1) + \frac{(-1)^m \mathcal{B}_{2m+2}}{2m+2}$$

The equation $\phi(0) = 0$ gives for $r = 2m+1$

$$C_0(r) = -\frac{(-1)^{\frac{r-1}{2}}\mathcal{B}_{r+1}}{r+1} = \frac{(-1)^{\frac{r+1}{2}}\mathcal{B}_{r+1}}{r+1} = \frac{\mathcal{B}_{r+1}}{r+1}\cos(\pi\frac{r+1}{2})$$

Thus for $r = 2m$

$$\frac{\mathcal{B}_r}{r}\cos(\pi\frac{r}{2}) = C_0(r-1) = \text{ constant of } \sum x^{r-1}$$

In Chapter V entry 4 Ramanujan extend this formula and replace r by $1-r$ this gives

$$\frac{\mathcal{B}_{1-r}}{1-r}\cos(\pi\frac{1-r}{2}) = \text{ constant of } \sum x^{-r}$$

and Ramanujan note that for $r > 1$ we have

$$\text{constant of } \sum x^{-r} = \sum_{n=1}^{+\infty}\frac{1}{n^r} = \frac{1}{2}\frac{(2\pi)^r}{r!}\mathcal{B}_r$$

(note that the constant is now $C_\infty(x^{-r})$) thus he obtain

$$\frac{\mathcal{B}_{1-r}}{1-r}\cos(\pi\frac{1-r}{2}) = \frac{1}{2}\frac{(2\pi)^r}{r!}\mathcal{B}_r$$

With this formal derivation Ramanujan get the functional equation

$$\frac{\mathcal{B}_{1-r}}{1-r}\sin(\pi\frac{r}{2}) = \frac{1}{2}\frac{(2\pi)^r}{r!}\mathcal{B}_r$$

With $\frac{1}{2}\frac{(2\pi)^r}{r!}\mathcal{B}_r = \zeta(r)$ this gives the classical functional equation for the zeta function

$$\zeta(1-r)2\Gamma(1-r)(2\pi)^{r-1}\sin(\pi\frac{r}{2}) = \zeta(r)$$

We shall give later a more rigorous proof of the functional equation.

1.2 Ramanujan summation

1.2.1 The functions φ_f and R_f

In his notebooks Ramanujan use the function $\varphi(x) = f(1) + ... + f(x)$, it seems he has in mind a *sort of unique interpolation function* φ_f of the partial sums of the series associated to f. This function must verifiy

$$\varphi_f(x) - \varphi_f(x-1) = f(x)$$

and Ramanujan gives the additional condition $\varphi_f(0) = 0$.

This gives for n integer ≥ 1

$$\varphi_f(n) - \varphi_f(0) = f(1) + f(2) + ... + f(n)$$

thus if the series $\sum f(n)$ is convergent we must have $\lim_{n\to+\infty}\varphi_f(n) = \sum_{n=1}^{+\infty} f(n)$.

Our usual Euler-Mclaurin formula

$$f(1) + ... + f(n) = C(f) + \int_1^n f(t)dt + \frac{f(n)}{2} + \sum_{k=1}^m \frac{B_{2k}}{(2k)!}\partial^{2k-1}f(n) - \int_n^{+\infty}\frac{b_{2m+1}(t)}{(2m+1)!}\partial^{2m+1}f(t)dt$$

writes

$$\boxed{\varphi_f(n) = C(f) + f(n) - R_f(n)}$$

with

$$R_f(n) = \frac{f(n)}{2} - \sum_{k=1}^{m} \frac{B_{2k}}{(2k)!} \partial^{2k-1} f(n) + \int_n^{+\infty} \frac{b_{2m+1}(t)}{(2m+1)!} \partial^{2m+1} f(t) dt - \int_1^n f(t) dt \tag{1.3}$$

and

$$C(f) = \frac{f(1)}{2} - \sum_{k=1}^{m} \frac{B_{2k}}{(2k)!} \partial^{2k-1} f(1) + \int_1^{+\infty} \frac{b_{2m+1}(x)}{(2m+1)!} \partial^{2m+1} f(x) dx = R_f(1)$$

The relations $\varphi_f(n) = C(f) + f(n) - R_f(n)$ and $f(n+1) = \varphi_f(n+1) - \varphi_f(n)$ gives the difference equation

$$R_f(n) - R_f(n+1) = f(n)$$

1.2.2 A difference equation

By the preceding section it seems natural to define Ramanujan summation of the series $\sum_{n\geq 1} f(n)$ by

$$\sum_{n\geq 1}^{\mathcal{R}} f(n) = R(1)$$

where the function R is solution of the difference equation

$$R(x) - R(x+1) = f(x)$$

But clearly that this equation is not sufficient to determine the function R, we need other conditions on the function R. Let us try to find these conditions.

We see by the preceding definition of $R = R_f$ (eq. 1.3) that if f and his derivatives are sufficiently decreasing at $+\infty$, we have

$$\lim_{n\to+\infty} R(n) = -\int_1^{+\infty} f(x) dx$$

This condition involves the integral $\int_1^{+\infty} f(x)dx$ which in the general case can be divergent thus we translated it in another form. Suppose we have a smooth function R_f solution of the difference equation

$$R_f(x) - R_f(x+1) = f(x) \textit{ for all } x > 0$$

Integrating between k and $k+1$ for all integer $k \geq 1$, and and summing on k we get

$$\int_1^{+\infty} f(x) dx = \int_1^2 R_f(x) dx - \lim_{x\to+\infty} R_f(x)$$

Thus the condition

$$\lim_{x\to+\infty} R_f(x) = -\int_1^{+\infty} f(x) dx$$

is equivalent for R_f to the condition

$$\int_1^2 R_f(x) dx = 0$$

Thus we can try to define the function R_f by the difference equation $R_f(x) - R_f(x+1) = f(x)$ with the preceeding condition. Unfortunately this does not specify the function R_f because we can add to R_f any combination of periodic functions $x \mapsto e^{2i\pi kx}$. To avoid this we add the hypothesis that R_f is analytic for $Re(x) > 0$ of exponential type $< 2\pi$.

Definition
A function g analytic for $Re(x) > a$ is of exponential type $< \alpha$ ($\alpha > 0$) if there exist $\beta < \alpha$ such that

$$|g(x)| \leq Ce^{\beta|x|} \text{ for } Re(x) > a$$

We define $\mathcal{O}^\alpha$ the space of functions g analytic for $Re(x) > a$ with some $a < 1$ and of exponential type $< \alpha$. We say that f is of *moderate growth* if $f \in \mathcal{O}^\varepsilon$ for all $\varepsilon > 0$.

Lemma 1
Let $R \in \mathcal{O}^{2\pi}$ solution of $R(x) - R(x+1) = 0$ with $\int_1^2 R(x)dx = 0$, then $R = 0$.

Proof
By the condition $R(x) - R(x+1) = 0$, we see that R can be extended to an entire function. And we can write

$$R(x) = R_0(e^{2i\pi x}),$$

with R_0 is the analytic funtion given in $\mathbb{C} - \{0\}$ defined by $R_0(z) = R(\frac{1}{2i\pi} Log(z))$ (where Log is defined by $Log(re^{i\theta}) = \ln(r) + i\theta$ with $0 \leq \theta < 2\pi$).
The Laurent expansion $R_0(z) = \sum_{n\in\mathbb{Z}} c_n z^n$ gives

$$R(x) = \sum_{n\in\mathbb{Z}} c_n e^{2i\pi nx},$$

where

$$c_n = \frac{1}{2\pi r^n}\int_0^{2\pi} R_0(re^{it})e^{-int}dt = \frac{1}{2\pi r^n}\int_0^{2\pi} R(\frac{t}{2\pi} + \frac{1}{2i\pi}\ln(r))e^{-int}dt \text{ for } r > 0.$$

The condition that R is of exponential type $< 2\pi$ gives

$$|c_n| \leq \frac{1}{r^n} Ce^{\frac{\alpha}{2\pi}|\ln(r)|} \text{ with } \frac{\alpha}{2\pi} < 1.$$

If we let $r \to 0$ we get $c_n = 0$ for $n < 0$ and if we let $r \to +\infty$ then we get $c_n = 0$ for $n > 0$. The condition $\int_1^2 R(x)dx = 0$ then gives $c_0 = 0$.
□

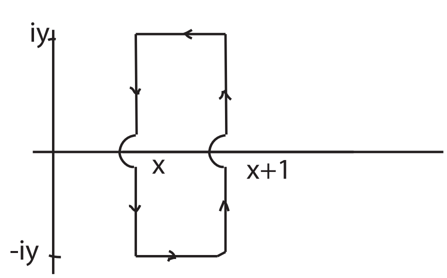
Theorem 1 *If $f \in \mathcal{O}^\alpha$ with $\alpha \leq 2\pi$ there exist a unique function $R_f \in \mathcal{O}^\alpha$ such that $R_f(x) - R_f(x+1) = f(x)$ with $\int_1^2 R_f(x)dx = 0$. This function is*

$$R_f(x) = -\int_1^x f(t)dt + \frac{f(x)}{2} + i\int_0^{+\infty} \frac{f(x+it) - f(x-it)}{e^{2\pi t} - 1}dt \tag{1.4}$$

Proof
a) The unicity is given by the preceding lemma.
b) The function R_f defined by (2.3) is clearly in $\mathcal{O}^\alpha$.
c) Let us prove that $R_f(x) - R_f(x+1) = f(x)$, by analyticity it is sufficient to prove this for real x. Consider the integral

$$\int_\gamma f(z)\frac{1}{2i}\cot(\pi(z-x))dz$$

with γ the path



By the residue theorem we have $\int_\gamma f(z)\frac{1}{2i}\cot(\pi(z-x))dz = f(x)$. To evaluate the different contributions of the integral we use the formulas:

$$\frac{1}{2i}\cot(\pi(z-x)) = -\frac{1}{2} - \frac{1}{e^{-2i\pi(z-x)}-1} \text{ when } Im(z) > 0$$

and

$$\frac{1}{2i}\cot(\pi(z-x)) = \frac{1}{2} + \frac{1}{e^{2i\pi(z-x)}-1} \text{ when } Im(z) < 0.$$

Let us examine the different contributions of the integral:

* the semicircular path at x and $x+1$ gives when $\varepsilon \to 0$

$$\frac{1}{2}f(x) - \frac{1}{2}f(x+1)$$

* the horizontal lines gives

$$-(-\frac{1}{2})\int_x^{x+1} f(t+iy)dt + \frac{1}{2}\int_x^{x+1} f(t-iy)dt$$

and two other terms which $\to 0$ when $y \to +\infty$ by the hypothesis that f of exponential type $< 2\pi$.

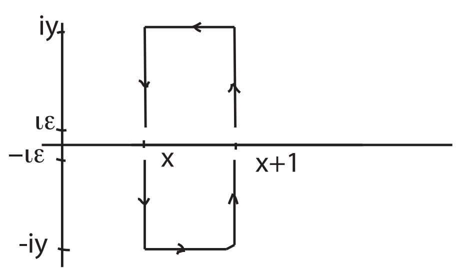
* the vertical lines gives

$$i\int_\varepsilon^y \frac{f(x+it)-f(x-it)}{e^{2\pi t}-1}dt - i\int_\varepsilon^y \frac{f(x+1+it)-f(x+1-it)}{e^{2\pi t}-1}dt$$

and

$$\frac{1}{2}\int_\varepsilon^y f(x+it)idt - \frac{1}{2}\int_\varepsilon^y f(x-it)idt - \frac{1}{2}\int_\varepsilon^y f(x+1+it)idt + \frac{1}{2}\int_\varepsilon^y f(x+1-it)idt$$

If we add this term with the contributions of the horizontal lines we obtain the sum of the integrals of f on the paths



By Cauchy theorem this sum is

$$\frac{1}{2}\int_x^{x+1} f(t+i\varepsilon)dt + \frac{1}{2}\int_x^{x+1} f(t-i\varepsilon)dt$$

which gives the contribution $\int_x^{x+1} f(t)dt$ when $\varepsilon \to 0$.

Finally we get when $\varepsilon \to 0$ and $y \to +\infty$

$$\begin{aligned} f(x) &= \frac{1}{2}f(x) - \frac{1}{2}f(x+1) \\ &\quad +i\int_0^{+\infty} \frac{f(x+it)-f(x-it)}{e^{2\pi t}-1}dt \\ &\quad -i\int_0^{+\infty} \frac{f(x+1+it)-f(x+1-it)}{e^{2\pi t}-1} \\ &\quad +\int_x^{x+1} f(t)dt \end{aligned}$$

This is $f(x) = R_f(x) - R_f(x+1)$ with R_f given by (1.3).

d) It remains to prove that $\int_1^2 R_f(x)dx = 0$. By Fubini's theorem

$$\int_1^2 \int_0^{+\infty} \frac{f(x+it) - f(x-it)}{e^{2\pi t} - 1} dt dx = \int_0^{+\infty} \int_1^2 (f(x+it) - f(x-it)) dx \frac{1}{e^{2\pi t} - 1} dt$$

We have for $1 < x < 2$

$$\int_1^2 f(x+it)dx - \int_1^2 f(x-it)dx \quad = \quad F(2+it) - F(2-it) - (F(1+it) - F(1-it))$$

where $F(x) = \int_1^x f(t)dt$. Thus

$$\int_1^2 \int_0^{+\infty} \frac{f(x+it) - f(x-it)}{e^{2\pi t} - 1} dt dx = \int_0^{+\infty} \frac{F(2+it) - F(2-it)}{e^{2\pi t} - 1} dt - \int_0^{+\infty} \frac{F(1+it) - F(1-it)}{e^{2\pi t} - 1} dt$$

By the preceding result (applied with F in place of f) we have

$$\begin{aligned} F(x) \quad &= \quad \frac{1}{2}F(x) - \frac{1}{2}F(x+1) \\ &+ i \int_0^{+\infty} \frac{F(x+it) - F(x-it)}{e^{2\pi t} - 1} dt \\ &- i \int_0^{+\infty} \frac{F(x+1+it) - F(x+1-it)}{e^{2\pi t} - 1} \\ &+ \int_x^{x+1} F(t)dt \end{aligned}$$

With $x = 1$ we get

$$i \int_1^2 \int_0^{+\infty} \frac{f(x+it) - f(x-it)}{e^{2\pi t} - 1} dt dx = -\frac{F(1) + F(2)}{2} + \int_1^2 F(t)dt$$

This gives

$$\int_1^2 R_f(x)dx = -\int_1^2 F(x)dx + \frac{1}{2}\int_1^2 f(x)dx - (\frac{F(1) + F(2)}{2}) + \int_1^2 F(t)dt = 0$$

□

Remark: the Plana formula

The relation $R_f(x) - R_f(x+1) = f(x)$ gives

$$\varphi_f(n) = f(1) + ... + f(n) = R_f(1) + f(n) - R_f(n)$$

thus by (1.4) we get **the Plana formula**

$$\begin{aligned} \varphi_f(n) \quad &= \quad \frac{f(1)}{2} + i \int_0^{+\infty} \frac{f(1+it) - f(1-it)}{e^{2\pi t} - 1} dt \\ &+ \frac{f(n)}{2} + \int_1^n f(t)dt - i \int_0^{+\infty} \frac{f(n+it) - f(n-it)}{e^{2\pi t} - 1} dt \end{aligned}$$

1.3 The summation

1.3.1 Definition and examples

Let $f \in \mathcal{O}^{2\pi}$ by the preceding theorem we can try to define the Ramanujan summation of the series $\sum_{n\geq 1} f(n)$ by

$$\sum_{n\geq 1}^{\mathcal{R}} f(n) = R_f(1)$$

where R_f is the unique solution in $\mathcal{O}^{2\pi}$ of

$$R_f(x) - R_f(x+1) = 0 \text{ with } \int_1^2 R_f(x)dx = 0.$$

With this definition let us look at the sum

$$\sum_{n\geq 1}^{\mathcal{R}} \sin(n\pi)$$

Let $f(x) = \sin(\pi x)$ then

$$\sin(\pi x) - \sin(\pi(x+1)) = 2\sin(\pi x)$$

thus

$$R_f(x) = \frac{\sin(\pi x)}{2} - \int_1^2 \frac{\sin(\pi x)}{2} dx = \frac{\sin(\pi x)}{2} + \frac{1}{\pi}$$

and we get the surprising result

$$\sum_{n\geq 1}^{\mathcal{R}} \sin(\pi n) = \frac{1}{\pi}$$

On the other hand we have $\sin(\pi n) = 0 = g(n)$ with the function $g = 0$. And we have $R_g = 0$ which gives

$$\sum_{n\geq 1}^{\mathcal{R}} 0 = 0$$

Thus we see that with the preceding definition the summation of $\sum_{n\geq 1} f(n)$ not only on the values $f(n)$ for integers $n \geq 1$ but specially of the interpolation function f. To avoid this phenomenon we set the condition that f is in $\mathcal{O}^{\pi}$, with this condition we can apply Carlson's theorem which gives that the interpolation function f is uniquely determined by values $f(n)$ for integers $n \geq 1$. Note that in this case the function R_f given by theorem 1 is also in $\mathcal{O}^{\pi}$.

Definition
If $f \in \mathcal{O}^{\pi}$, then there exist a unique solution $R_f \in \mathcal{O}^{\pi}$ of

$$R_f(x) - R_f(x+1) = f(x) \text{ with } \int_1^2 R_f(x)dx = 0$$

and we set

$$\sum_{n\geq 1}^{\mathcal{R}} f(n) = R_f(1)$$

We have

$$\sum_{n\geq 1}^{\mathcal{R}} f(n) = \frac{f(1)}{2} + i\int_0^{+\infty} \frac{f(1+it) - f(1-it)}{e^{2\pi t} - 1} dt$$

We call this procedure, the Ramanujan summation of series $\sum_{n\geq 1} f(n)$.

Linearity

Note that if a and b are complex numbers and f and g are in $\mathcal{O}^{\pi}$, then we verify Ramanujan summation has the property of linearity

$$\sum_{n\geq 1}^{\mathcal{R}} af(n)+bg(n)=a\sum_{n\geq 1}^{\mathcal{R}} f(n)+b\sum_{n\geq 1}^{\mathcal{R}} g(n)$$

Real and imaginary

Let $g\in\mathcal{O}^{\pi}$ is such that $g(x)\in\mathbb{R}$ if $x\in\mathbb{R}$.

Then for all $t>0$ we have by the reflection principle $g(1-it)=\overline{g(1+it)}$ thus $i(g(1+it)-g(1-it))\in\mathbb{R}$ and we get

$$\sum_{n\geq 1}^{\mathcal{R}} g(n)\in\mathbb{R}$$

For $f\in\mathcal{O}^{\pi}$ let the functions $Rf: x\mapsto Re(f(x))$ and $If: x\mapsto Im(f(x))$ defined for $x\in\mathbb{R}$. If the functions Rf and If have an analytic continuations that are in $\mathcal{O}^{\pi}$ then we can define the sums $\sum_{n\geq 1}^{\mathcal{R}} Re(f(n))$ and $\sum_{n\geq 1}^{\mathcal{R}} Im(f(n))$ by

$$\sum_{n\geq 1}^{\mathcal{R}} Re(f(n))=\sum_{n\geq 1}^{\mathcal{R}} Rf(n)$$

$$\sum_{n\geq 1}^{\mathcal{R}} Im(f(n))=\sum_{n\geq 1}^{\mathcal{R}} If(n)$$

By linearity we have

$$\sum_{n\geq 1}^{\mathcal{R}} f(n)=\sum_{n\geq 1}^{\mathcal{R}} Rf(n))+i\ If(n))=\sum_{n\geq 1}^{\mathcal{R}} Rf(n)+i\sum_{n\geq 1}^{\mathcal{R}} If(n)=\sum_{n\geq 1}^{\mathcal{R}} Re(f(n))+i\sum_{n\geq 1}^{\mathcal{R}} Im(f(n))$$

Since $Rf(x)$ and $If(x)$ are real for $x\in\mathbb{R}$ then we get

$$Re(\sum_{n\geq 1}^{\mathcal{R}} f(n))=\sum_{n\geq 1}^{\mathcal{R}} Re(f(n))$$

$$Im(\sum_{n\geq 1}^{\mathcal{R}} f(n))=\sum_{n\geq 1}^{\mathcal{R}} Im(f(n))$$

Remark

Note that generally we can not write $\overline{\sum_{n\geq 1}^{\mathcal{R}} f(n)}=\sum_{n\geq 1}^{\mathcal{R}} \overline{f}(n)$ since the function $\overline{f}$ is not analytic (if f is non constant). For example if $f(z)=\frac{1}{z+i}$ then $\overline{f}(z)=\frac{1}{\overline{z}+i}$ is not analytic but $Rf(z)=\frac{z}{z^2+1}$ and $If(z)=\frac{-1}{z^2+1}$ are analytic functions in $\mathcal{O}^{\pi}$. Thus

$$Re(\sum_{n\geq 1}^{\mathcal{R}} \frac{1}{n+i})=\sum_{n\geq 1}^{\mathcal{R}} \frac{n}{n^2+1}$$

$$Im(\sum_{n\geq 1}^{\mathcal{R}} \frac{1}{n+i})=\sum_{n\geq 1}^{\mathcal{R}} \frac{-1}{n^2+1}$$

Examples

1) If $f(x)=e^{ax}$ with $a\neq 0$ we have

$$\frac{e^{ax}}{1-e^a}-\frac{e^{a(x+1)}}{1-e^a}=e^{ax}$$

thus $R_f(x)=\frac{e^{ax}}{1-e^a}-\int_1^2 \frac{e^{ax}}{1-e^a}dx$ this gives

$$R_f(x)=\frac{e^{ax}}{1-e^a}+\frac{e^a}{a}$$

If $a \in \mathbb{R}$ and $a < \pi$ or $a \in \mathbb{C}$ and $|a| < \pi$ we get for $a \neq 0$

$$\boxed{\sum_{n\geq 1}^{\mathcal{R}} e^{an} = \frac{e^a}{1-e^a} + \frac{e^a}{a}} \tag{1.5}$$

Note that for $a = 0$ we have

$$\sum_{n\geq 1}^{\mathcal{R}} e^{an} = \sum_{n\geq 1}^{\mathcal{R}} 1 = \frac{1}{2} = \lim_{a\to 0}\left(\frac{e^a}{1-e^a} + \frac{e^a}{a}\right)$$

Then if $|t| < 1$ we get

$$\sum_{n\geq 1}^{\mathcal{R}} e^{i\pi nt} = \frac{1}{2} i \frac{e^{i\frac{\pi t}{2}}}{\sin(\frac{\pi t}{2})} - i\frac{e^{i\pi t}}{\pi t}$$

Thus if $-1 < t < 1$

$$\boxed{\sum_{n\geq 1}^{\mathcal{R}} \cos(\pi nt) = \frac{\sin(\pi t)}{\pi t} - \frac{1}{2}}$$

$$\boxed{\sum_{n\geq 1}^{\mathcal{R}} \sin(\pi nt) = \frac{1}{2}\cot(\frac{\pi t}{2}) - \frac{\cos(\pi t)}{\pi t}}$$

2) By the definition $\sum_n \frac{B_n(x)}{n!} z^n = \frac{ze^{xz}}{e^z - 1}$ we verify that the Bernoulli polynomials are solution of

$$\frac{B_{k+1}(x+1)}{k+1} - \frac{B_{k+1}(x)}{k+1} = x^k$$

and we have

$$\int_1^2 \frac{B_{k+1}(x)}{k+1} dx = \int_0^1 \frac{B_{k+1}(x+1)}{k+1} dx = \int_0^1 x^k dx = \frac{1}{k+1}$$

Thus if $f(x) = x^k$ where k is an integer ≥ 0 then

$$R_f(x) = \frac{1 - B_{k+1}(x)}{k+1}$$

Thus

$$\boxed{\sum_{n\geq 1}^{\mathcal{R}} n^k = \frac{1 - B_{k+1}}{k+1} \text{ if } k \geq 1 \text{ and } \sum_{n\geq 1}^{\mathcal{R}} 1 = \frac{1}{2}}$$

Thus $\sum_{n\geq 1}^{\mathcal{R}} f(n)$ can be evaluated in terms of Bernoulli numbers for any polynomial f.

3) Let $f(x) = \frac{1}{x^z}$ for $Re(z) > 1$ then

$$R_f(x) = \sum_{n=0}^{+\infty} \frac{1}{(n+x)^z} - \int_1^2 \sum_{n=0}^{+\infty} \frac{1}{(n+x)^z} dx$$

The series $\sum_{n=0}^{+\infty} \frac{1}{(n+x)^z}$ is uniformly convergent for $x \in [1, 2]$ then

$$\int_1^2 \sum_{n=0}^{+\infty} \frac{1}{(n+x)^z} dx = \sum_{n=0}^{+\infty} \int_1^2 \frac{1}{(n+x)^z} dx = \frac{1}{z-1} \sum_{n=0}^{+\infty} \left(\frac{1}{(n+1)^{z-1}} - \frac{1}{(n+2)^{z-1}}\right)$$

thus

$$R_f(x) = \sum_{n=0}^{+\infty} \frac{1}{(n+x)^z} - \frac{1}{z-1} = \zeta(z,x) - \frac{1}{z-1}$$

with $(x,z) \mapsto \zeta(z,x)$ the Hurwitz zeta function.

Note that for all $z \neq 1$ the Hurwitz zeta function $(x,z) \mapsto \zeta(z,x)$ is defined and we have

$$\zeta(z,x) - \zeta(z,x+1) = \frac{1}{x^z}$$

and $\int_1^2 \zeta(z,x)dx = \frac{1}{z-1}$. Thus

$$R_{1/x^z}(x) = \zeta(z,x) - \frac{1}{z-1}$$

and for all $z \neq 1$

$$\boxed{\sum_{n\geq 1}^{\mathcal{R}} \frac{1}{n^z} = \zeta(z) - \frac{1}{z-1}}$$

4) If $f(x) = \frac{1}{x}$ then

$$R_f(x) = \sum_{n=0}^{+\infty}(\frac{1}{n+x} - \frac{1}{n+1}) - \int_1^2 \sum_{n=0}^{+\infty}(\frac{1}{n+x} - \frac{1}{n+1})dx$$

This last integral is

$$\sum_{n=1}^{+\infty}(Log(n+1) - Log(n) - \frac{1}{n}) = \lim_{N\to+\infty} Log(N+1) - \sum_{n=1}^{N}\frac{1}{n} = -\gamma$$

Thus

$$R_f(x) = \sum_{n=0}^{+\infty}(\frac{1}{n+x} - \frac{1}{n+1}) + \gamma$$

and

$$\boxed{\sum_{n\geq 1}^{\mathcal{R}} \frac{1}{n} = \gamma}$$

Note that the function $\Psi = \Gamma'/\Gamma$ verify

$$\Psi(x+1) - \Psi(x) = \frac{1}{x}$$

with $\int_1^2 \Psi(x)dx = 0$. Thus

$$R_{\frac{1}{x}} = -\Psi(x)$$

If $f(x) = \frac{1}{x^{k+1}}$ with k integer > 0 then

$$\partial^k\Psi(x+1) - \partial^k\Psi(x) = (-1)^k k! \frac{1}{x^{k+1}}$$

with

$$\int_1^2 \partial^k\Psi(x)dx = \partial^{k-1}\psi(2) - \partial^{k-1}\psi(1) = (-1)^{k-1}(k-1)!$$

Thus

$$R_{\frac{1}{x^{k+1}}} = \frac{(-1)^{k-1}}{k!}\partial^k\Psi - \frac{1}{k}$$

5) If $f(x) = Log(x)$ then $R_f = -Log\ \Gamma + \int_1^2 Log\ \Gamma(t)dt$ and $\int_1^2 Log\ \Gamma(t)dt = -1 + Log(\sqrt{2\pi})$ thus

$$R_{Log}(x) = -Log(\Gamma(x)) + Log(\sqrt{2\pi}) - 1$$

and

$$\boxed{\sum_{n\geq 1}^{\mathcal{R}} Log(n) = Log(\sqrt{2\pi}) - 1}$$

1.3.2 The φ_f function

Let $f \in \mathcal{O}^\pi$ and a function φ analytic for $Re(x) > a$ with $-1 < a < 0$ and of exponential type $< \pi$ solution of

$$\varphi(x) - \varphi(x-1) = f(x) \text{ with } \varphi(0) = 0$$

If we set

$$R(x) = -\varphi(x-1) + \int_0^1 \varphi(x)dx$$

then R is an analytic function for $Re(x) > a+1$ and of exponential type $< \pi$ solution of $R(x) - R(x+1) = f(x)$ with $\int_1^2 R(x)dx = 0$, thus $R = R_f$ and we get

$$\varphi(x) = \int_0^1 \varphi(x)dx - R_f(x+1)$$

The condition $\varphi(0) = 0$ gives $\int_0^1 \varphi(x)dx = R_f(1)$. Conversely if $f \in \mathcal{O}^\pi$ and if we set

$$\varphi_f(x) = R_f(1) - R_f(x+1)$$

then φ_f is analytic for $Re(x) > a$ with $-1 < a < 0$ and of exponential type $< \pi$ and is solution of

$$\varphi_f(x) - \varphi_f(x-1) = f(x) \text{ with } \varphi_f(0) = 0$$

Thus we get the

Equivalent definition of Ramanujan summation:
If $f \in \mathcal{O}^\pi$ then there exist a unique analytic function $x \mapsto \varphi_f(x)$ for $Re(x) > a$ with $-1 < a < 0$ and of exponential type $< \pi$ solution of

$$\varphi_f(x) - \varphi_f(x-1) = f(x) \text{ with } \varphi_f(0) = 0$$

And we set

$$\sum_{n\geq 1}^{\mathcal{R}} f(n) = \int_0^1 \varphi_f(x)dx$$

Remark
The relation $\varphi_f(x) = R_f(1) - R_f(x+1)$ gives

$$\boxed{\varphi_f(x) = \sum_{n\geq 1}^{\mathcal{R}} f(n) - R_f(x) + f(x)}$$

We have

$$\varphi_f(n) = R_f(1) - R_f(n+1) = f(1) + f(2) + ... + f(n)$$

Thus the function $\varphi_f(x)$ is an interpolation function of the partial sums of the series $\sum_{n\geq 1} f(n)$, Ramanujan write

$$\boxed{\varphi_f(x) = f(1) + f(2) + ... + f(x)}$$

The sum $\sum_{n\geq 1}^{\mathcal{R}} f(n)$ is then the constant term $C(f)$ in the MacLaurin expansion

$$\varphi_f(x) = C(f) + \int_1^x f(t)dt + \frac{f(x)}{2} + \sum_{k=1}^m \frac{B_{2k}}{(2k)!}\partial^{2k-1} f(x) - \int_x^{+\infty} \frac{b_{2m+1}(t)}{(2m+1)!}\partial^{2m+1} f(t)dt$$

Examples
1) For $z \neq 1$ we have

$$\varphi_{1/x^z}(x) = \sum_{n\geq 1}^{\mathcal{R}} \frac{1}{n^z} - (\zeta(z,x) - \frac{1}{z-1}) + \frac{1}{x^z}$$

thus

$$\varphi_{1/x^z}(x) = \zeta(z) - \zeta(z,x) + \frac{1}{x^z}$$

For $z = 1$ we have

$$1 + ... + \frac{1}{x} = \varphi_{\frac{1}{x}}(x) = \sum_{n\geq 1}^{\mathcal{R}} \frac{1}{n} - R_{\frac{1}{x}}(x+1) = \gamma + \Psi(x+1)$$

2) We have $Log(\Gamma(x+1)) - Log(\Gamma(x)) = Log(x)$ and $Log(\Gamma(1)) = 0$ thus

$$\varphi_{Log}(x) = Log(\Gamma(x+1))$$

1.3.3 Relation to usual summation.

By the definition of R_f we have $R_f(1) - R_f(n) = \sum_{k=1}^{n-1} f(k)$. Thus the series $\sum_{n\geq 1} f(n)$ is convergent if and only if $R_f(n)$ has a finite limit when $n \to +\infty$ and in this case

$$\sum_{n\geq 1}^{\mathcal{R}} f(n) = \sum_{n\geq 1}^{\infty} f(n) + \lim_{n\to+\infty} R_f(n)$$

In some cases $\lim_{n\to+\infty} R_f(n)$ is simply related to the integral of the function f. We have

$$R_f(n) = \frac{f(n)}{2} - \int_1^n f(t)dt + i\int_0^{+\infty} \frac{f(n+it) - f(n-it)}{e^{2\pi t} - 1} dt$$

If $f(z) \to 0$ when $Re(z) \to +\infty$ then by the dominated convergence theorem we see that

$$\lim_{n\to+\infty} \int_0^{+\infty} \frac{f(n+it) - f(n-it)}{e^{2\pi t} - 1} dt = 0 \qquad (1.6)$$

If the integral $\int_1^{+\infty} f(t)dt$ is convergent then $R_f(n)$ has a finite limit when $n \to +\infty$

$$\lim_{n\to+\infty} R_f(n) = -\int_1^{+\infty} f(t)dt$$

Conclusion

Let $f \in \mathcal{O}^\pi$ with $f(z) \to 0$ when $Re(z) \to +\infty$. If the integral $\int_1^{+\infty} f(t)dt$ is convergent then the series $\sum_{n\geq 1} f(n)$ is convergent and we have

$$\boxed{\sum_{n\geq 1}^{\mathcal{R}} f(n) = \sum_{n\geq 1}^{\infty} f(n) - \int_1^{+\infty} f(x)dx} \qquad (1.7)$$

Remark

This relation can be obtained with other hypothesis on f:
if the series $\sum_{n\geq 1} f(x+n)$ is convergent for $Re(x) > 0$ we have

$$\sum_{n\geq 0}^{\infty} f(x+n) - \sum_{n\geq 0}^{\infty} f(x+1+n) = f(x)$$

thus if $x \mapsto \sum_{n\geq 0}^{\infty} f(x+n)$ is in $\mathcal{O}^\pi$ we get

$$R_f(x) = \sum_{n\geq 0}^{\infty} f(x+n) - \int_1^2 \sum_{n\geq 0}^{\infty} f(x+n)dx$$

If

$$\int_1^2 \sum_{n\geq 0}^{\infty} f(x+n)dx = \sum_{n\geq 0}^{\infty} \int_1^2 f(x+n)dx = \int_1^{+\infty} f(x)dx$$

then

$$R_f(x) = \sum_{n\geq 0}^{\infty} f(x+n) - \int_1^{+\infty} f(x)dx. \tag{1.8}$$

In this case we have

$$\varphi_f(x) = R_f(1) - R_f(x+1) = \sum_{n\geq 1}^{+\infty}(f(n) - f(n+x))$$

Note that if this last series is convergent and if $\lim_{x\to+\infty} f(x) = 0$ then we get

$$\sum_{n\geq 1}^{+\infty}(f(n) - f(n+x)) - \sum_{n\geq 1}^{+\infty}(f(n) - f(n+x-1)) = f(x)$$

and $\sum_{n\geq 1}^{+\infty}(f(n) - f(n+0)) = 0$ thus we get

$$\varphi_f(x) = \sum_{n\geq 1}^{+\infty}(f(n) - f(n+x))$$

a formula that Ramanujan uses in some places.

Example
We have

$$\sum_{n\geq 1}^{+\infty}(\frac{1}{n} - \frac{1}{n+x}) = \varphi_{\frac{1}{x}}(x) = \Psi(x+1) + \gamma$$

Thus

$$Log(\Gamma(x+1)) = -\gamma x - \sum_{n=1}^{+\infty}(Log(1+\frac{x}{n}) - \frac{x}{n})$$

that is

$$\frac{1}{\Gamma(x+1)} = e^{\gamma x}\prod_{n=1}^{+\infty}(1+\frac{x}{n})e^{-\frac{x}{n}}$$

Chapter 2

Properties of the Ramanujan summation

2.1 Some elementary properties

2.1.1 The unusual property of the shift.

Let $f \in \mathcal{O}^\pi$ by linearity we have

$$\sum_{n\geq 1}^{\mathcal{R}} f(n+1) = \sum_{n\geq 1}^{\mathcal{R}} f(n) - \sum_{n\geq 1}^{\mathcal{R}} (f(n) - f(n+1))$$

Let $g(x) = f(x) - f(x+1)$ then $R_g(x) = f(x) - \int_1^2 f(x)dx$ thus

$$\sum_{n\geq 1}^{\mathcal{R}} (f(n) - f(n+1)) = f(1) - \int_1^2 f(x)dx$$

We get

$$\boxed{\sum_{n\geq 1}^{\mathcal{R}} f(n+1) = \sum_{n\geq 1}^{\mathcal{R}} f(n) - f(1) + \int_1^2 f(x)dx}$$

We see that the usual properties of summation of convergent series

$$\sum_{n=1}^{+\infty} f(n+1) = \sum_{n=1}^{+\infty} f(n) - f(1)$$

is not satisfied by Ramanujan summation.

Let $f \in \mathcal{O}^\pi$ and $x > 0$ if $g(u) = f(u+x)$ then we verify immediately that

$$R_g(u) = R_f(u+x) - \int_{x+1}^{x+2} R_f(u)du$$

thus

$$\sum_{n\geq 1}^{\mathcal{R}} g(n) = \sum_{n\geq 1}^{\mathcal{R}} f(n+x) = R_f(x+1) - \int_{x+1}^{x+2} R_f(u)du$$

Integrating the equation $R_f(u) - R_f(u+1) = f(u)$ between 1 and $x+1$ we find that

$$\int_1^{x+1} f(u)du = -\int_{x+1}^{x+2} R_f(u)du \tag{2.1}$$

thus without any hypothesis of convergence we have

$$\sum_{n\geq 1}^{\mathcal{R}} f(n+x) = R_f(x+1) + \int_1^{x+1} f(u)du = R_f(x) - f(x) + \int_1^{x+1} f(t)dt \tag{2.2}$$

For a positive integer $x = m$ we have $R_f(m+1) = R_f(1) - \sum_{n=1}^{m} f(n)$ thus

$$\sum_{n\geq 1}^{\mathcal{R}} f(n+m) \quad = \quad R_f(1) - \sum_{n=1}^{m} f(n) + \int_1^{m+1} f(u)du$$

Conclusion
If m is a positive integer we have the *shift property*

$$\sum_{n\geq 1}^{\mathcal{R}} f(n+m) = \sum_{n\geq 1}^{\mathcal{R}} f(n) - \sum_{n=1}^{m} f(n) + \int_1^{m+1} f(x)dx \tag{2.3}$$

Examples
1) Let $f(x) = \frac{1}{x}$ and $H_m = \sum_{k=1}^{m} \frac{1}{k}$ we have

$$\sum_{n\geq 1}^{\mathcal{R}} \frac{1}{n+m} \quad = \quad \sum_{n\geq 1}^{\mathcal{R}} \frac{1}{n} - H_m + \int_1^{m+1} \frac{1}{x}dx = \gamma - H_m + Log(m+1)$$

Since $R_{\frac{1}{x}} = -\Psi(x)$ the formula (2.2) gives more generally

$$\boxed{\sum_{n\geq 1}^{\mathcal{R}} \frac{1}{n+x} = -\Psi(x+1) + Log(x+1)} \tag{2.4}$$

If $0 < p < q$ are integers then

$$\sum_{n\geq 1}^{\mathcal{R}} \frac{1}{n+\frac{p}{q}} = -\Psi(\frac{p}{q}) - \frac{q}{p} + Log(\frac{p}{q}+1)$$

Thus by Gauss formula (c.f. Lehmer) we get

$$\sum_{n\geq 1}^{\mathcal{R}} \frac{1}{p+nq} = \frac{1}{q}\gamma + \frac{1}{q}Log(\frac{q}{2}) + \frac{\pi}{2q}\cot(\pi\frac{p}{q}) - \frac{2}{q}\sum_{0<k<q/2} \cos(2\pi k\frac{p}{q})Log(\sin(\pi\frac{k}{q})) - \frac{1}{p} + \frac{1}{q}Log(\frac{p}{q}+1)$$

2) For $z \neq 1$ the formula (2.2) gives

$$\sum_{n\geq 1}^{\mathcal{R}} \frac{1}{(n+x)^z} = \zeta(z,x) - \frac{1}{x^z} + \frac{1}{z-1}\frac{1}{(x+1)^{z-1}}$$

where ζ is the Hurwitz zeta function.
3) For $f(x) = Log(x)$ we get by (2.2)

$$\sum_{n\geq 1}^{\mathcal{R}} Log(n+x) = -Log(\Gamma(x)) + Log(\sqrt{2\pi}) - 1 - Log(x) + (x+1)Log(x+1) - x$$

This gives

$$Log(\Gamma(x)) = -\sum_{n\geq 1}^{\mathcal{R}} Log(1+\frac{n}{x}) - \frac{1}{2}Log(x) + Log(\sqrt{2\pi}) + xLog(x) - x + (x+1)Log(1+\frac{1}{x}) - 1$$

thus we get the Stirling formula

$$Log(\Gamma(x+1)) = \frac{1}{2}Log(x) + Log(\sqrt{2\pi}) + xLog(x) - x + R(x)$$

with

$$R(x) = (x+1)Log(1+\frac{1}{x}) - 1 - \sum_{n\geq 1}^{\mathcal{R}} Log(1+\frac{n}{x})$$

Remark
The relation $\varphi_f(x) = R_f(1) - R_f(x+1)$ gives

$$\boxed{\varphi_f(x) = \sum_{n\geq 1}^{\mathcal{R}} f(n) - \sum_{n\geq 1}^{\mathcal{R}} f(n+x) + \int_1^{x+1} f(u)du}$$

2.1.2 Functional relations for φ_f

Theorem 2 *Let $f \in \mathcal{O}^\pi$ and an integer $N > 1$ then*

$$R_{f(x/N)}(x) = \sum_{k=0}^{N-1} R_f(\frac{x+k}{N}) - N\int_{1/N}^1 f(x)dx \tag{2.5}$$

We get

$$\sum_{n\geq 1}^{\mathcal{R}} f(\frac{n}{N}) = \sum_{k=0}^{N-1} [\sum_{n\geq 1}^{\mathcal{R}} f(n-\frac{k}{N})] - \sum_{k=1}^{N-1}\int_1^{k/N} f(x)dx - N\int_{1/N}^1 f(x)dx \tag{2.6}$$

Proof
The function

$$R(x) = \sum_{k=0}^{N-1} R_f(\frac{x+k}{N}) = R_f(\frac{x}{N}) + R_f(\frac{x+1}{N}) + ... + R_f(\frac{x+N-1}{N})$$

satisfies

$$R(x) - R(x+1) = R_f(\frac{x}{N}) - R_f(\frac{x}{N}+1) = f(\frac{x}{N})$$

thus

$$R_{f(\frac{x}{N})}(x) = \sum_{k=0}^{N-1} R_f(\frac{x+k}{N}) - \int_1^2 \sum_{k=0}^{N-1} R_f(\frac{x+k}{N})dx$$

but we have from (2.1)

$$\int_1^2 \sum_{k=0}^{N-1} R_f(\frac{x+k}{N})dx = N\int_{\frac{1}{N}}^{\frac{1}{N}+1} R_f(x)dx = N\int_{1/N}^1 f(x)dx$$

Thus

$$R_{f(x/N)}(x) = \sum_{k=0}^{N-1} R_f(\frac{x+k}{N}) - N\int_{1/N}^1 f(x)dx$$

and

$$\sum_{n\geq 1}^{\mathcal{R}} f(\frac{n}{N}) = \sum_{k=0}^{N-1} R_f(\frac{k+1}{N}) - N\int_{1/N}^1 f(x)dx$$

We have by (2.2)

$$R_f(\frac{k+1}{N}) = \sum_{n\geq 1}^{\mathcal{R}} f(n-1+\frac{k+1}{N}) - \int_1^{(k+1)/N} f(x)dx$$

thus we get

$$\sum_{n\geq 1}^{\mathcal{R}} f(\frac{n}{N}) = \sum_{k=0}^{N-1}\sum_{n\geq 1}^{\mathcal{R}} f(n-1+\frac{k+1}{N}) - \sum_{k=1}^{N-1}\int_1^{k/N} f(x)dx - N\int_{1/N}^1 f(x)dx$$

which is

$$\sum_{n\geq 1}^{\mathcal{R}} f(\frac{n}{N}) = \sum_{k=0}^{N-1}\sum_{n\geq 1}^{\mathcal{R}} f(n-\frac{k}{N}) - \sum_{k=1}^{N-1}\int_1^{k/N} f(x)dx - N\int_{1/N}^1 f(x)dx$$

□

Remark
If $g\in\mathcal{O}^{\pi/N}$ then with $f: x\mapsto g(Nx)$ we have

$$\sum_{n\geq 1}^{\mathcal{R}} g(n) = \sum_{k=0}^{N-1}\Big[\sum_{n\geq 1}^{\mathcal{R}} g(Nn-k)\Big] + \sum_{k=1}^{N-1}\frac{1}{N}\int_k^N g(x)dx - \int_1^N g(x)dx$$

Thus if $f\in\mathcal{O}^{\pi/2}$ then we get

$$\sum_{n\geq 1}^{\mathcal{R}} f(n) = \sum_{n\geq 1}^{\mathcal{R}} f(2n) + \sum_{n\geq 1}^{\mathcal{R}} f(2n-1) - \frac{1}{2}\int_1^2 f(x)dx \tag{2.7}$$

For $f(x)=\frac{1}{x}$ we get

$$\sum_{n\geq 1}^{\mathcal{R}}\frac{1}{n} = \frac{1}{2}\sum_{n\geq 1}^{\mathcal{R}}\frac{1}{n} + \sum_{n\geq 1}^{\mathcal{R}}\frac{1}{2n-1} - \frac{1}{2}Log(2)$$

thus $\sum_{n\geq 1}^{\mathcal{R}}\frac{1}{2n-1} = \frac{\gamma}{2}+\frac{1}{2}Log(2)$.

Theorem 3 *Let $f\in\mathcal{O}^{\pi}$ and an integer $N>1$ then*

$$\varphi_{f(x/N)}(x) = \sum_{j=0}^{N-1}\varphi_f(\frac{x-j}{N}) + \sum_{n\geq 1}^{\mathcal{R}} f(\frac{n}{N}) - N\sum_{n\geq 1}^{\mathcal{R}} f(n) + N\int_{1/N}^1 f(x)dx$$

which is the entry 7 Ch VI of Ramanujan Notebook corrected with the integral term.

Proof
We can write eq (2.5) in the form

$$R_{f(x/N)}(x+1) = R_f(\frac{x+1}{N}) + R_f(\frac{x+2}{N}) + ... + R_f(\frac{x+N}{N}) - N\int_{1/N}^1 f(x)dx$$

with $R_{f(x/N)}(x+1) = \sum_{n\geq 1}^{\mathcal{R}} f(\frac{n}{N}) - \varphi_{f(x/N)}(x)$ we get

$$\sum_{n\geq 1}^{\mathcal{R}} f(\frac{n}{N}) - \varphi_{f(x/N)}(x) = N\sum_{n\geq 1}^{\mathcal{R}} f(n) - \big[\varphi_f(\frac{x+1-N}{N}) + \varphi_f(\frac{x+2-N}{N}) + ... + \varphi_f(\frac{x+N-N}{N})\big] - N\int_{1/N}^1 f(x)dx$$

□

Corollary
Since $\varphi_{f(x/N)}(0)=0$ and $\varphi_f(0)=0$ then we get

$$\sum_{k=1}^{N-1}\varphi_f(\frac{-k}{N}) = N\sum_{n\geq 1}^{\mathcal{R}} f(n) - \sum_{n\geq 1}^{\mathcal{R}} f(\frac{n}{N}) - N\int_{1/N}^1 f(x)dx$$

a formula that Ramanujan gives without the correcting integral term.

Examples

1) If $f(x) = \frac{1}{x}$ we have $\varphi_f(x) = \gamma + \Psi(x+1)$ and in this case we have also $f(x/N) = Nf(x)$ thus $\varphi_{f(x/N)}(x) = N\varphi_f(x)$ and the preceding theorem gives

$$N(\gamma + \Psi(x+1)) = \gamma + \Psi(\frac{x}{N}+1) + ... + \gamma + \Psi(\frac{x+1}{N}) + N\gamma - N\gamma + NLog(N)$$

thus

$$N\Psi(x+1) = \Psi(\frac{x+1}{N}) + ... + \Psi(\frac{x+N}{N}) + NLog(N)$$

we get the well known formula

$$\Psi(x) = \frac{1}{N}\sum_{k=0}^{N-1}\Psi(\frac{x-k}{N}) + Log(N)$$

2) If $f(x) = Log(x)$ then $\varphi_f(x) = Log(\Gamma(x+1))$ and $f(x/N) = Log(x) - Log(N)$ thus

$$\varphi_{f(x/N)}(x) = \varphi_{Log}(x) - \varphi_{Log(N)}(x) = Log(\Gamma(x+1)) - xLog(N)$$

With the preceding theorem we get

$$Log(\Gamma(x+1)) = \sum_{k=1}^{N} Log(\frac{\Gamma(\frac{x+k}{N})}{\sqrt{2\pi}}) + (x+\frac{1}{2})Log(N) + Log(\sqrt{2\pi})$$

taking the exponential we get the Gauss formula for the Gamma function.

3) If $f(x) = \frac{Log(x)}{x}$ then $f(x/N) = N\frac{Log(x)}{x} - NLog(N)\frac{1}{x}$, thus

$$\varphi_{f(x/N)}(x) = N\varphi_f(x) - NLog(N)(\gamma + \Psi(x+1))$$

and we get

$$N\varphi_f(x) - NLog(N)\Psi(x+1) = \sum_{j=0}^{N-1}\varphi_f(\frac{x-j}{N}) + N\int_{1/N}^{1} f(x)dx$$

this gives

$$\varphi_{\frac{Log(x)}{x}}(x) = \frac{1}{N}\sum_{j=0}^{N-1}\varphi_{\frac{Log(x)}{x}}(\frac{x-j}{N}) + Log(N)\Psi(x+1) - \frac{1}{2}Log^2(N)$$

which is entry 17 of chapter 8.

4) If $f(x) = Log^2(x)$ then $f(x/N) = Log^2(x) - 2Log(x)Log(N) + Log^2(N)$, thus

$$\varphi_{f(x/N)}(x) = \varphi_f(x) - 2Log(N)Log(\Gamma(x+1)) + xLog^2(N)$$

and we get

$$\begin{aligned}\varphi_f(x) - 2Log(N)Log(\Gamma(x+1)) + xLog^2(N) &= \sum_{j=0}^{N-1}\varphi_f(\frac{x-j}{N}) \\ &+ (1-N)\sum_{n\geq 1}^{\mathcal{R}} Log^2(n) - 2Log(N)(Log(\sqrt{2\pi}) - 1) + \frac{1}{2}Log^2(N) \\ &+ N\int_{1/N}^{1} f(x)dx\end{aligned}$$

this gives

$$\varphi_f(x) = \sum_{j=0}^{N-1}\varphi_f(\frac{x-j}{N}) + 2Log(N)Log(\frac{\Gamma(x+1)}{\sqrt{2\pi}}) - (\frac{1}{2}+x)Log^2(N) - (N-1)(\sum_{n\geq 1}^{\mathcal{R}} Log^2(n) - 2)$$

this is entry 18(ii) of chapter 8 with $C = \sum_{n\geq 1}^{\mathcal{R}} Log^2(n) - 2$ (note that the constant C in entry 18 of Ramanujan is $C_0(f)$ and that $\sum_{n\geq 1}^{\mathcal{R}} Log^2(n)$ is $C_1(f)$ and $C_1(f) - C_0(f) = \int_0^1 Log^2(x)dx = 2$)

2.1.3 Summation on $\mathbb{Z}$

Let $f \in \mathcal{O}^\pi$ such that the function $x \mapsto f(-x+1)$ is also in $\mathcal{O}^\pi$. Then we can try to define $\sum_{n\in\mathbb{Z}}^{\mathcal{R}} f(n)$ by breaking the sum in two parts

$$\sum_{n\in\mathbb{Z}}^{\mathcal{R}} f(n) = \sum_{n\geq 1}^{\mathcal{R}} f(n) + \sum_{n\geq 1}^{\mathcal{R}} f(-n+1)$$

But to get a coherent definition this sum must be independent on the breaking point, thus we must have

$$\sum_{n\in\mathbb{Z}}^{\mathcal{R}} f(n) = \sum_{n\geq 1}^{\mathcal{R}} f(n+m) + \sum_{n\geq 1}^{\mathcal{R}} f(-n+1+m)$$

Now by the shift property we find easily that

$$\sum_{n\geq 1}^{\mathcal{R}} f(n+m) + \sum_{n\geq 1}^{\mathcal{R}} f(-n+1+m) - \int_m^{m+1} f(x)dx = \sum_{n\geq 1}^{\mathcal{R}} f(n) + \sum_{n\geq 1}^{\mathcal{R}} f(-n+1) - \int_0^1 f(x)dx$$

Thus we get the following definition:

Definition
Let $f \in \mathcal{O}^\pi$ such that the function $x \mapsto f(-x+1)$ is also in $\mathcal{O}^\pi$. Then we define $\sum_{n\in\mathbb{Z}}^{\mathcal{R}} f(n)$ by

$$\sum_{n\in\mathbb{Z}}^{\mathcal{R}} f(n) = \sum_{n\geq 1}^{\mathcal{R}} f(n) + \sum_{n\geq 1}^{\mathcal{R}} f(-n+1) - \int_0^1 f(x)dx$$

Remark
With this definition in a case of convergence we have

$$\sum_{n\in\mathbb{Z}}^{\mathcal{R}} f(n) = \sum_{n=1}^{+\infty} f(n) + \sum_{n=1}^{+\infty} f(-n+1) - \int_{-\infty}^{+\infty} f(x)dx$$

Examples
1) Let $a \in \mathbb{C}$ and $|a| < \pi$ and $a \neq 0$. Let $f(x) = e^{ax}$ then the divergent calculation

$$\sum_{n\in\mathbb{Z}} e^{an} = \sum_{n\geq 1} e^{an} + e^a \sum_{n\geq 1} e^{-an} = \frac{e^a}{1-e^a} + \frac{1}{1-e^{-a}} = 0$$

is perfectly rigorous if we take our preceding definition since

$$\begin{aligned}
\sum_{n\in\mathbb{Z}}^{\mathcal{R}} e^{an} &= \sum_{n\geq 1}^{\mathcal{R}} e^{an} + e^a \sum_{n\geq 1}^{\mathcal{R}} e^{-an} - \int_0^1 e^{ax}dx \\
&= \left(\frac{e^a}{1-e^a} + \frac{e^a}{a}\right) + \left(\frac{1}{1-e^{-a}} - \frac{1}{a}\right) - \frac{e^a - 1}{a} \\
&= 0
\end{aligned}$$

2.2 Summation and derivation

Let $f \in \mathcal{O}^\pi$ there is a very simple relation between ∂R_f and $R_{\partial f}$. This is a consequence of the fact that

$$\partial R_f(x) - \partial R_f(x+1) = \partial f(x)$$

thus $R_{\partial f} = \partial R_f - \int_1^2 \partial R_f(x)dx$ by $\int_1^2 \partial R_f(x)dx = -f(1)$ we get

$$\boxed{R_{\partial f}(x) = \partial R_f(x) + f(1)} \qquad (2.8)$$

The relation $\varphi_f(x) = \sum_{n\geq 1}^{\mathcal{R}} f(n) - R_f(x+1)$ gives also

$$\boxed{\varphi_{\partial f}(x) = \partial\varphi_f(x) - f(1) + \sum_{n\geq 1}^{\mathcal{R}} \partial f(n)} \tag{2.9}$$

Theorem 4 *Let $f \in \mathcal{O}^\pi$ then*

$$\sum_{n\geq 1}^{\mathcal{R}} f(n) = -\sum_{k=1}^{m} \frac{B_k}{k!}\partial^{k-1}f(1) + (-1)^{m+1}\int_0^1 R_{\partial^m f}(t+1)\frac{B_m(t)}{m!}dt \tag{2.10}$$

Thus if $F(x) = F_1(x) = \int_1^x f(t)dt$ and for $k \geq 2$ let $F_k(x) = \int_1^x F_{k-1}(t)dt$, then we have

$$\sum_{n\geq 1}^{\mathcal{R}} F_m(n) = (-1)^{m+1}\int_0^1 R_f(t+1)\frac{B_m(t)}{m!}dt$$

For $m = 1$ we have

$$\sum_{n\geq 1}^{\mathcal{R}} F(n) = \int_1^2 tR_f(t)dt$$

Proof

We have $B_1(t) = t - \frac{1}{2}$ thus

$$0 = \int_0^1 R_f(t+1)dt = \int_0^1 R_f(t+1)\partial B_1(t)dt$$

integrating by parts we get with the relation (2.8)

$$R_f(1) = \frac{1}{2}f(1) + \int_0^1 R_{\partial f}(t+1)B_1(t)dt$$

If we continue integration by parts we find

$$\sum_{n\geq 1}^{\mathcal{R}} f(n) = -\sum_{k=1}^{m} \frac{B_k}{k!}\partial^{k-1}f(1) + (1)^{m+1}\int_0^1 R_{\partial^m f}(t+1)\frac{B_m(t)}{m!}dt$$

□

Remark

If f is a polynomial of degree N then $\partial^{N+1}f = 0$ thus

$$\sum_{n\geq 1}^{\mathcal{R}} f(n) = -\sum_{k=0}^{N} \frac{B_{k+1}}{(k+1)!}\partial^k f(1)$$

Example

Let $f(x) = 1/x$. We have $F_1(x) = Log(x)$, $F_2(x) = xLog(x) - x + 1$, $F_3(x) = \frac{x^2}{2}Log(x) - \frac{3}{4}x^2 + x - \frac{1}{4}$ and more generally we have

$$F_k(x) = \frac{x^{k-1}}{(k-1)!}Log(x) + P_k(x)$$

where the P_k are the polynomials defined by $P_1 = 0$ and

$$\begin{aligned} P_k'(x) &= P_{k-1}(x) - \frac{x^{k-2}}{(k-1)!} \text{ if } k \geq 2 \\ P_k(1) &= 0 \end{aligned}$$

Thus

$$\sum_{n\geq 1}^{\mathcal{R}} Log(n) = -\int_0^1 \psi(t+1)B_1(t)dt$$

$$\sum_{n\geq 1}^{\mathcal{R}} nLog(n) = \int_0^1 \psi(t+1)\frac{B_2(t)}{2!}dt + \sum_{n\geq 1}^{\mathcal{R}} n - \sum_{n\geq 1}^{\mathcal{R}} 1$$

$$\sum_{n\geq 1}^{\mathcal{R}} \frac{n^2}{2}Log(n) = -\int_0^1 \psi(t+1)\frac{B_3(t)}{3!}dt + \frac{3}{4}\sum_{n\geq 1}^{\mathcal{R}} n^2 - \sum_{n\geq 1}^{\mathcal{R}} n + \frac{1}{4}\sum_{n\geq 1}^{\mathcal{R}} 1$$

and

$$\frac{1}{k!}\sum_{n\geq 1}^{\mathcal{R}} n^k Log(n) = (-1)^{k-1}\int_0^1 \psi(t+1)\frac{B_{k+1}(t)}{(k+1)!}dt - \sum_{n\geq 1}^{\mathcal{R}} P_{k+1}(n)$$

Theorem 5 *Let $f \in \mathcal{O}^{\pi}$ then*

$$\sum_{n\geq 1}^{\mathcal{R}}\sum_{k=1}^{n} f(k) = \frac{3}{2}\sum_{n\geq 1}^{\mathcal{R}} f(n) - \sum_{n\geq 1}^{\mathcal{R}} nf(n) - \sum_{n\geq 1}^{\mathcal{R}} F(n) \text{ with } F(x) = \int_1^x f(t)dt$$

Proof
We use

$$xR_f(x) - (x+1)R_f(x+1) = xf(x) - R_f(x+1)$$

thus

$$\sum_{n\geq 1}^{\mathcal{R}} nf(n) - R_f(n+1) = 1R_f(1) - \int_1^2 xR_f(x)dx$$

Since $\int_1^2 xR_f(x)dx = \sum_{n\geq 1}^{\mathcal{R}} F(n)$ we get

$$\sum_{n\geq 1}^{\mathcal{R}} nf(n) - \sum_{n\geq 1}^{\mathcal{R}} R_f(n+1) = \sum_{n\geq 1}^{\mathcal{R}} f(n) - \int_1^2 xR_f(x)dx = \sum_{n\geq 1}^{\mathcal{R}} f(n) - \sum_{n\geq 1}^{\mathcal{R}} F(n)$$

With $R_f(n+1) = R_f(1) - \sum_{k=1}^{n} f(k)$ we get

$$\sum_{n\geq 1}^{\mathcal{R}} nf(n) - \sum_{n\geq 1}^{\mathcal{R}} R_f(1) + \sum_{n\geq 1}^{\mathcal{R}}\sum_{k=1}^{n} f(k) = \sum_{n\geq 1}^{\mathcal{R}} f(n) - \sum_{n\geq 1}^{\mathcal{R}} F(n)$$

since $\sum_{n\geq 1}^{\mathcal{R}} R_f(1) = \frac{1}{2}R_f(1) = \frac{1}{2}\sum_{n\geq 1}^{\mathcal{R}} f(n)$ we get our assertion.
□

Example
We have for all s

$$\sum_{n\geq 1}^{\mathcal{R}}\sum_{k=1}^{n} \frac{1}{k^s} = \frac{3}{2}\sum_{n\geq 1}^{\mathcal{R}} \frac{1}{n^s} - \sum_{n\geq 1}^{\mathcal{R}} \frac{1}{n^{s-1}} - \sum_{n\geq 1}^{\mathcal{R}} \int_1^n t^{-s}dt$$

For $Re(s) > 2$ we have

$$\sum_{n\geq 1}^{\mathcal{R}}\sum_{k=1}^{n} \frac{1}{k^s} = \frac{3}{2}\zeta(s) - \frac{3}{2}\frac{1}{s-1} - \zeta(s-1) + \frac{1}{s-2} - \frac{1}{s-1}\sum_{n\geq 1}^{\mathcal{R}}(1 - \frac{1}{n^{s-1}})$$

Let $H_n^{(s)} = \sum_{k=1}^{n} \frac{1}{k^s}$ then

$$\boxed{\sum_{n\geq 1}^{\mathcal{R}} H_n^{(s)} = \frac{3}{2}\zeta(s) - \frac{s-2}{s-1}\zeta(s-1) - \frac{1}{s-1}}$$

For $s = 1$ we have

$$\sum_{n\geq 1}^{\mathcal{R}} \sum_{k=1}^{n} \frac{1}{k} = \frac{3}{2}\sum_{n\geq 1}^{\mathcal{R}} \frac{1}{n} - \sum_{n\geq 1}^{\mathcal{R}} 1 - \sum_{n\geq 1}^{\mathcal{R}} log(n)$$

thus

$$\boxed{\sum_{n\geq 1}^{\mathcal{R}} H_n = \frac{3}{2}\gamma + \frac{1}{2} - Log(\sqrt{2\pi})}$$

For $s = 2$ we have

$$\sum_{n\geq 1}^{\mathcal{R}} \sum_{k=1}^{n} \frac{1}{k^2} = \frac{3}{2}\sum_{n\geq 1}^{\mathcal{R}} \frac{1}{n^2} - \sum_{n\geq 1}^{\mathcal{R}} \frac{1}{n} + \sum_{n\geq 1}^{\mathcal{R}} (\frac{1}{n} - 1)$$

thus

$$\boxed{\sum_{n\geq 1}^{\mathcal{R}} H_n^{(2)} = \frac{3}{2}\zeta(2) - 2}$$

Remark. We can write the result of the theorem in the form

$$\sum_{n\geq 1}^{\mathcal{R}} \varphi_f(n) = \frac{3}{2}\sum_{n\geq 1}^{\mathcal{R}} f(n) - \sum_{n\geq 1}^{\mathcal{R}} nf(n) - \int_1^2 xR_f(x)dx$$

This can be generalized to the sums $\sum_{n\geq 1}^{\mathcal{R}} f(n)g(n)$ where f and g are of moderate growth. We have

$$R_f(x)R_g(x) - R_f(x+1)R_g(x+1) = R_f(x)g(x) + f(x)(R_g(x) - g(x))$$

and we get

$$\sum_{n\geq 1}^{\mathcal{R}} \varphi_f(n)g(n) + \sum_{n\geq 1}^{\mathcal{R}} \varphi_g(n)f(n) = \sum_{n\geq 1}^{\mathcal{R}} f(n)\sum_{n\geq 1}^{\mathcal{R}} g(n) + \sum_{n\geq 1}^{\mathcal{R}} f(n)g(n) + \int_1^2 R_f(x)R_g(x)dx$$

For $g = \partial f$ we have

$$\int_1^2 R_f(x)R_\partial f(x)dx = \int_1^2 R_f(x)\partial R_f(x)dx = \frac{1}{2}(R_f(2))^2 - R_f(1))^2) = \frac{1}{2}f(1)^2 - f(1)R_f(1)$$

thus we get

$$\sum_{n\geq 1}^{\mathcal{R}} \varphi_f(n)\partial f(n) + \sum_{n\geq 1}^{\mathcal{R}} \varphi_{\partial f}(n)f(n) = \sum_{n\geq 1}^{\mathcal{R}} f(n)\sum_{n\geq 1}^{\mathcal{R}} \partial f(n) + \sum_{n\geq 1}^{\mathcal{R}} f(n)\partial f(n) + \frac{1}{2}f(1)^2 - f(1)\sum_{n\geq 1}^{\mathcal{R}} f(n)$$

2.3 The case of an entire function

Theorem 6 *Let f an entire function defined by*

$$f(x) = \sum_{k=0}^{+\infty} \frac{c_k}{k!}x^k \text{ with } |c_k| \leq C\tau^k$$

where $\tau < \pi$. *Then*

$$\sum_{n\geq 1}^{\mathcal{R}} f(n) = \int_0^1 f(x)dx - \frac{1}{2}c_0 - \sum_{k=1}^{+\infty} c_k \frac{B_{k+1}}{(k+1)!}$$

By the fact that $B_{2k+1} = 0$ *for* $k \geq 1$ *we have*

$$\sum_{n\geq 1}^{\mathcal{R}} f(n) = \int_0^1 f(x)dx - \frac{1}{2}f(0) - \sum_{k=1}^{+\infty} c_{2k-1} \frac{B_{2k}}{(2k)!}$$

Thus in a case of convergence we have

$$\sum_{n\geq 1}^{+\infty} f(n) = \int_0^{+\infty} f(x)dx - \frac{1}{2}f(0) - \sum_{k=1}^{+\infty} c_{2k-1} \frac{B_{2k}}{(2k)!}$$

Proof

Let us evaluate R_f. We have

$$R_{x^k} = \frac{1 - B_{k+1}(x)}{k+1}$$

thus we consider the function

$$x \mapsto \sum_{k=0}^{+\infty} \frac{c_k}{(k+1)!} - \sum_{k=0}^{+\infty} c_k \frac{B_{k+1}(x)}{(k+1)!}$$

By $\frac{te^{xz}}{e^z - 1} = \sum_{n\geq 0} \frac{B_n(x)}{n!} z^n$ and the Cauchy integral formula we have for $0 < r < 2\pi$

$$\frac{B_{k+1}(x)}{(k+1)!} = \frac{1}{2\pi r^k} \int_0^{2\pi} \frac{e^{xre^{it}}}{e^{re^{it}} - 1} e^{-ikt} dt$$

thus for $Re(x) > 0$ we get

$$|\frac{B_{k+1}(x)}{(k+1)!}| \leq \frac{1}{2\pi r^k} e^{r|x|} \int_0^{2\pi} \frac{1}{|e^{re^{it}} - 1|} dt = C_r r^{-k} e^{r|x|}$$

For $\tau < r < 2\pi$ this prove that the series $\sum_{k=0}^{+\infty} c_k \frac{B_{k+1}(x)}{(k+1)!}$ is uniformly convergent and define an analytic function of exponential type $< 2\pi$ for $Re(x) > 0$. By

$$\int_1^2 \sum_{k=0}^{+\infty} c_k \frac{1 - B_{k+1}(x)}{(k+1)!} dx = \sum_{k=0}^{+\infty} c_k \int_1^2 \frac{1 - B_{k+1}(x)}{(k+1)!} dx = 0$$

we have

$$R_f(x) = \sum_{k=0}^{+\infty} c_k \frac{1 - B_{k+1}(x)}{(k+1)!}$$

Thus

$$R_f(1) = \sum_{k=0}^{+\infty} c_k \frac{1 - B_{k+1}(1)}{(k+1)!} = \sum_{k=0}^{+\infty} \frac{c_k}{(k+1)!} - c_0 B_1(1) - \sum_{k=1}^{+\infty} c_k \frac{B_{k+1}(1)}{(k+1)!}$$

this gives

$$\sum_{n\geq 1}^{\mathcal{R}} f(n) = \int_0^1 f(x)dx - \frac{1}{2}c_0 - \sum_{k=1}^{+\infty} c_k \frac{B_{k+1}}{(k+1)!}$$

Thus

$$\sum_{n\geq 1}^{\mathcal{R}} f(n) = \int_0^1 f(x)dx - \frac{1}{2}f(0) - \sum_{k=1}^{+\infty} c_k \frac{B_{k+1}}{(k+1)!}$$

□

Remarks

1) If we write $f(x)=\sum_{k=0}^{+\infty}\frac{\partial^k f(0)}{k!}x^k$ the preceding theorem is nothing else than

$$\sum_{n\geq 1}^{\mathcal{R}} f(n)=\sum_{k=0}^{+\infty}\frac{\partial^k f(0)}{k!}\sum_{n\geq 1}^{\mathcal{R}} n^k$$

If we use the expansion of f at 1 then we find

$$\sum_{n\geq 1}^{\mathcal{R}} f(n)=\sum_{k=0}^{+\infty}\frac{\partial^k f(1)}{k!}\sum_{n\geq 1}^{\mathcal{R}}(n-1)^k=\frac{1}{2}f(1)-\sum_{k=1}^{+\infty}\partial^k f(1)\frac{B_{k+1}}{(k+1)!}=-\sum_{k=1}^{+\infty}\partial^{k-1} f(1)\frac{B_k}{k!}$$

2) The preceding theorem is not valid if f is not entire: take $f(x)=\frac{1}{1+x^2t^2}$ then if we apply the preceding result we find

$$\int_0^{+\infty}\frac{1}{1+x^2t^2}dx-\frac{1}{2}=\frac{\pi}{2t}-\frac{1}{2}\quad\text{which is not}\quad\sum_{n\geq 1}^{+\infty}\frac{1}{1+n^2t^2}=\frac{\pi}{2t}-\frac{1}{2}+\frac{\pi}{t}\frac{1}{e^{2\pi/t}-1}$$

Example

With p integer >0 and $0<t<\pi/p$ let $f(x)=\frac{\sin^p(xt)}{x^p}$ for $x\neq 0$ and $f(0)=t^p$. This fonction is entire and even thus

$$\boxed{\sum_{n\geq 1}^{\mathcal{R}}\frac{\sin^p(nt)}{n^p}=\int_0^1\frac{\sin^p(xt)}{x^p}dx-\frac{1}{2}t^p}$$

We are in a case of convergence thus

$$\sum_{n=1}^{+\infty}\frac{\sin^p(nt)}{n^p}=\sum_{n\geq 1}^{\mathcal{R}}\frac{\sin^p(nt)}{n^p}+\int_1^{+\infty}\frac{\sin^p(xt)}{x^p}dx=\int_0^{+\infty}\frac{\sin^p(xt)}{x^p}dx-\frac{1}{2}t^p$$

this gives

$$\sum_{n=1}^{+\infty}\frac{\sin^p(nt)}{n^p}=t^{p-1}\int_0^{+\infty}\frac{\sin^p(x)}{x^p}dx-\frac{1}{2}t^p$$

With $p=1$ we get for $0<t<\pi$

$$\sum_{n=1}^{+\infty}\frac{\sin(nt)}{n}=\int_0^{+\infty}\frac{\sin(x)}{x}dx-\frac{1}{2}t$$

Note that for $t=\frac{\pi}{2}$ we find easily the value of $\int_0^{+\infty}\frac{\sin(x)}{x}dx$

$$\int_0^{+\infty}\frac{\sin(x)}{x}dx=\sum_{n=1}^{+\infty}\frac{\sin(n\frac{\pi}{2})}{n}+\frac{\pi}{4}=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1}-\frac{\pi}{4}=\frac{\pi}{2}$$

Remark

The preceding theorem gives easily the sum of some trigonometric series.

Let $0<t<1$ and f the entire function

$$f(x)=\frac{\cos(\pi xt)-1}{x^2}=\sum_{k=1}^{+\infty}(-1)^k\frac{\pi^{2k}t^{2k}}{2k(2k-1)}\frac{x^{2k-2}}{(2k-2)!}$$

This function is even and we have $f(0)=-\frac{\pi^2t^2}{2}$.

Then by the preceding theorem we have

$$\sum_{n\geq 1}^{\mathcal{R}}\frac{\cos(\pi nt)-1}{n^2}=\int_0^1\frac{\cos(\pi xt)-1}{x^2}dx+\frac{\pi^2t^2}{4}$$

thus

$$\sum_{n\geq 1}^{\mathcal{R}} \frac{\cos(\pi nt)}{n^2} = \sum_{n\geq 1}^{\mathcal{R}} \frac{1}{n^2} + \int_0^1 \frac{\cos(\pi xt)-1}{x^2}dx + \frac{\pi^2 t^2}{4} \tag{2.11}$$

Since we are in a case of convergence then

$$\sum_{n\geq 1}^{+\infty} \frac{\cos(\pi nt)}{n^2} = \sum_{n\geq 1}^{+\infty} \frac{1}{n^2} + \int_0^1 \frac{\cos(\pi xt)-1}{x^2}dx + \int_1^{+\infty} \frac{\cos(\pi xt)}{x^2}dx - 1 + \frac{\pi^2 t^2}{4}$$

Integrating by parts we have

$$\int_0^1 \frac{\cos(\pi xt)-1}{x^2}dx + \int_1^{+\infty} \frac{\cos(\pi xt)}{x^2}dx = 1 - \pi t \int_0^{+\infty} \frac{\sin(\pi xt)}{x}dx = 1 - \frac{1}{2}\pi^2 t$$

Thus

$$\sum_{n\geq 1}^{+\infty} \frac{\cos(\pi nt)}{n^2} = \frac{\pi^2}{6} - \frac{1}{2}\pi^2 t + \frac{1}{4}\pi^2 t^2 \tag{2.12}$$

2.3.1 Expression of Catalan's constant

The Catalan's constant G is defined by

$$G = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^2} = \sum_{n=1}^{+\infty} \frac{\sin(\frac{\pi}{2}n)}{n^2}$$

Let the entire function

$$f(x) = \frac{1}{x^2}\sin(\frac{\pi}{2}x) - \frac{\pi}{2x} = \sum_{k=1}^{+\infty}(-1)^k(\frac{\pi}{2})^{2k+1}\frac{x^{2k-1}}{(2k+1)!}$$

We have by the preceding theorem

$$\sum_{n\geq 1}^{\mathcal{R}} \frac{\sin(\frac{\pi}{2}n)}{n^2} - \frac{\pi}{2n} = \int_0^1 (\frac{\sin(\frac{\pi}{2}t)}{t^2} - \frac{\pi}{2t})dt - \sum_{k=1}^{+\infty}(-1)^k(\frac{\pi}{2})^{2k+1}\frac{B_{2k}}{2k(2k+1)!}$$

Since we are in a case of convergence then

$$\sum_{n\geq 1}^{\mathcal{R}} \frac{\sin(\frac{\pi}{2}n)}{n^2} = \sum_{n\geq 1}^{+\infty} \frac{\sin(\frac{\pi}{2}n)}{n^2} - \int_1^{+\infty} \frac{\sin(\frac{\pi}{2}t)}{t^2}dt = G - \int_1^{+\infty} \frac{\sin(\frac{\pi}{2}t)}{t^2}dt$$

thus

$$G - \frac{\pi}{2}\gamma = \int_0^1 (\frac{\sin(\frac{\pi}{2}t)}{t^2} - \frac{\pi}{2t})dt + \int_1^{+\infty} \frac{\sin(\frac{\pi}{2}t)}{t^2}dt - \sum_{k=1}^{+\infty}(-1)^k(\frac{\pi}{2})^{2k+1}\frac{B_{2k}}{2k(2k+1)!}$$

With

$$\int_0^1 (\frac{\sin(\frac{\pi}{2}t)}{t^2} - \frac{\pi}{2t})dt + \int_1^{+\infty} \frac{\sin(\frac{\pi}{2}t)}{t^2}dt = \frac{\pi}{2} - \frac{\pi}{2}Log(\frac{\pi}{2}) - \frac{\pi}{2}\gamma$$

we get

$$\boxed{G = \frac{\pi}{2} - \frac{\pi}{2}Log(\frac{\pi}{2}) - \sum_{k=1}^{+\infty}(-1)^k(\frac{\pi}{2})^{2k+1}\frac{B_{2k}}{2k(2k+1)!}}$$

2.3.2 The sum $\sum_{n\geq 1}^{\mathcal{R}} \frac{e^{-zn}}{n}$

For $0 < z < \pi$ let the entire function

$$f(x) = \frac{e^{-zx}-1}{x} = \sum_{k=0}^{+\infty} \frac{(-1)^{k+1}}{k!}\frac{z^{k+1}}{k+1}x^k$$

We have

$$\sum_{n\geq 1}^{\mathcal{R}} \frac{e^{-zn}-1}{n} = \int_0^1 \frac{e^{-zx}-1}{x}dx + \frac{1}{2}z - \sum_{k=1}^{+\infty}(-1)^{k+1}\frac{z^{k+1}}{k+1}\frac{B_{k+1}}{(k+1)!}$$

Thus

$$\sum_{n\geq 1}^{\mathcal{R}} \frac{e^{-zn}}{n} = \gamma + \int_0^1 \frac{e^{-zx}-1}{x}dx + \frac{1}{2}z - \sum_{k=2}^{+\infty}\frac{(-1)^k}{k!}z^k\frac{B_k}{k}$$

Since $B_{2j+1} = 0$ for $j \geq 1$ this last sum is $\sum_{k=2}^{+\infty}\frac{1}{k!}z^k\frac{B_k}{k}$ and his derivative is

$$\partial(\sum_{k=2}^{+\infty}\frac{1}{k!}z^k\frac{B_k}{k}) = \frac{1}{z}\sum_{k=2}^{+\infty}\frac{z^k}{k!}B_k = \frac{1}{z}(\frac{z}{e^z-1} - 1 + \frac{1}{2}z) = \frac{e^{-z}}{1-e^{-z}} - \frac{1}{z} + \frac{1}{2}$$

which gives

$$\sum_{k=2}^{+\infty}\frac{(-1)^k}{k!}z^k\frac{B_k}{k} = Log(1-e^{-z}) - Log(z) + \frac{1}{2}z$$

Finally we get for $0 < z < \pi$

$$\sum_{n\geq 1}^{\mathcal{R}} \frac{e^{-zn}}{n} = \gamma + \int_0^1 \frac{e^{-zx}-1}{x}dx - Log(1-e^{-z}) + Log(z) \tag{2.13}$$

Since we are in a case of convergence then

$$\sum_{n\geq 1}^{\mathcal{R}} \frac{e^{-zn}}{n} = \sum_{n=1}^{+\infty}\frac{e^{-zn}}{n} - \int_1^{+\infty}\frac{e^{-zx}}{x}dx = -Log(1-e^{-z}) - \int_1^{+\infty}\frac{e^{-zx}}{x}dx$$

thus we get

$$\int_1^{+\infty}\frac{e^{-zx}}{x}dx + \int_0^1\frac{e^{-zx}-1}{x}dx = -\gamma - Log(z) \tag{2.14}$$

Remark

Let $\sum_{n=1}^{+\infty} e^{-tn^\alpha}$ with $0 < \alpha \leq 1$ and $0 < t < \pi$. Then we have

$$\sum_{n=1}^{+\infty} e^{-tn^\alpha} = \sum_{n\geq 1}^{\mathcal{R}} e^{-tn^\alpha} + \int_1^{+\infty} e^{-tx^\alpha}dx$$

With

$$e^{-tn^\alpha} = \sum_{k=0}^{+\infty}\frac{(-1)^k t^k}{k!}n^{\alpha k}$$

and $R_{x^{\alpha k}}(x) = \zeta(-\alpha k, x) + \frac{1}{\alpha k+1}$ (with $(x,z) \mapsto \zeta(z,x)$ the Hurwitz zeta function) we get

$$R_{e^{-tx^\alpha}}(x) = \sum_{k=0}^{+\infty}\frac{(-1)^k t^k}{k!}(\zeta(-\alpha k, x) + \frac{1}{\alpha k+1})$$

Thus

$$\sum_{n=1}^{+\infty} e^{-tn^\alpha} = \sum_{k=0}^{+\infty} \frac{(-1)^k t^k}{k!}\zeta(-\alpha k) + \sum_{k=0}^{+\infty} \frac{(-1)^k t^k}{k!}\frac{1}{\alpha k+1} + \int_1^{+\infty} e^{-tx^\alpha}dx$$

and we get

$$\sum_{n=1}^{+\infty} e^{-tn^\alpha} = \sum_{k=0}^{+\infty} \frac{(-1)^k t^k}{k!}\zeta(-\alpha k) + \int_0^{+\infty} e^{-tx^\alpha}dx = \sum_{k=0}^{+\infty} \frac{(-1)^k t^k}{k!}\zeta(-\alpha k) + \frac{1}{\alpha}\Gamma(\frac{1}{\alpha})t^{-\frac{1}{\alpha}}$$

Note that this formula is not valid for $\alpha = 2$ since it gives

$$\sum_{n=1}^{+\infty} e^{-tn^2} = \zeta(0) + \frac{1}{2}\Gamma(\frac{1}{2})t^{-\frac{1}{2}}$$

but the true formula is known to involves exponentially small terms when $t \to 0$ (cf. Bellman)

$$\sum_{n=1}^{+\infty} e^{-tn^2} = \zeta(0) + \frac{1}{2}\Gamma(\frac{1}{2})t^{-\frac{1}{2}} + \sqrt{\pi}t^{-\frac{1}{2}}\sum_{n=1}^{+\infty} e^{-\pi^2 n^2/t}$$

2.4 A surprising relation

Let f of moderate growth then for $0 < z < \pi$ then we prove the "surprising relation"

$$\boxed{\sum_{n\geq 1}^{\mathcal{R}} e^{-nz}\varphi_f(n) = \frac{1}{1-e^{-z}}\sum_{n\geq 1}^{\mathcal{R}} e^{-nz}f(n) - \frac{e^{-z}}{z}\sum_{n\geq 1}^{\mathcal{R}} f(n) - e^{-z}\sum_{n\geq 1}^{\mathcal{R}} e^{nz}\int_1^n e^{-zt}f(t)dt}$$

Proof

By the shift property we can write for $|z| < \pi$

$$\begin{aligned}\sum_{n\geq 1}^{\mathcal{R}} e^{-nz}\varphi_f(n) &= e^z\sum_{n\geq 1}^{\mathcal{R}} e^{-(n+1)z}\varphi_f(n+1) - e^z\sum_{n\geq 1} e^{-(n+1)z}f(n+1)\\ &= e^z\sum_{n\geq 1}^{\mathcal{R}} e^{-nz}\varphi_f(n) - \varphi_f(1) + e^z\int_1^2 e^{-zx}\varphi_f(x)dx\\ &\quad -e^z\sum_{n\geq 1}^{\mathcal{R}} e^{-nz}f(n) + f(1) - e^z\int_1^2 e^{-zx}f(x)dx\end{aligned}$$

Thus

$$\begin{aligned}\sum_{n\geq 1}^{\mathcal{R}} e^{-nz}\varphi_f(n) &= \frac{e^z}{1-e^z}\int_1^2 e^{-zx}(\varphi_f(x) - f(x))dx - \frac{e^z}{1-e^z}\sum_{n\geq 1}^{\mathcal{R}} e^{-nz}f(n)\\ &= \frac{e^z}{1-e^z}\int_1^2 e^{-zx}(\sum_{n\geq 1}^{\mathcal{R}} f(n) - R_f(x))dx - \frac{e^z}{1-e^z}\sum_{n\geq 1}^{\mathcal{R}} e^{-nz}f(n)\end{aligned}$$

We get

$$\sum_{n\geq 1}^{\mathcal{R}} e^{-nz}\varphi_f(n) = -\frac{e^{-z}}{z}\sum_{n\geq 1}^{\mathcal{R}} f(n) - \frac{e^z}{1-e^z}\sum_{n\geq 1}^{\mathcal{R}} e^{-nz}f(n) - \frac{e^z}{1-e^z}\int_1^2 e^{-zx}R_f(x)dx$$

It remains to prove that

$$\sum_{n\geq 1}^{\mathcal{R}} e^{nz}\int_1^n e^{-zt}f(t)dt = \frac{e^{2z}}{1-e^z}\int_1^2 e^{-zt}R_f(t)dt$$

Let G the function defined by

$$G(x,z)=e^{zx}\int_1^x e^{-zt}f(t)dt$$

We can evaluate $\sum_{n\geq 1}^{\mathcal{R}} G(n,z)$ by observing that the function G is the solution of the differential equation

$$\partial_x G - zG = f \text{ with } G(1)=0$$

The condition $G(1)=0$ gives $R_{\partial_x G}=\partial_x R_G$ thus the function R_G is the solution of the differential equation

$$\partial_x R_G - zR_G = R_f$$

This gives

$$R_G(x,z)=Ke^{zx}+e^{zx}\int_1^x e^{-zt}R_f(t)dt$$

With the condition $\int_1^2 R_G(x)dx=0$ and integration by parts we get gives

$$K=-\frac{e^z}{e^z-1}\int_1^2 e^{-zt}R_f(t)dt$$

thus $R_G(1,z)=Ke^z$ gives

$$\sum_{n\geq 1}^{\mathcal{R}} e^{nz}\int_1^n e^{-zt}f(t)dt=\frac{e^{2z}}{1-e^z}\int_1^2 e^{-zt}R_f(t)dt$$

□

Remark
Let $F_0=f$ and for $k\geq 1$ let $F_k(x)=\int_1^x F_{k-1}(t)dt$ we have

$$\sum_{k\geq 1}F_k(x)z^{k-1}=\sum_{k\geq 0}z^k\int_1^x\frac{(x-t)^k}{k!}f(t)dt=e^{zx}\int_1^x e^{-zt}f(t)dt$$

thus

$$\sum_{n\geq 1}^{\mathcal{R}} e^{nz}\int_1^n e^{-zt}f(t)dt=\sum_{k\geq 1}(\sum_{n\geq 1}^{\mathcal{R}}F_k(n))z^{k-1}$$

and we the "surprising relation" is simply

$$\boxed{\sum_{n\geq 1}^{\mathcal{R}} e^{-nz}\varphi_f(n)=\frac{1}{1-e^{-z}}\sum_{n\geq 1}^{\mathcal{R}} e^{-nz}f(n)-\frac{e^{-z}}{z}\sum_{k\geq 0}(\sum_{n\geq 1}^{\mathcal{R}}F_k(n))z^k}$$

that gives a relation between the sums $\sum_{n\geq 1}^{\mathcal{R}} n^k\varphi_f(n)$, $\sum_{n\geq 1}^{\mathcal{R}} n^k f(n)$, and $C_k=\sum_{n\geq 1}^{\mathcal{R}}F_k(n)$:

$$\begin{aligned}
\sum_{n\geq 1}^{\mathcal{R}}\varphi_f(n) &= \frac{3}{2}\sum_{n\geq 1}^{\mathcal{R}}f(n)-\sum_{n\geq 1}^{\mathcal{R}}nf(n)-C_1\\
\sum_{n\geq 1}^{\mathcal{R}}n\varphi_f(n) &= \frac{5}{12}\sum_{n\geq 1}^{\mathcal{R}}f(n)+\frac{1}{2}\sum_{n\geq 1}^{\mathcal{R}}nf(n)-\frac{1}{2}\sum_{n\geq 1}^{\mathcal{R}}n^2f(n)-C_1+C_2\\
\sum_{n\geq 1}^{\mathcal{R}}n^2\varphi_f(n) &= \frac{1}{3}\sum_{n\geq 1}^{\mathcal{R}}f(n)-\frac{1}{6}\sum_{n\geq 1}^{\mathcal{R}}nf(n)+\frac{1}{2}\sum_{n\geq 1}^{\mathcal{R}}n^2f(n)-\frac{1}{3}\sum_{n\geq 1}^{\mathcal{R}}n^3f(n)-C_1+2C_2-2C_3\\
&\dots
\end{aligned}$$

Example

For $f(x) = 1/x$ we have $F_1(x) = Log(x)$, $F_2(x) = xLog(x) - x + 1$, $F_3(x) = \frac{x^2}{2}Log(x) - \frac{3}{4}x^2 + x - \frac{1}{4}$ and we get the sums $\sum_{n\geq k}^{\mathcal{R}} n^k H_n$ in terms of the constants γ and $\zeta'(-j)$

$$\begin{aligned}
\sum_{n\geq 1}^{\mathcal{R}} H_n &= \frac{3}{2}\gamma + \frac{1}{2} - Log(\sqrt{2\pi}) \\
\sum_{n\geq 1}^{\mathcal{R}} nH_n &= \frac{5}{12}\gamma + \frac{7}{8} - Log(\sqrt{2\pi}) - \zeta'(-1) \\
\sum_{n\geq 1}^{\mathcal{R}} n^2 H_n &= \frac{1}{3}\gamma + \frac{17}{34} - Log(\sqrt{2\pi}) - 2\zeta'(-1) + \zeta'(-2)
\end{aligned}$$

More generally we have (Cf. Candelpergher, Gadiyar, Padma)

$$\boxed{\sum_{m\geq 1}^{\mathcal{R}} m^p H_m = \frac{1 - B_{p+1}}{p+1}\gamma + \sum_{k=1}^{p}(-1)^k C_p^k \zeta'(-k) - Log(\sqrt{2\pi}) + r_p \text{ with } r_p \in \mathbb{Q}}$$

2.5 The case of a Laplace transform

Theorem 7 *Let $\hat{f}$ is a continuous function on $[0, +\infty[$ such that*

$$|\hat{f}(\xi)| \leq Ce^{a\xi} \text{ with } a < 1$$

and f his Laplace transform defined for $Re(x) > a$ by

$$f(x) = \int_0^{+\infty} e^{-x\xi}\ \hat{f}(\xi)d\xi$$

Then f is analytic for $Re(x) > a$ and is of moderate growth, we have

$$\sum_{n\geq 1}^{\mathcal{R}} f(n) = R_f(1) = \int_0^{+\infty} e^{-\xi}\left(\frac{1}{1 - e^{-\xi}} - \frac{1}{\xi}\right) \hat{f}(\xi)d\xi\,.$$

Proof

The function f is analytic for $\mathrm{Re}(x) > a$ and $|f(x)| \leq C\frac{1}{\mathrm{Re}(x) - a}$, thus f is of moderate growth. Now we have

$$R_f(x) = -\int_1^x f(t)dt + \int_0^{+\infty} e^{-x\xi}\left(\frac{1}{1 - e^{-\xi}} - \frac{1}{\xi}\right) \hat{f}(\xi)d\xi$$

To prove this assertion write

$$R_f(x) - R_f(x+1) = \int_x^{x+1} f(t)dt + \int_0^{+\infty} e^{-x\xi}(1 - e^{-\xi})\left(\frac{1}{1 - e^{-\xi}} - \frac{1}{\xi}\right) \hat{f}(\xi)d\xi$$

thus

$$R_f(x) - R_f(x+1) = f(x) + \int_x^{x+1} f(t)dt - \int_0^{+\infty} e^{-x\xi}\frac{1 - e^{-\xi}}{\xi}\hat{f}(\xi)d\xi$$

but we have by Fubini's theorem

$$\int_x^{x+1} f(t)dt = \int_0^{+\infty}\int_x^{x+1} e^{-t\xi}\ \hat{f}(\xi)dt\ d\xi = \int_0^{+\infty} e^{-x\xi}\frac{1 - e^{-\xi}}{\xi}\hat{f}(\xi)d\xi$$

Thus $R_f(x) - R_f(x+1) = f(x)$ and we can verify that $\int_1^2 R_f(x)dx = 0$, then

$$\sum_{n\geq 1}^{\mathcal{R}} f(n) = R_f(1) = \int_0^{+\infty} e^{-\xi}\left(\frac{1}{1 - e^{-\xi}} - \frac{1}{\xi}\right) \hat{f}(\xi)d\xi\,.$$

□

Example
Let $f(x)=\frac{1}{x^k}$ for $k>0$ then $f(x)=\int_0^{+\infty}e^{-x\xi}\frac{\xi^{k-1}}{(k-1)!}d\xi$ and we get for $k\neq 1$

$$\zeta(k)-\frac{1}{k-1}=\sum_{n\geq 1}^{\mathcal{R}}\frac{1}{n^k}=\int_0^{+\infty}e^{-\xi}(\frac{1}{1-e^{-\xi}}-\frac{1}{\xi})\frac{\xi^{k-1}}{(k-1)!}d\xi\,.$$

and

$$\gamma=\sum_{n\geq 1}^{\mathcal{R}}\frac{1}{n}=\int_0^{+\infty}e^{-\xi}(\frac{1}{1-e^{-\xi}}-\frac{1}{\xi})\,d\xi\,.$$

Theorem 8 *Let $\sum_{k\geq 1}c_kx^k$ a power series with radius $\rho>1$ and let $f(x)=\sum_{k=1}^{+\infty}c_k\frac{1}{x^k}$ then the function f is analytic in $\{Re(x)>1/\rho\}$ and is the Laplace transform of the entire function*

$$\hat{f}(\xi)=\sum_{k=1}^{+\infty}c_k\frac{\xi^{k-1}}{(k-1)!}$$

We have

$$\sum_{n\geq 1}^{\mathcal{R}}f(n)=c_1\gamma+\sum_{k=2}^{+\infty}c_k(\zeta(k)-\frac{1}{k-1})$$

Proof
By hypothesis we have $|c_k|\leq Mr^k$ with $0<r=\frac{1}{\rho-\varepsilon}<1$. The series $\sum_{k\geq 1}c_k\frac{\xi^{k-1}}{(k-1)!}$ is convergent for all $\xi\in\mathbb{C}$ and the function

$$\hat{f}(\xi)=\sum_{k=1}^{+\infty}c_k\frac{\xi^{k-1}}{(k-1)!}$$

is an entire function with

$$|\hat{f}(\xi)|\leq Ce^{r|\xi|}$$

For $\mathrm{Re}(x)>r$ we have

$$\int_0^{+\infty}e^{-x\xi}\ \hat{f}(\xi)d\xi=\int_0^{+\infty}\sum_{k=1}^{+\infty}c_ke^{-x\xi}\frac{\xi^{k-1}}{(k-1)!}d\xi=\sum_{k=1}^{+\infty}c_k\int_0^{+\infty}e^{-x\xi}\frac{\xi^{k-1}}{(k-1)!}d\xi=\sum_{k=1}^{+\infty}c_k\frac{1}{x^k}=f(x)$$

the permutation is justified by

$$\int_0^{+\infty}\sum_{k=1}^{+\infty}|c_k|e^{-\mathrm{Re}(x)\xi}\frac{\xi^{k-1}}{(k-1)!}d\xi\leq M\int_0^{+\infty}\sum_{k=1}^{+\infty}r^ke^{-\mathrm{Re}(x)\xi}\frac{\xi^{k-1}}{(k-1)!}\leq Mr\int_0^{+\infty}e^{-(\mathrm{Re}(x)-r)\xi}d\xi<+\infty$$

Thus by the preceding theorem

$$\sum_{n\geq 1}^{\mathcal{R}}f(n)=\int_0^{+\infty}e^{-\xi}(\frac{1}{1-e^{-\xi}}-\frac{1}{\xi})\,\hat{f}(\xi)d\xi=\int_0^{+\infty}e^{-\xi}(\frac{1}{1-e^{-\xi}}-\frac{1}{\xi})\sum_{k\geq 1}c_k\frac{\xi^{k-1}}{(k-1)!}d\xi.$$

interchanging $\int$ et $\sum$ we get

$$\sum_{n\geq 1}^{\mathcal{R}}f(n)=\sum_{k=1}^{\infty}c_k\int_0^{+\infty}e^{-\xi}(\frac{1}{1-e^{-\xi}}-\frac{1}{\xi})\frac{\xi^{k-1}}{(k-1)!}d\xi\,.$$

and by the preceding example

$$\sum_{n\geq 1}^{\mathcal{R}}f(n)=c_1\gamma+\sum_{k=2}^{+\infty}c_k(\zeta(k)-\frac{1}{k-1})$$

□

Remark
Since

$$\sum_{n\geq 1}^{\mathcal{R}} \frac{1}{n} = \gamma \text{ and } \sum_{n\geq 1}^{\mathcal{R}} \frac{1}{n^k} = \zeta(k) - \frac{1}{k-1} \text{ for } k \neq 1$$

the preceding theorem can be stated in the form of the result of a permutation of signs $\sum_{n\geq 1}^{\mathcal{R}}$ and $\sum_{k=1}^{+\infty}$

$$\boxed{\sum_{n\geq 1}^{\mathcal{R}} \sum_{k=1}^{+\infty} c_k \frac{1}{n^k} = \sum_{k=1}^{+\infty} c_k \sum_{n\geq 1}^{\mathcal{R}} \frac{1}{n^k}}$$

Examples
1) We have

$$\sum_{k=2}^{+\infty} \frac{(-1)^k}{k} \sum_{n\geq 1}^{\mathcal{R}} \frac{1}{n^k} = \sum_{n\geq 1}^{\mathcal{R}} \sum_{k=2}^{+\infty} \frac{(-1)^k}{k} \frac{1}{n^k}$$

But

$$\sum_{k=2}^{+\infty} \frac{(-1)^k}{k} \frac{1}{n^k} = -Log(1+\frac{1}{n}) + \frac{1}{n} = Log(n) - Log(n+1) + \frac{1}{n}$$

Thus

$$\sum_{n\geq 1}^{\mathcal{R}} \sum_{k=2}^{+\infty} \frac{(-1)^k}{k} \frac{1}{n^k} = \sum_{n\geq 1}^{\mathcal{R}} (Log(n) - Log(n+1)) + \gamma = -2Log(2) + 1 + \gamma$$

this gives

$$\sum_{k=2}^{+\infty} \frac{(-1)^k}{k} (\zeta(k) - \frac{1}{k-1}) = -2Log(2) + 1 + \gamma$$

With

$$\sum_{k=2}^{+\infty} \frac{(-1)^k}{k} \frac{1}{k-1} = \sum_{k=2}^{+\infty} \frac{(-1)^k}{k-1} - \sum_{k=2}^{+\infty} \frac{(-1)^k}{k} = 2Log(2) - 1$$

we get finally

$$\boxed{\sum_{k=2}^{+\infty} \frac{(-1)^k}{k} \zeta(k) = \gamma}$$

2) Let $z \in \mathbb{C}$ we have

$$\sum_{n\geq 1}^{\mathcal{R}} \frac{e^{-z/n}}{n} = \sum_{n\geq 1}^{\mathcal{R}} \sum_{k=0}^{+\infty} \frac{(-1)^k z^k}{k!} \frac{1}{n^{k+1}} = \sum_{k=0}^{+\infty} \frac{(-1)^k z^k}{k!} \sum_{n\geq 1}^{\mathcal{R}} \frac{1}{n^{k+1}}$$

thus

$$\boxed{\sum_{n\geq 1}^{\mathcal{R}} \frac{e^{-z/n}}{n} = \gamma + \sum_{k\geq 1} \frac{(-1)^k}{k!} (\zeta(k+1) - \frac{1}{k}) z^k \qquad (2.15)}$$

2.6 Analyticity with respect to a parameter

2.6.1 The theorem of analyticity

It is well known that the simple convergence of a series $\sum_{n\geq 1} f(n,z)$ of functions $z \mapsto f(n,z)$ analytic in a domain U does not imply that the sum $\sum_{n\geq 1}^{+\infty} f(n,z)$ is analytic in U. A very important property of Ramanujan summation is that analyticity of the terms imply analyticity of the sum. We have an illustration of this fact with

$$\sum_{n\geq 1}^{\mathcal{R}} \frac{1}{n^z} = \zeta(z) - \frac{1}{z-1}$$

where we see that the pole of *zeta* is removed.

Definition
Let $(x,z) \mapsto f(x,z)$ a function defined for $Re(x) > 0$ and $z \in U \subset \mathbb{C}$ such that $z \mapsto f(x,z)$ is analytic for $z \in U$. We say that f is *locally uniformly in* $\mathcal{O}^{\pi}$ if
a) for all $z \in U$ the function $x \mapsto f(x,z)$ is analytic for $Re(x) > 0$
b) for any K compact of U there exist $\alpha < \pi$ and $C > 0$ such that for $Re(x) > 0$ and $z \in K$

$$|f(x,z)| \leq Ce^{\alpha|x|}$$

By Cauchy formula there is the same type of inequality for the derivatives $\partial_z^k f$ thus $\partial_z^k f$ is locally uniformly in $\mathcal{O}^{\pi}$.

Theorem 9 ***Analyticity of*** $z \mapsto \sum_{n\geq 1}^{\mathcal{R}} f(n,z)$
Let $(x,z) \mapsto f(x,z)$ a function defined for $Re(x) > 0$ and $z \in U \subset \mathbb{C}$ such that $z \mapsto f(x,z)$ is analytic for $z \in U$ and f is locally uniformly in $\mathcal{O}^{\pi}$. Then the function

$$z \mapsto \sum_{n\geq 1}^{\mathcal{R}} f(n,z)$$

is analytic in U and

$$\partial_z^k \sum_{n\geq 1}^{\mathcal{R}} f(n,z) = \sum_{n\geq 1}^{\mathcal{R}} \partial_z^k f(n,z)$$

Thus if $z_0 \in U$ and

$$f(n,z) = \sum_{k=0}^{+\infty} a_k(n)(z-z_0)^k \text{ for } |z-z_0| < \rho \text{ and } n \geq 1$$

then

$$\sum_{n\geq 1}^{\mathcal{R}} f(n,z) = \sum_{k=0}^{+\infty} [\sum_{n\geq 1}^{\mathcal{R}} a_k(n)](z-z_0)^k$$

Proof
Let $z \in K \subset U$ then we have

$$\sum_{n\geq 1}^{\mathcal{R}} f(n,z) = \frac{f(1,z)}{2} + i\int_0^{+\infty} \frac{f(1+it,z) - f(1-it,z)}{e^{2\pi t}-1} dt$$

The function

$$h : z \mapsto \frac{f(1+it,z) - f(1-it,z)}{e^{2\pi t}-1}$$

is analytic in U for all $t \in]0,+\infty[$ and if $z \in K$

$$|\frac{f(1+it,z) - f(1-it,z)}{e^{2\pi t}-1}| \leq \frac{Ae^{\alpha t}}{e^{2\pi t}-1}$$

then by the analyticity theorem of an integral depending on a parameter we get the analyticity of h in U and for $t > 1$ we have

$$\partial^k h(z) = \int_0^{+\infty} \frac{\partial_z^k f(1+it,z) - \partial_z^k f(1-it,z)}{e^{2\pi t}-1} dt$$

Thus the function $z \mapsto \sum_{n\geq 1}^{\mathcal{R}} f(n,z)$ is analytic in U and

$$\partial_z^k \sum_{n\geq 1}^{\mathcal{R}} f(n,z) = \sum_{n\geq 1}^{\mathcal{R}} \partial_z^k f(n,z)$$

For $z_0 \in U$ let

$$f(n,z) = \sum_{k=0}^{+\infty} a_k(n)(z-z_0)^k \text{ for } |z-z_0| < \rho \text{ and } n \geq 1$$

then $a_k(n) = \frac{1}{k!}\partial_z^k f(n,z_0)$ and

$$\sum_{n\geq 1}^{\mathcal{R}} f(n,z) = \sum_{k=0}^{+\infty} \frac{1}{k!}\partial_z^k \sum_{n\geq 1}^{\mathcal{R}} f(n,z_0)(z-z_0)^k = \sum_{k=0}^{+\infty}\sum_{n\geq 1}^{\mathcal{R}} \frac{1}{k!}\partial_z^k f(n,z_0)(z-z_0)^k = \sum_{k=0}^{+\infty}[\sum_{n\geq 1}^{\mathcal{R}} a_k(n)](z-z_0)^k$$

□

Examples

1) Let $f(x,z) = \frac{1}{z+x}$ and $U = \{|z| < 1\}$ we have for $z \in U$ and $x \geq 1$

$$f(x,z) = \frac{1}{x}\frac{1}{1+\frac{z}{x}} = \frac{1}{x} + \sum_{k=1}^{+\infty} \frac{(-1)^k}{x^{k+1}} z^k$$

thus

$$\sum_{n\geq 1}^{\mathcal{R}} \frac{1}{z+n} = \gamma + \sum_{k=1}^{+\infty} (-1)^k (\zeta(k+1) - \frac{1}{k}) z^k$$

By $\sum_{n\geq 1}^{\mathcal{R}} \frac{1}{z+n} = -\Psi(z+1) + Log(z+1)$ we get

$$\boxed{\Psi(z+1) = -\gamma + \sum_{k=1}^{+\infty} (-1)^{k-1}\zeta(k+1) z^k}$$

Integrating we get

$$Log(\Gamma(z+1)) = -\gamma z + \sum_{k=2}^{+\infty} (-1)^k \zeta(k)\frac{z^k}{k}$$

Note that on the other hand we have formally

$$\sum_{n\geq 1}^{\mathcal{R}} \frac{1}{n+z} = \frac{1}{z}\sum_{n\geq 1}^{\mathcal{R}} \frac{1}{1+\frac{n}{z}} = \sum_{n\geq 1}^{\mathcal{R}}\sum_{k\geq 0} (-1)^k \frac{n^k}{z^{k+1}}$$

thus

$$\begin{aligned}\sum_{n\geq 1}^{\mathcal{R}} \frac{1}{n+z} &= \frac{1}{2z} + \sum_{k\geq 1} (-1)^k \frac{1}{z^{k+1}}(\frac{1-B_{k+1}}{k+1}) \\ &= -\frac{1}{2z} + Log(1+\frac{1}{z}) + \sum_{k\geq 1} \frac{B_{2k}}{2k}\frac{1}{z^{2k}}\end{aligned}$$

This gives the asymptotic expansion

$$\Psi(z+1) = Log(z) + \frac{1}{2z} - \sum_{k\geq 1} \frac{B_{2k}}{2k}\frac{1}{z^{2k}}$$

the last series is divergent but the sum can be defined by the Borel summation procedure.

2) Let the function $f(x,s) = \frac{1}{x^s}$ the preceding theorem the function $s \mapsto \sum_{n\geq 1}^{\mathcal{R}} \frac{1}{n^s}$ is an entire function. We have seen in chapter 1 that $\sum_{n\geq 1}^{\mathcal{R}} \frac{1}{n^s} = \zeta(s) - \frac{1}{s-1}$ for $Re(s) > 1$ thus by analytic continuation we get

$$\sum_{n\geq 1}^{\mathcal{R}} \frac{1}{n^s} = \zeta(s) - \frac{1}{s-1} \text{ for } s \neq 1$$

If $s = -k$ with k integer ≥ 1 we get

$$\zeta(-k) + \frac{1}{k+1} = \sum_{n\geq 1}^{\mathcal{R}} n^k = \frac{1 - B_{k+1}}{k+1} \text{ if } k \geq 1$$

thus

$$\zeta(-k) = -\frac{B_{k+1}}{k+1} \text{ if } k \geq 1$$

for $k = 0$ we get $\zeta(0) = -1 + \sum_{n\geq 1}^{\mathcal{R}} 1 = -\frac{1}{2}$.
By derivation we get

$$\partial \sum_{n\geq 1}^{\mathcal{R}} \frac{1}{n^s} = -\sum_{n\geq 1}^{\mathcal{R}} \frac{Log(n)}{n^s}$$

this gives

$$\boxed{\sum_{n\geq 1}^{\mathcal{R}} \frac{Log(n)}{n^s} = -\zeta'(s) - \frac{1}{(s-1)^2} \text{ for } s \neq 1}$$

Thus for example $\zeta'(0) = -1 - \sum_{n\geq 1}^{\mathcal{R}} Log(n) = -Log(\sqrt{2\pi})$ and $\sum_{n\geq 1}^{\mathcal{R}} nLog(n) = -\zeta'(-1) - \frac{1}{4}$ but we have seen that $\sum_{n\geq 1}^{\mathcal{R}} nLog(n) = Log(A) - \frac{1}{3}$ where A is the Glaisher-Kinkelin constant, thus we get

$$-\zeta'(-1) - \frac{1}{4} = Log(A) - \frac{1}{3}$$

More generally

$$\boxed{\sum_{n\geq 1}^{\mathcal{R}} \frac{Log^k(n)}{n^s} = (-1)^k \partial^k \zeta(s) - \frac{k!}{(s-1)^{k+1}} \text{ for } s \neq 1.}$$

For $s = 1$ we have the sums $\sum_{n\geq 1}^{\mathcal{R}} \frac{Log^k(n)}{n}$ which are related to the *Stieltjes constants* γ_k defined by the Laurent expansion of ζ at 1

$$\zeta(s+1) = \frac{1}{s} + \sum_{n\geq 0} \frac{(-1)^k}{k!}\gamma_k s^k$$

We have

$$\sum_{n\geq 1}^{\mathcal{R}} \frac{1}{n^{s+1}} = \zeta(s+1) - \frac{1}{s}$$

and the expansion

$$\frac{1}{n^{s+1}} = \sum_{k\geq 0} \frac{(-1)^k}{k!} s^k \frac{Log^k(n)}{n}$$

gives

$$\sum_{n\geq 1}^{\mathcal{R}} \frac{1}{n^{s+1}} = \sum_{k\geq 0} \frac{(-1)^k}{k!} s^k \sum_{n\geq 1}^{\mathcal{R}} \frac{Log^k(n)}{n}$$

thus

$$\gamma_k = \sum_{n\geq 1}^{\mathcal{R}} \frac{Log^k(n)}{n} \tag{2.16}$$

3) With $f(x,z) = Log^2(z+x)$ we get

$$\sum_{n\geq 1}^{\mathcal{R}} Log^2(z+n) = \sum_{n\geq 1}^{\mathcal{R}} Log^2(n) + 2\sum_{n\geq 1}^{\mathcal{R}} Log(n)Log(1+\frac{z}{n}) + \sum_{n\geq 1}^{\mathcal{R}} Log^2(1+\frac{z}{n})$$

We have

$$\sum_{n\geq 1}^{\mathcal{R}} Log(n)Log(1+\frac{z}{n}) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} z^k \sum_{n\geq 1}^{\mathcal{R}} \frac{Log(n)}{n^k} = \gamma_1 z + \sum_{k=2}^{\infty} \frac{(-1)^k}{k}(\zeta'(k) + \frac{1}{(k-1)^2})z^k$$

and

$$\sum_{n\geq 1}^{\mathcal{R}} Log^2(1+\frac{z}{n}) = \sum_{k=2}^{\infty} (-1)^k(\zeta(k) - \frac{1}{k-1})\sigma_k z^k \text{ with } \sigma_k = \sum_{i=1}^{k-1} \frac{1}{i(k-i)} = \frac{2}{k} H_{k-1}$$

By the shift property we have if z is an integer

$$\sum_{n\geq 1}^{\mathcal{R}} Log^2(z+n) = \sum_{n\geq 1}^{\mathcal{R}} Log^2(n) - \sum_{k=1}^{z} Log^2(k) + (z+1)Log^2(z+1) - 2(z+1)Log(z+1) + 2z$$

The function $\varphi_{Log^2 x}(z) = \sum_{k=1}^{z} Log^2(k)$ is introduced by Ramanujan in his notebooks (chapter 8 entry 18). Since

$$(z+1)Log^2(z+1) - 2(z+1)Log(z+1) + 2z = 2\sum_{k=2}^{\infty} \frac{(-1)^k}{k} \frac{1}{(k-1)^2} z^k - \sum_{k=2}^{\infty} (-1)^k \frac{1}{k-1} \frac{2}{k} H_{k-1} z^k$$

we get for z integer

$$\varphi_{Log^2 x}(z) = -2\gamma_1 z - 2\sum_{k=2}^{\infty} \frac{(-1)^k}{k} \zeta'(k) z^k - 2\sum_{k=2}^{\infty} \frac{(-1)^k}{k} \zeta(k) H_{k-1} z^k \tag{2.17}$$

Since $\partial Log^2 x = 2\frac{Log(x)}{x}$ and using the relation

$$\varphi_{\partial f}(x) = \partial\varphi_f(x) + \sum_{n\geq 1}^{\mathcal{R}} \partial f(n) - f(1)$$

we get

$$\varphi_{\frac{Log(x)}{x}}(z) = -\gamma_1 - \sum_{k=2}^{\infty} (-1)^k \zeta'(k) z^{k-1} - \sum_{k=2}^{\infty} (-1)^k \zeta(k) H_k z^{k-1} + \sum_{n\geq 1}^{\mathcal{R}} \frac{Log(n)}{n}$$

thus

$$\varphi_{\frac{Log(x)}{x}}(z) = \sum_{k=1}^{\infty} (-1)^k \zeta'(k+1) z^k + \sum_{k=1}^{\infty} (-1)^k \zeta(k+1) H_k z^k$$

4) Let $f \in \mathcal{O}^0$ and $0 < z < \pi$, since we are in a case of convergence for the series $\sum_{n\geq 1} e^{-zn} f(n)$ we have

$$\sum_{n\geq 1}^{\mathcal{R}} e^{-zn} f(n) = \sum_{n=1}^{+\infty} e^{-zn} f(n) - \int_1^{+\infty} e^{-zx} f(x)dx$$

Let $F(x) = \int_1^x f(t)dt$, if $e^{-zx}F(x) = O(\frac{1}{x^\alpha})$ with $\alpha > 1$, we have by integration by parts

$$\int_1^{+\infty} e^{-zx} f(x)dx = z \int_1^{+\infty} e^{-zx} F(x)dx$$

With

$$\sum_{n=1}^{+\infty} e^{-zn} F(n) = \sum_{n\geq 1}^{\mathcal{R}} e^{-zn} F(n) + \int_1^{+\infty} e^{-zx} F(x)dx$$

we get

$$\sum_{n=1}^{+\infty} e^{-zn} F(n) = \sum_{n\geq 1}^{\mathcal{R}} e^{-zn} F(n) + \frac{1}{z} \int_1^{+\infty} e^{-zx} f(x)dx$$

If $f(x) = \frac{1}{x}$ ans $0 < z < \pi$ then

$$\begin{aligned} \sum_{n=1}^{+\infty} e^{-zn} Log(n) &= \sum_{n\geq 1}^{\mathcal{R}} e^{-zn} Log(n) + \frac{1}{z} \int_1^{+\infty} \frac{e^{-zx}}{x} dx \\ &= \sum_{n\geq 1}^{\mathcal{R}} e^{-zn} Log(n) - \sum_{k=1}^{+\infty} \frac{(-1)^k}{k!} \frac{z^{k-1}}{k} - \frac{\gamma + Log(z)}{z} \end{aligned}$$

But we have by the theorem of analyticity

$$\sum_{n\geq 1}^{\mathcal{R}} e^{-zn} Log(n) = \sum_{k=0}^{+\infty} \frac{(-1)^k}{k!} z^k \sum_{n\geq 1}^{\mathcal{R}} n^k Log(n) = \sum_{k=0}^{+\infty} \frac{(-1)^{k-1}}{k!} z^k (\zeta'(-k) + \frac{1}{(k+1)^2})$$

thus

$$\boxed{\sum_{n=1}^{+\infty} e^{-zn} Log(n) = \sum_{k=0}^{+\infty} \frac{(-1)^{k-1}}{k!} z^k \zeta'(-k) - \frac{\gamma + Log(z)}{z}}$$

Corollary
As a consequence of the preceding theorem and

$$R_{f(x,z)}(x) = \sum_{n\geq 1}^{\mathcal{R}} f(n+x, z) + f(x,z) - \int_1^{x+1} f(t,z)dt$$

$$\varphi_{f(x,z)}(x) = \sum_{n\geq 1}^{\mathcal{R}} f(n,z) - \sum_{n\geq 1}^{\mathcal{R}} f(n+x,z) + \int_1^{x+1} f(u,z)du$$

we have with the same hypothesis as in the preceding theorem the analyticity of these functions of z and by derivation with respect to z we get

$$\partial_z R_{f(x,z)}(x) = R_{\partial_z f(x,z)}(x)$$

$$\partial_z \varphi_{f(x,z)}(x) = \varphi_{\partial_z f(x,z)}(x)$$

Example If $f(x,z)=\frac{1}{x^z}$ with $z\neq 1$ then

$$\varphi_{\frac{Log(x)}{x^z}}=-\partial_z\varphi_{\frac{1}{x^z}}=-\zeta'(z)+\partial_z\zeta(z,x)-\partial_z\frac{1}{x^z}$$

we get

$$\varphi_{\frac{Log(x)}{x^z}}=-\zeta'(z)+\partial_z\zeta(z,x)+\frac{Log(x)}{x^z}$$

For $z=0$ then

$$\varphi_{Log}(x)=-\zeta'(0)+\partial_z\zeta(0,x)+Log(x)$$

but we know that $\varphi_{Log}(x)=Log(\Gamma(x+1))$ thus we get the Lerch formula (c.f Berndt)

$$Log(\Gamma(x))=-\zeta'(0)+\partial_z\zeta(0,x)$$

2.6.2 Analytic continuation of Dirichlet series

Let $x\mapsto c(x)$ a function analytic for $\text{Re}(x)>0$ such that we have the asymptotic expansion at infinity

$$c(x)=\sum_{k\geq 0}\alpha_k\frac{1}{x^{j_k}}$$

where $\text{Re}(j_0)<\text{Re}(j_1)<\text{Re}(j_2)<...<\text{Re}(j_k)<...$ Then let

$$h(s)=\sum_{n\geq 1}\frac{c(n)}{n^s}$$

this function is analytic for $\text{Re}(s)>1-\text{Re}(j_0)$ and we have

$$h(s)=\sum_{n\geq 1}^{\mathcal{R}}\frac{c(n)}{n^s}+\int_1^{+\infty}c(x)x^{-s}dx$$

The function $s\mapsto\sum_{n\geq 1}^{\mathcal{R}}\frac{c(n)}{n^s}$ is an entire function, thus the singularities of the funtion h are given by the integral term

$$\int_1^{+\infty}c(x)x^{-s}dx=\int_1^{+\infty}\Big(\sum_{k\geq 0}^{N-1}\alpha_k x^{-s-j_k}+O(x^{-s-j_N})\Big)dx=\sum_{k\geq 0}^{N-1}\alpha_k\frac{1}{s+j_k-1}+R_N(s)$$

Thus we get simple poles for the function h at the points $s=1-j_k$ with residues α_k.

Examples
1) Let $h(s)=\sum_{n=1}^{+\infty}\frac{1}{(n+1)n^s}$ for $\text{Re}(s)>1$ then

$$\sum_{n=1}^{+\infty}\frac{1}{(n+1)n^s}=\sum_{n\geq 1}^{\mathcal{R}}\frac{1}{(n+1)n^s}+\int_1^{+\infty}\frac{1}{(x+1)x^s}dx$$

The function $s\mapsto\sum_{n\geq 1}^{\mathcal{R}}\frac{1}{(n+1)n^s}$ is an entire function, thus the singularities of the (analytic continuation of the) function h are the singularities of the function $s\mapsto\int_1^{+\infty}\frac{1}{(x+1)x^s}dx$. Since for $x>1$ we have

$$\frac{1}{(x+1)x^s}=\sum_{k=0}^{+\infty}\frac{(-1)^k}{x^{s+k+1}}$$

we get by the dominated convergence

$$\int_1^{+\infty}\frac{1}{(x+1)x^s}dx=\sum_{k=0}^{+\infty}(-1)^k\int_1^{+\infty}x^{-s-k-1}dx=\sum_{k=0}^{+\infty}\frac{(-1)^k}{s+k}$$

Thus the function h has simple poles at $s = -k, k = 0, 1, 2, ...$ and

$$h(s) = \frac{(-1)^k}{s+k} + \sum_{n\geq 1}^{\mathcal{R}} \frac{1}{(n+1)n^s} + \sum_{j\neq k}^{+\infty} \frac{(-1)^j}{s+j}$$

We observe that

$$\lim_{s=-k} (h(s) - \frac{(-1)^k}{s+k}) = \sum_{n\geq 1}^{\mathcal{R}} \frac{n^k}{n+1} + \sum_{j\neq k}^{+\infty} \frac{(-1)^j}{j-k}$$

2) Let $c(x) = \Psi(x+1) + \gamma$ then $\Psi(n) = H_n$ and for $\mathrm{Re}(s) > 1$ let $h(s) = \sum_{n=1}^{+\infty} \frac{H_n}{n^s}$.
We have for $\mathrm{Re}(s) > 1$

$$h(s) = \sum_{n\geq 1}^{\mathcal{R}} \frac{H_n}{n^s} + \int_1^{+\infty} (\Psi(x+1) + \gamma) x^{-s} dx$$

The asymptotic expansion at infinity

$$\Psi(x+1) + \gamma = Log(x) + \gamma + \frac{1}{2x} - \sum_{k\geq 1} \frac{B_{2k}}{2k} \frac{1}{x^{2k}}$$

gives

$$h(s) = \sum_{n\geq 1}^{\mathcal{R}} \frac{H_n}{n^s} + \int_1^{+\infty} Log(x) x^{-s} dx + \frac{\gamma}{s-1} + \frac{1}{2s} - \sum_{k\geq 1}^{N} \frac{B_{2k}}{2k} \frac{1}{s+2k-1} + R_N(s)$$

The integral term

$$\int_1^{+\infty} Log(x) x^{-s} dx = \frac{1}{(s-1)^2}$$

gives for h a pole of order 2 at s=1 with residue γ. We have a simple pole at $s = 0$ with residue 1/2 and simple poles at $s = 1 - 2k$ with residues $-\frac{B_{2k}}{2k}$.

2.6.3 The zeta function associated to Laplacian on the sphere S^2

Let $A = -\Delta$ the Laplacian on the sphere S^2 the zeta function associated to this operator is (cf. Birmingham and Sen)

$$\zeta_A(s) = \sum_{n=1}^{+\infty} \frac{2n+1}{n^s(n+1)^s}$$

we have

$$\zeta_A(s) = \sum_{n\geq 1}^{\mathcal{R}} \frac{2n+1}{n^s(n+1)^s} + \int_1^{+\infty} \frac{2x+1}{x^s(x+1)^s} dx$$

The function $s \mapsto \sum_{n\geq 1}^{\mathcal{R}} \frac{2n+1}{n^s(n+1)^s}$ is an entire function, thus the singularities of ζ_A are the singularities of the function $s \mapsto \int_1^{+\infty} \frac{2x+1}{x^s(x+1)^s} dx$. Since for $x > 1$ we have

$$\frac{1}{(x+1)^s x^s} = \sum_{k=0}^{+\infty} \frac{(-1)^k}{k!} s(s+1)...(s+k-1) x^{-2s-k}$$

we get by the dominated convergence

$$\int_1^{+\infty} \frac{2x+1}{x^s(x+1)^s} dx = \frac{1}{s-1} + \sum_{k=0}^{+\infty} \frac{(-1)^{k+1}}{(k+1)!} s(s+1)...(s+k-1)$$

Thus the function ζ_A has only a simple pole at $s = 1$. By Mellin inversion we have with $c > 1$

$$Tr(e^{-At}) = \frac{1}{2i\pi} \int_{c-i\infty}^{c+i\infty} \zeta_A(s) \Gamma(s) t^{-s} ds$$

and by the residues theorem we get the asymptotic expansion

$$Tr(e^{-At}) \sim \frac{1}{t} + \sum_{p\geq 0} \frac{(-1)^p}{p!}\zeta_A(-p)t^p$$

The evaluation of $\zeta_A(-p)$ is easily done by the Ramanujan summation

$$\zeta_A(-p) = \sum_{n\geq 1}^{\mathcal{R}} (2n+1)n^p(n+1)^p - \frac{1}{p+1} + \sum_{k=0}^{p} \frac{(-1)^{k+1}}{(k+1)!}(-p)(-p+1)...(-p+k-1)$$

We find for example

$$\zeta_A(0) = \sum_{n\geq 1}^{\mathcal{R}} (2n+1) - 2 = -\frac{2}{3}$$

$$\zeta_A(-1) = \sum_{n\geq 1}^{\mathcal{R}} (2n+1)n(n+1) - 2 = -\frac{1}{15}$$

We see that Ramanujan summation gives a simple alternative way to the way of Mellin summation technique proposed by Birmingham an Sen.

2.6.4 Zeta regularization of divergent products

Let a function $x \mapsto a(x)$ such that $x \mapsto Log(a(x))$ is a function of moderate growth, we can use the Ramanujan summation to give the following definition of the *Ramanujan product*

$$\prod_{n\geq 1}^{\mathcal{R}} a(n) = e^{\sum_{n\geq 1}^{\mathcal{R}} Log(a(n))}$$

Thus we get for example

$$\prod_{n\geq 1}^{\mathcal{R}} n = e^{\sum_{n\geq 1}^{\mathcal{R}} Log(n)} = e^{Log(\sqrt{2\pi})-1} = \frac{\sqrt{2\pi}}{e}$$

Note that with this definition of the Ramanujan product we have for any positive constant C

$$\prod_{n\geq 1}^{\mathcal{R}} (a(n)C) = e^{\sum_{n\geq 1}^{\mathcal{R}} Log(a(n))+log(C)} = e^{\sum_{n\geq 1}^{\mathcal{R}} Log(a(n))} e^{\sum_{n\geq 1}^{\mathcal{R}} Log(C)} = \Big(\prod_{n\geq 1}^{\mathcal{R}} a(n)\Big) e^{\frac{1}{2}Log(C)}$$

thus we get the strange relation

$$\prod_{n\geq 1}^{\mathcal{R}} (a(n)C) = \sqrt{C} \prod_{n\geq 1}^{\mathcal{R}} a(n)$$

There is a well-known procedure to define infinite divergent products which avoid such strange property. That is the *zeta-regularization* of divergents products (cf. Quine Heydari Song), defined by

$$\prod_{n\geq 1}^{reg} a(n) = e^{-Z_a'(0)} \text{ with } Z_a \text{ defined analytic continuation of } s \mapsto \sum_{n\geq 1}^{+\infty} \frac{1}{(a(n))^s}$$

it is assumed that $s \mapsto \sum_{n\geq 1}^{+\infty} \frac{1}{(a(n))^s}$ is defined for $\mathrm{Re}(s) > \alpha$ and Z_a is defined near 0 by analytic continuation.

There is a simple relation of this Ramaujan product with the *zeta-regularization* of divergents products.
For $\mathrm{Re}(s) > \alpha$ we have

$$\sum_{n\geq 1}^{+\infty} \frac{1}{(a(n))^s} = \sum_{n\geq 1}^{\mathcal{R}} \frac{1}{(a(n))^s} + \int_1^{+\infty} \frac{1}{(a(x))^s} dx$$

The function $\mapsto \sum_{n\geq 1}^{\mathcal{R}} \frac{1}{(a(n))^s}$ is an entire function, we assume that the function

$$Y_a : s \mapsto \int_1^{+\infty} \frac{1}{(a(x))^s} dx$$

has an analytic continuation near 0. Thus

$$Z_a'(s) = -\sum_{n\geq 1}^{\mathcal{R}} \frac{Log(a(n))}{(a(n))^s} + Y_a'(s)$$

and we get

$$\prod_{n\geq 1}^{reg} a(n) = e^{-Z_a'(0)} = e^{\sum_{n\geq 1}^{\mathcal{R}} Log(a(n))} e^{-Y_a'(0)} = e^{-Y_a'(0)} \prod_{n\geq 1}^{\mathcal{R}} a(n)$$

Thus we can use Ramanujan summation to evaluate zeta-regularized products.

Example
We have for $\mathrm{Re}(z) > 0$

$$\prod_{n\geq 1}^{\mathcal{R}} (n+z) = e^{\sum_{n\geq 1}^{\mathcal{R}} Log(n+z)} = \frac{\sqrt{2\pi}}{\Gamma(z+1)} (x+1)^{x+1} e^{-(x+1)}$$

and for $\mathrm{Re}(s) > 1$

$$Y(s) = \int_1^{+\infty} \frac{1}{(x+z)^s} dx = \frac{(z+1)^{-s+1}}{s-1}$$

thus

$$Y'(0) = (z+1)Log(z+1) - (z+1)$$

and we get

$$\prod_{n\geq 1}^{reg} (n+z) = e^{-Y'(0)} \prod_{n\geq 1}^{\mathcal{R}} (n+z) = \frac{\sqrt{2\pi}}{\Gamma(z+1)}$$

2.7 Integration with respect to a parameter

2.7.1 Interchanging $\sum_{n\geq 1}^{\mathcal{R}}$ and $\int_I$

Theorem 10 *Let $(x,u) \mapsto f(x,u)$ defined for $Re(x) > 0$ and $u \in I$ where I is an interval $I \subset \mathbb{R}$. We suppose that*
a) for all $Re(x) > 0$ the function $u \mapsto f(x,u)$ is integrable on I
b) f is in $\mathcal{O}^\pi$ uniformly in the parameter $u \in I$ in an interval $I \subset \mathbb{R}$:
there is $\alpha < \pi$ such that

$$|f(x,u)| \leq Ce^{\alpha|x|} \text{ for all } Re(x) > 0 \text{ and all } u \in I$$

Then

$$\int_I \sum_{n\geq 1}^{\mathcal{R}} f(n,u)du = \sum_{n\geq 1}^{\mathcal{R}} \int_I f(n,u)du$$

Proof
We have

$$\sum_{n\geq 1}^{\mathcal{R}} f(n,u) = \frac{f(1,u)}{2} + i\int_0^{+\infty} \frac{f(1+it,u) - f(1-it,u)}{e^{2\pi t}-1} dt$$

It suffices to prove that

$$\int_I (\int_0^{+\infty} \frac{f(1+it,u) - f(1-it,u)}{e^{2\pi t}-1} dt) du = \int_0^{+\infty} \frac{\int_I f(1+it,u)du - \int_I f(1-it,u)du}{e^{2\pi t}-1} dt$$

This is a consequence of the Fubini theorem since

$$\frac{|f(1+it,u)-f(1-it,u)|}{e^{2\pi t}-1} \leq A\frac{e^{\alpha t}}{e^{2\pi t}-1}$$

□

Examples

1) We use the equation

$$\sum_{n\geq 1}^{\mathcal{R}} \cos(\pi n t) = \frac{\sin(\pi t)}{\pi t} - \frac{1}{2} \text{ for } t \in [0,1[.$$

and

$$\int_0^{1/2} (t-\frac{1}{2})^2 \cos(\pi n t)dt = -\frac{2\sin\frac{1}{2}\pi n}{\pi^3 n^3} + \frac{\pi n}{\pi^3 n^3}$$

Thus we have

$$\sum_{n\geq 1}^{\mathcal{R}} \int_0^{1/2} (t-\frac{1}{2})^2 \cos(\pi n t)dt = -2\sum_{n\geq 1}^{\mathcal{R}} \frac{\sin\frac{1}{2}\pi n}{\pi^3 n^3} + \sum_{n\geq 1}^{\mathcal{R}} \frac{1}{\pi^2 n^2}$$

Since we are in a case of convergence

$$\sum_{n\geq 1}^{\mathcal{R}} \frac{\sin\frac{1}{2}\pi n}{\pi^3 n^3} = \sum_{n\geq 1}^{+\infty} \frac{\sin\frac{1}{2}\pi n}{\pi^3 n^3} - \int_1^{+\infty} \frac{\sin\frac{1}{2}\pi x}{\pi^3 x^3}dx = \frac{1}{\pi^3}\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^3} - \int_1^{+\infty} \frac{\sin\frac{1}{2}\pi x}{\pi^3 x^3}dx$$

By the preceding theorem

$$\sum_{n\geq 1}^{\mathcal{R}} \int_0^{1/2} (t-\frac{1}{2})^2 \cos(\pi n t)dt = \int_0^{1/2} (t-\frac{1}{2})^2 \sum_{n\geq 1}^{\mathcal{R}} \cos(\pi n t)dt = \int_0^{1/2} (t-\frac{1}{2})^2 (\frac{\sin(\pi t)}{\pi t} - \frac{1}{2})dt$$

Integrating by part we have

$$\int_1^{+\infty} \frac{\sin\frac{1}{2}\pi x}{\pi^3 x^3}dx = \frac{1}{2\pi^3} - \frac{1}{8}\int_1^{+\infty} \frac{\sin\frac{1}{2}\pi x}{\pi x}dx$$

Thus we get

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^3} = \frac{1}{32}\pi^3$$

2) We use the equation

$$\sum_{n\geq 1}^{\mathcal{R}} \sin(\pi n t) = -\frac{\cos(\pi t)}{\pi t} + \frac{1}{2}\cot(\frac{\pi t}{2}) \text{ for } t \in [0,1[.$$

and

$$\int_0^{1/2} t\sin(\pi n t)dt = -\frac{\cos\frac{1}{2}\pi n}{2\pi n} + \frac{\sin(\frac{1}{2}\pi n)}{\pi^2 n^2}$$

Thus

$$\sum_{n\geq 1}^{\mathcal{R}} \int_0^{1/2} t\sin(\pi n t)dt = \sum_{n\geq 1}^{\mathcal{R}} (-\frac{\cos\frac{1}{2}\pi n}{2\pi n} + \frac{\sin(\frac{1}{2}\pi n)}{\pi^2 n^2}) = \frac{1}{4\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} + \int_1^{+\infty} \frac{\cos\frac{1}{2}\pi x}{2\pi x}dx$$

$$+\frac{1}{\pi^2}\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^2} - \int_1^{+\infty} \frac{\sin\frac{1}{2}\pi x}{\pi^2 x^2}dx$$

By the preceding theorem this is

$$\int_0^{1/2}\sum_{n\geq 1}^{\mathcal{R}} t\sin(\pi nt)dt=\int_0^{1/2}(-\frac{\cos(\pi t)}{\pi}+\frac{1}{2}t\cot(\frac{\pi t}{2}))dt$$

Finally after an integration by parts we get

$$\int_0^{\pi/2} x\cot(\frac{x}{2})dx=2G+\frac{1}{2}\pi Log(2)$$

where $G=\sum_{n=1}^{+\infty}\frac{(-1)^{n-1}}{(2n-1)^2}$ is the Catalan's constant.

And by same type of calculation

$$\int_0^{\pi/2} x^2\cot(\frac{x}{2})dx=2\pi G+\frac{1}{4}\pi^2 Log(2)-\frac{35}{8}\zeta(3)$$

3) By the shift property we have for $x>0$ and $x\neq 1$

$$\sum_{n\geq 1}^{\mathcal{R}}\frac{1}{(n+1)^x}=\zeta(x)-1-\frac{2^{-(x-1)}}{x-1}$$

if $x=1$ then this formula is extended analytically by $\sum_{n\geq 1}^{\mathcal{R}}\frac{1}{n+1}=\gamma-1+Log(2)$. Since

$$\int_0^{+\infty}\sum_{n\geq 1}^{\mathcal{R}}\frac{1}{(n+1)^x}dx=\sum_{n\geq 1}^{\mathcal{R}}\int_0^{+\infty} e^{-xLog(n+1)}dx=\sum_{n\geq 1}^{\mathcal{R}}\frac{1}{Log(n+1)}$$

We get

$$\sum_{n\geq 1}^{\mathcal{R}}\frac{1}{Log(n+1)}=\int_0^{+\infty}\left[\zeta(x)-1-\frac{2^{-(x-1)}}{x-1}\right]dx$$

2.7.2 The functional equation for zeta

By the formula

$$\sum_{n\geq 1}^{\mathcal{R}} f(n)=\frac{f(1)}{2}+i\int_0^{+\infty}\frac{f(1+it)-f(1-it)}{e^{2\pi t}-1}dt$$

we get with $x>0$ and $f(u)=1/(u-1+x)$

$$\sum_{n\geq 1}^{\mathcal{R}}\frac{1}{n-1+x}=\frac{1}{2x}+i\int_0^{+\infty}\frac{1/(x+it)-1/(x-it)}{e^{2\pi t}-1}dt$$

and with the shift property we get

$$\sum_{n\geq 1}^{\mathcal{R}}\frac{1}{n+x}=Log(1+\frac{1}{x})-\frac{1}{2x}+2\int_0^{+\infty}\frac{t/(x^2+t^2)}{e^{2\pi t}-1}dt$$

this gives

$$\boxed{\sum_{n\geq 1}^{\mathcal{R}}\frac{1}{n+x}=Log(1+\frac{1}{x})+2\int_0^{+\infty}(\frac{1}{e^{2\pi t}-1}-\frac{1}{2\pi t})\frac{t}{x^2+t^2}dt}$$

Taking the Mellin transform we get for $0 < s < 1$

a) for the left side

$$\int_0^{+\infty} x^{s-1} \sum_{n\geq 1}^{\mathcal{R}} \frac{1}{n+x}\, dx = \sum_{n\geq 1}^{\mathcal{R}} \int_0^{+\infty} x^{s-1} \frac{1}{n+x}\, dx = \sum_{n\geq 1}^{\mathcal{R}} n^{s-1} \int_0^{+\infty} x^{s-1} \frac{1}{1+x}\, dx = \frac{\pi}{\sin \pi s} \sum_{n\geq 1}^{\mathcal{R}} n^{s-1}$$

b) for the right side

$$\int_0^{+\infty} x^{s-1} Log(1+\frac{1}{x}) + 2\int_0^{+\infty} x^{s-1} \int_0^{+\infty} (\frac{1}{e^{2\pi t}-1} - \frac{1}{2\pi t}) \frac{t}{x^2+t^2} dt\ dx$$

since

$$\int_0^{+\infty} x^{s-1} Log(1+\frac{1}{x}) = \frac{\pi}{\sin \pi s} \frac{1}{s}$$

and

$$\begin{aligned} \int_0^{+\infty} x^{s-1} \int_0^{+\infty} (\frac{1}{e^{2\pi t}-1} - \frac{1}{2\pi t}) \frac{t}{x^2+t^2} dt\ dx &= \int_0^{+\infty} (\frac{1}{e^{2\pi t}-1} - \frac{1}{2\pi t}) \int_0^{+\infty} x^{s-1} \frac{t}{x^2+t^2} dx\ dt \\ &= \int_0^{+\infty} (\frac{1}{e^{2\pi t}-1} - \frac{1}{2\pi t}) t^{s-1} \frac{\pi/2}{\sin \pi s/2} \\ &= (2\pi)^{-s} \Gamma(s) \zeta(s) \frac{\pi/2}{\sin \pi s/2} \end{aligned}$$

Thus we get

$$\frac{\pi}{\sin \pi s} \sum_{n\geq 1}^{\mathcal{R}} n^{s-1} = 2(2\pi)^{-s} \Gamma(s)\zeta(s) \frac{\pi/2}{\sin \pi s/2} + \frac{\pi}{\sin \pi s}$$

since $\sum_{n\geq 1}^{\mathcal{R}} n^{s-1} - \frac{1}{s} = \zeta(1-s)$ we get

$$\frac{\pi}{\sin \pi s} \zeta(1-s) = 2(2\pi)^{-s} \Gamma(s)\zeta(s) \frac{\pi/2}{\sin(\pi s/2)}$$

this is the Riemann functional equation.

2.7.3 The Muntz formula

Let a function f integrable on $[0,+\infty[$ such that $f(x) = O(\frac{1}{|x|^\alpha})$ with $\alpha > 1$, then for $0 <\mathrm{Re}(s) < 1$. Then by interchanging $\int_I \sum_{n\geq 1}^{\mathcal{R}} = \sum_{n\geq 1}^{\mathcal{R}} \int_I$ we get

$$\int_0^{+\infty} x^{s-1} \sum_{n\geq 1}^{\mathcal{R}} f(nx) dx = \sum_{n\geq 1}^{\mathcal{R}} \int_0^{+\infty} x^{s-1} f(nx) dx$$

thus

$$\int_0^{+\infty} x^{s-1} \sum_{n\geq 1}^{\mathcal{R}} f(nx) dx = (\sum_{n\geq 1}^{\mathcal{R}} \frac{1}{n^s}) \int_0^{+\infty} x^{s-1} f(x) dx$$

Then since we are in a case of convergence we have

$$\sum_{n\geq 1}^{\mathcal{R}} f(nx) = \sum_{n\geq 1}^{+\infty} f(nx) - \frac{1}{x} \int_x^{+\infty} f(t) dt$$

thus

$$\sum_{n\geq 1}^{\mathcal{R}} f(nx) - \frac{1}{x} \int_0^{x} f(t) dt = \sum_{n\geq 1}^{+\infty} f(nx) - \frac{1}{x} \int_0^{+\infty} f(t) dt$$

We get

$$\int_0^{+\infty} x^{s-1}\big(\sum_{n\geq 1}^{+\infty} f(nx) - \frac{1}{x}\int_0^{+\infty} f(t)dt\big) = \big(\sum_{n\geq 1}^{\mathcal{R}} \frac{1}{n^s}\big)\int_0^{+\infty} x^{s-1}f(x)dx - \int_0^{+\infty} x^{s-1}\big(\frac{1}{x}\int_0^x f(t)dt\big)dx$$

For 0 <Re(s) < 1 we get by integration by parts

$$\int_0^{+\infty} x^{s-1}\big(\frac{1}{x}\int_0^x f(t)dt\big)dx = -\frac{1}{s-1}\int_0^{+\infty} x^{s-1}f(x)dx$$

Since $\sum_{n\geq 1}^{\mathcal{R}} \frac{1}{n^s} = \zeta(s) - \frac{1}{s-1}$ we obtain the Muntz formula (cf. E.C.Titchmarsh and D.R. Heath-Brown p.28)

$$\int_0^{+\infty} x^{s-1}\big(\sum_{n\geq 1}^{+\infty} f(nx) - \frac{1}{x}\int_0^{+\infty} f(t)dt\big)\ dx = \zeta(s)\int_0^{+\infty} x^{s-1}f(x)dx$$

Remark

Let a function f such that $(x,t)\mapsto x^\alpha f(xt)$ satisfies the hypothesis of the preceding theorem with $I=]0,1[$ then

$$\int_0^1 \sum_{n\geq 1}^{\mathcal{R}} n^\alpha f(nt)dt = \sum_{n\geq 1}^{\mathcal{R}}\int_0^1 n^\alpha f(nt)dt = \sum_{n\geq 1}^{\mathcal{R}} n^{\alpha-1}\int_0^n f(u)du$$

thus if $F_0(x) = \int_0^x f(t)dt$ then

$$\sum_{n\geq 1}^{\mathcal{R}} n^{\alpha-1}F_0(n) = \int_0^1 \sum_{n\geq 1}^{\mathcal{R}} n^\alpha f(nt)dt$$

With $\alpha = 0$ and $f(x) = \frac{1}{1+x}$ we get

$$\sum_{n\geq 1}^{\mathcal{R}} \frac{Log(n+1)}{n} = \int_0^1 \sum_{n\geq 1}^{\mathcal{R}} \frac{1}{1+nt}dt = \int_0^1 \frac{1}{t}\sum_{n\geq 1}^{\mathcal{R}} \frac{1}{n+\frac{1}{t}}dt = \int_1^{+\infty} \frac{Log(1+u)-\Psi(1+u)}{u}du$$

With $\alpha = 1$ and $f(x) = \frac{1}{1+x^2}$ we get

$$\sum_{n\geq 1}^{\mathcal{R}} Arctg(n) = \int_0^1 \sum_{n\geq 1}^{\mathcal{R}} \frac{n}{1+n^2t^2}dt = \int_0^1 \frac{1}{t^2}\sum_{n\geq 1}^{\mathcal{R}} \frac{n}{n^2+\frac{1}{t^2}}dt$$

since

$$\sum_{n\geq 1}^{\mathcal{R}} \frac{n}{n^2+a^2} = \frac{1}{2}\big(\sum_{n\geq 1}^{\mathcal{R}} \frac{1}{n+ia} + \sum_{n\geq 1}^{\mathcal{R}} \frac{1}{n-ia}\big) = Log(\sqrt{1+a^2}) - \frac{1}{2}(\Psi(1+ia)+\Psi(1-ia))$$

we get

$$\sum_{n\geq 1}^{\mathcal{R}} Arctg(n) = \int_1^{+\infty} Log(\sqrt{1+u^2}) - \frac{1}{2}(\Psi(1+iu)+\Psi(1-iu))\ du$$

Note that the same calculations with $I=]0,+\infty[$ gives

$$\int_0^{+\infty} \sum_{n\geq 1}^{\mathcal{R}} n^\alpha f(nt)dt = \big[\sum_{n\geq 1}^{\mathcal{R}} n^{\alpha-1}\big]\int_0^{+\infty} f(t)dt$$

2.8 Heat equation

Let $x=(x_1,x_2)\in\mathbb{R}^2$ and $t\geq 0$, the function

$$U(x,t)=\sum_{n\geq 1}^{\mathcal{R}}\frac{1}{n+t}e^{-\frac{|x|^2}{4(n+t)}}$$

is solution of heat equation

$$\partial_t U=\partial^2_{x_1x_1}U+\partial^2_{x_2x_2}U$$

Let

$$f(z)=\sum_{n\geq 1}^{\mathcal{R}}\frac{1}{n}e^{-\frac{z}{n}}=\gamma+\sum_{k=1}^{\infty}(-1)^k(\zeta(k+1)-\frac{1}{k})\frac{z^k}{k!}$$

then

$$f(\frac{|x|^2}{4})=\sum_{n\geq 1}\frac{1}{n}e^{-\frac{|x|^2}{4n}}=U(x,0)$$

With the heat kernel we get

$$\begin{aligned} U(x,t) &= \int_{\mathbb{R}^2}\frac{1}{4\pi t}e^{-\frac{|x-y|^2}{4t}}U(y,0)dy \\ &= e^{-\frac{|x|^2}{4t}}\frac{1}{4\pi t}\int_{\mathbb{R}^2}e^{\frac{x.y}{2t}}e^{-\frac{|y|^2}{4t}}f(\frac{|y|^2}{4})dy \end{aligned}$$

Let $x=(r,0)$ with polar coordinates we get

$$U(x,t)=e^{-\frac{|x|^2}{4t}}\frac{1}{4\pi t}\int_0^{+\infty}\int_0^{2\pi}e^{\frac{1}{2t}r\rho\cos(\theta)}\rho e^{-\frac{\rho^2}{4t}}f(\frac{\rho^2}{4})d\rho d\theta$$

We have for $t>0$

$$g(r\rho/2t)=\int_0^{2\pi}e^{\frac{1}{2t}r\rho\cos(\theta)}d\theta=2\pi I_0(r\rho/2t)$$

where I_0 is the Bessel function $I_0(z)=\sum_{k\geq 0}\frac{1}{(k!)^2}(z/2)^{2k}$
Thus

$$U(x,t)=e^{-\frac{r^2}{4t}}\frac{1}{4\pi t}2\pi\int_0^{+\infty}I_0(r\rho/2t)\rho e^{-\frac{\rho^2}{4t}}f(\frac{\rho^2}{4})d\rho$$

this gives

$$\sum_{n\geq 1}^{\mathcal{R}}\frac{1}{(n+t)}e^{-\frac{r^2}{4(n+t)}}=e^{-\frac{r^2}{4t}}\frac{1}{2t}\int_0^{+\infty}I_0(r\rho/2t)\rho e^{-\frac{\rho^2}{4t}}(\sum_{n\geq 1}^{\mathcal{R}}\frac{1}{n}e^{-\frac{\rho^2}{4n}})d\rho$$

With $z=\frac{r^2}{4}$ and $u=\frac{\rho^2}{4}$ we get

$$\sum_{n\geq 1}^{\mathcal{R}}\frac{1}{n+t}e^{-\frac{z}{n+t}}=e^{-z/t}\frac{1}{t}\int_0^{+\infty}I_0(2\sqrt{zu}/t)e^{-\frac{u}{t}}(\sum_{n\geq 1}^{\mathcal{R}}\frac{1}{n}e^{-\frac{u}{n}})du$$

If t is a positive integer we have the shift property

$$\sum_{n\geq 1}^{\mathcal{R}}\frac{1}{n+t}e^{-\frac{z}{n+t}} = \sum_{n\geq 1}^{\mathcal{R}}\frac{1}{n}e^{-\frac{z}{n}}-\sum_{n\geq 1}^{t}\frac{1}{n}e^{-\frac{z}{n}}+\int_1^{t+1}\frac{1}{x}e^{-z/x}dx$$

with $t=1$ we get the integral equation

$$f(z) = e^{-z} - \int_1^2 \frac{1}{x} e^{-z/x} dx + e^{-z} \int_0^{+\infty} I_0(2\sqrt{zu}) e^{-u} f(u) du \tag{2.18}$$

Note that

$$f(z) = \sum_{n\geq 1}^{\mathcal{R}} \frac{e^{-\frac{z}{n}} - 1}{n} + \gamma = \gamma + \sum_{n\geq 1}^{+\infty} \frac{e^{-\frac{z}{n}} - 1}{n} - \int_1^{+\infty} \frac{e^{-\frac{z}{t}} - 1}{t} dt$$

Let

$$g(z) = \gamma + \sum_{n\geq 1}^{+\infty} \frac{e^{-\frac{z}{n}} - 1}{n}$$

this function is the exponential generating function of the zeta values

$$g(z) = \gamma + \sum_{k\geq 1} \zeta(k+1)(-1)^k \frac{z^k}{k!}$$

Since $f(z) = g(z) - \int_1^{+\infty} \frac{e^{-\frac{z}{t}}-1}{t} dt$ it is easy to prove that the integral equation (2.19) gives now a simpler integral equation for this generating function g that is

$$g(z) = e^{-z} + e^{-z} \int_0^{\infty} e^{-u} I_0(2\sqrt{zu}) g(u) du$$

2.9 Link with Borel summation

2.9.1 Ramanujan summation in term of Bernoulli numbers

Let a function f given by the Borel sum

$$f(x) = \sum_{k\geq 0}^{\mathcal{B}} \alpha_k x^k$$

that is for $\mathrm{Re}(x) > 0$

$$f(x) = \int_0^{+\infty} e^{-\xi} (\sum_{k\geq 0} \alpha_k x^k \frac{\xi^k}{k!}) d\xi$$

We assume that the function $(x,\xi) \mapsto \sum_{k\geq 0} \alpha_k x^k \frac{\xi^k}{k!}$ *satisfies the hypothesis of theorem 9.*
Then we have

$$\begin{aligned} \sum_{n\geq 1}^{\mathcal{R}} f(n) &= \sum_{n\geq 1}^{\mathcal{R}} \int_0^{+\infty} e^{-\xi} \sum_{k\geq 0} \alpha_k n^k \frac{\xi^k}{k!} d\xi \\ &= \int_0^{+\infty} e^{-\xi} \sum_{n\geq 1}^{\mathcal{R}} \sum_{k\geq 0} \alpha_k n^k \frac{\xi^k}{k!} d\xi \end{aligned}$$

Now *we assume that*

$$\sum_{n\geq 1}^{\mathcal{R}} \sum_{k\geq 0} \alpha_k n^k \frac{\xi^k}{k!} = \sum_{k\geq 0} \alpha_k \frac{\xi^k}{k!} \sum_{n\geq 1}^{\mathcal{R}} n^k$$

then

$$\begin{aligned} \sum_{n\geq 1}^{\mathcal{R}} f(n) &= \int_0^{+\infty} e^{-\xi} \sum_{k\geq 0} \alpha_k \sum_{n\geq 1}^{\mathcal{R}} n^k \frac{\xi^k}{k!} d\xi \\ &= \int_0^{+\infty} e^{-\xi} (\frac{\alpha_0}{2} + \sum_{k\geq 1} \alpha_k (\frac{1-B_{k+1}}{k+1}) \frac{\xi^k}{k!} d\xi \end{aligned}$$

thus we get by definition of Borel summation

$$\boxed{\sum_{n\geq 1}^{\mathcal{R}} f(n) = \frac{\alpha_0}{2} + \sum_{k\geq 1}^{\mathcal{B}} \alpha_k (\frac{1-B_{k+1}}{k+1})} \tag{2.19}$$

Note that this formula is nothing else than

$$\sum_{n\geq 1}^{\mathcal{R}} \sum_{k\geq 1}^{\mathcal{B}} \alpha_k n^k = \sum_{k\geq 1}^{\mathcal{B}} \sum_{n\geq 1}^{\mathcal{R}} \alpha_k n^k$$

which we recall is valid under the strong hypothesis: $\sum_{n\geq 1}^{\mathcal{R}} \sum_{k\geq 0} \alpha_k n^k \frac{\xi^k}{k!} = \sum_{k\geq 0} \alpha_k \frac{\xi^k}{k!} \sum_{n\geq 1}^{\mathcal{R}} n^k$.

Example

Let $f(x) = \frac{1}{1+x} = \sum_{k\geq 0}^{\mathcal{B}} (-1)^k x^k$ that is for $\mathrm{Re}(x) > 0$ we have $f(x) = \int_0^{+\infty} e^{-\xi} e^{-x\xi} d\xi$. We have

$$\sum_{n\geq 1}^{\mathcal{R}} \sum_{k\geq 0} (-1)^k n^k \frac{\xi^k}{k!} = \sum_{n\geq 1}^{\mathcal{R}} e^{-n\xi} = \frac{e^{-\xi}}{1-e^{-\xi}} + \frac{e^{-\xi}}{-\xi} = \frac{1}{2} + \sum_{k\geq 1} (-1)^k (\frac{1-B_{k+1}}{k+1}) \frac{\xi^k}{k!} = \sum_{k\geq 0} (-1)^k \frac{\xi^k}{k!} \sum_{n\geq 1}^{\mathcal{R}} n^k$$

thus

$$\sum_{n\geq 1}^{\mathcal{R}} \frac{1}{n+1} = \frac{1}{2} + \sum_{k\geq 1}^{\mathcal{B}} (-1)^k (\frac{1-B_{k+1}}{k+1})$$

but $\sum_{n\geq 1}^{\mathcal{R}} \frac{1}{n+1} = \gamma - 1 + Log(2)$ and $\sum_{k\geq 1}^{\mathcal{B}} \frac{(-1)^k}{k+1} = Log(2) - 1$ thus we get

$$\boxed{\gamma = \frac{1}{2} + \sum_{k\geq 1}^{\mathcal{B}} (-1)^{k-1} \frac{B_{k+1}}{k+1}}$$

More generally we have

$$\sum_{n\geq 1}^{\mathcal{R}} \frac{1}{n+z} = \frac{1}{z} \sum_{n\geq 1}^{\mathcal{R}} \frac{1}{1+\frac{n}{z}} = \sum_{n\geq 1}^{\mathcal{R}} \sum_{k\geq 0}^{\mathcal{B}} (-1)^k \frac{n^k}{z^{k+1}}$$

thus

$$\sum_{n\geq 1}^{\mathcal{R}} \frac{1}{n+z} = \frac{1}{2z} + \sum_{k\geq 1}^{\mathcal{B}} (-1)^k \frac{1}{z^{k+1}} (\frac{1-B_{k+1}}{k+1}) = -\frac{1}{2z} + Log(1+\frac{1}{z}) + \sum_{k\geq 1}^{\mathcal{B}} \frac{B_{2k}}{2k} \frac{1}{z^{2k}}$$

this gives by (2.4)

$$\Psi(z+1) = Log(z) + \frac{1}{2z} - \sum_{k\geq 1}^{\mathcal{B}} \frac{B_{2k}}{2k} \frac{1}{z^{2k}}$$

Note that we cannot make the same calculation for $\sum_{n\geq 1} \frac{1}{1+n^2}$. We have $\frac{1}{1+x^2} = \sum_{k\geq 0}^{\mathcal{B}} (-1)^k x^{2k}$ but $\sum_{n\geq 1}^{\mathcal{R}} \sum_{k\geq 0} (-1)^k n^{2k} \frac{\xi^k}{k!} = \sum_{n\geq 1}^{\mathcal{R}} e^{-n^2\xi}$ is not defined.

Remark

The formula (2.19) is not always valid.

Let $f(x) = \frac{xy}{e^{xy}-1}$ for $\mathrm{Re}(x) > 0$ and $y > 0$. We have $f(x) = \sum_{k\geq 0}^{\mathcal{B}} \frac{B_k}{k!} x^k y^k$.

thus if we apply (2.19) then

$$\sum_{n\geq 1}^{\mathcal{R}} \frac{ny}{e^{ny}-1} = \frac{B_0}{2} + \sum_{k\geq 1}^{\mathcal{B}} \frac{B_k}{k!} (\frac{1-B_{k+1}}{k+1}) y^k$$

Since $B_k B_{k+1} = 0$ if $k \geq 2$ we get

$$\sum_{n\geq 1}^{\mathcal{R}} \frac{ny}{e^{ny}-1} = \frac{1}{2} + \frac{1}{y}\sum_{k\geq 1}^{\mathcal{B}} \frac{B_k}{(k+1)!} y^{k+1} - \frac{B_1 B_2}{2} y$$

this gives

$$\sum_{n\geq 1}^{\mathcal{R}} \frac{ny}{e^{ny}-1} = \frac{1}{y}\int_0^y \frac{t}{e^t-1}dt - \frac{1}{2} - \frac{B_1 B_2}{2} y$$

Since we are in a case of convergence we get

$$\sum_{n\geq 1}^{+\infty} \frac{ny}{e^{ny}-1} = \frac{1}{y}\int_0^{+\infty} \frac{t}{e^t-1}dt - \frac{1}{2} - \frac{B_1 B_2}{2} y$$

that is

$$\sum_{n\geq 1}^{+\infty} \frac{ny}{e^{ny}-1} = \frac{\pi^2}{6y} - \frac{1}{2} + \frac{y}{24}$$

but unfortunately this formula is not true.

2.9.2 An integral formula

Let a function f given by the Borel sum $f(x) = \sum_{k\geq 0}^{\mathcal{B}} \alpha_k x^k$ that is for $\mathrm{Re}(x) > 0$

$$f(x) = \int_0^{+\infty} e^{-\xi}(\sum_{k\geq 0} \alpha_k x^k \frac{\xi^k}{k!})d\xi = \int_0^{+\infty} \frac{e^{-u/x}}{x}\widehat{f}(u)du$$

with

$$\widehat{f}(u) = \sum_{k\geq 0} \alpha_k \frac{u^k}{k!}$$

Then by interchange of $\int$ and $\sum_{n\geq 1}^{\mathcal{R}}$ we get

$$\sum_{n\geq 1}^{\mathcal{R}} f(n) \quad = \quad \int_0^{+\infty} \sum_{n\geq 1}^{\mathcal{R}} \frac{e^{-u/n}}{n}\widehat{f}(u)du$$

But we have

$$\sum_{n\geq 1}^{\mathcal{R}} \frac{e^{-u/n}}{n} = \gamma + \sum_{k\geq 1} \frac{(-1)^k}{k!}(\zeta(k+1) - \frac{1}{k})u^k$$

thus

$$\sum_{n\geq 1}^{\mathcal{R}} f(n) = \int_0^{+\infty} \left[\gamma + \sum_{k\geq 1} \frac{(-1)^k}{k!}(\zeta(k+1) - \frac{1}{k})u^k\right] \widehat{f}(u)du$$

Example

Let $f : x \mapsto \frac{1}{x+z}$ with $x > 0$ and $z > 0$ this function is given by the Borel sum $f(x) = \sum_{k\geq 0}^{\mathcal{B}} \frac{(-1)^k}{z^{k+1}} x^k$ then

$$f(x) = \int_0^{+\infty} \frac{e^{-u/x}}{x}\widehat{f}(u)du \text{ with } \widehat{f}(u) = \sum_{k\geq 0} \frac{(-1)^k}{z^{k+1}}\frac{u^k}{k!} = \frac{1}{z}e^{-u/z}$$

thus

$$\begin{aligned}\sum_{n\geq 1}^{\mathcal{R}} \frac{1}{n+z} &= \int_0^{+\infty} \left[\gamma + \sum_{k\geq 1} \frac{(-1)^k}{k!}(\zeta(k+1) - \frac{1}{k})u^k\right] \frac{1}{z} e^{-u/z} du \\ &= \gamma + \sum_{k\geq 1}^{\mathcal{B}} \left[(-1)^k(\zeta(k+1) - \frac{1}{k})\right] z^k \\ &= \gamma + \sum_{k\geq 1}^{\mathcal{B}} \left[(-1)^k\zeta(k+1)\right] z^k + Log(1+z)\end{aligned}$$

Since $\sum_{n\geq 1}^{\mathcal{R}} \frac{1}{z+n} = -\Psi(z+1) + Log(z+1)$ we get for $z > 0$

$$\Psi(z+1) = -\gamma + \sum_{k\geq 1}^{\mathcal{B}} \left[(-1)^{k+1}\zeta(k+1)\right] z^k$$

2.10 Double Ramanujan sums

2.10.1 Definitions and properties

We study iterate Ramanujan summation

$$\sum_{n\geq 1}^{\mathcal{R}} \sum_{m\geq 1}^{\mathcal{R}} f(m,n)$$

Theorem 11 *Let a function $(x,y) \mapsto f(x,y)$ analytic for $Re(x) > 0$ and $Re(y) > 0$ with*

$$x \mapsto f(x,y) \text{ in } \mathcal{O}^\pi \text{ for all } Re(y) > 0$$

$$y \mapsto f(x,y) \text{ in } \mathcal{O}^\pi \text{ for all } Re(x) > 0$$

If there exist a function W analytic for $Re(x) > 0$ and $Re(y) > 0$ such that

$$W(x,y) - W(x,y+1) - W(x+1,y) + W(x+1,y+1) = f(x,y)$$

with $x \longmapsto W(x,y)$ in $\mathcal{O}^\pi(P)$ for all $Re(y) > 0$ and $y \longmapsto W(x,y)$ in $\mathcal{O}^\pi(P)$ for all $Re(x) > 0$ then we have

$$\sum_{n\geq 1}^{\mathcal{R}} \sum_{m\geq 1}^{\mathcal{R}} f(m,n) = \sum_{m\geq 1}^{\mathcal{R}} \sum_{n\geq 1}^{\mathcal{R}} f(m,n)$$

Proof
Let $R(x,y) = W(x,y) - W(x,y+1)$ then

$$R(x,y) - R(x+1,y) = f(x,y)$$

thus

$$\sum_{m\geq 1}^{\mathcal{R}} f(m,y) = R(1,y) - (\int_1^2 W(x,y)dx - \int_1^2 W(x,y+1)dx)$$

For $x = 1$ the equation $W(1,y) - W(1,y+1) = R(1,y)$ gives

$$\sum_{n\geq 1}^{\mathcal{R}} R(1,n) = W(1,1) - \int_1^2 W(1,y)dy$$

thus

$$\begin{aligned}\sum_{n\geq 1}^{\mathcal{R}}\sum_{m\geq 1}^{\mathcal{R}} f(m,n) &= W(1,1)-\int_1^2 W(1,y)dy-\sum_{n\geq 1}^{\mathcal{R}}\int_1^2 W(x,n)dx+\sum_{n\geq 1}^{\mathcal{R}}\int_1^2 W(x,n+1)dx\\ &= W(1,1)-\int_1^2 W(1,y)dy-\int_1^2 W(x,1)dx+\int_1^2\int_1^2 W(x,y)dxdy\end{aligned}$$

Evaluation of $\sum_{m\geq 1}^{\mathcal{R}}\sum_{n\geq 1}^{\mathcal{R}} f(m,n)$
We have

$$W(x,y)-W(x+1,y)-(W(x,y+1)-W(x+1,y+1))=f(x,y)$$

$$\sum_{n\geq 1}^{\mathcal{R}} f(m,n)=W(m,1)-W(m+1,1)-(\int_1^2 W(m,y)dy-\int_1^2 W(m+1,y)dy)$$

thus

$$\begin{aligned}\sum_{m\geq 1}^{\mathcal{R}}\sum_{n\geq 1}^{\mathcal{R}} f(m,n) &= \sum_{m\geq 1}^{\mathcal{R}} W(m,1)-W(m+1,1)-\sum_{m\geq 1}^{\mathcal{R}}(\int_1^2 W(m,y)dy-\int_1^2 W(m+1,y)dy)\\ &= \sum_{m\geq 1}^{\mathcal{R}} W(m,1)-(\sum_{m\geq 1}^{\mathcal{R}} W(m,1)-W(1,1)+\int_1^2 W(x,1)dx)\\ &\quad -\int_1^2 W(1,y)dy+\int_1^2\int_1^2 W(x,y)dydx\\ &= W(1,1)-\int_1^2 W(x,1)dx-\int_1^2 W(1,y)dy+\int_1^2\int_1^2 W(x,y)dydx\end{aligned}$$

Conclusion

$$\begin{aligned}\sum_{n\geq 1}^{\mathcal{R}}\sum_{m\geq 1}^{\mathcal{R}} f(m,n) &= W(1,1)-\int_1^2 W(1,y)dy-\int_1^2 W(x,1)dx+\int_1^2\int_1^2 W(x,y)dxdy\\ \sum_{m\geq 1}^{\mathcal{R}}\sum_{n\geq 1}^{\mathcal{R}} f(m,n) &= W(1,1)-\int_1^2 W(x,1)dx-\int_1^2 W(1,y)dy+\int_1^2\int_1^2 W(x,y)dydx\end{aligned}$$

By Fubini's theorem we have

$$\int_1^2\int_1^2 W(x,y)dxdy=\int_1^2\int_1^2 W(x,y)dydx$$

donc $\sum_{n\geq 1}^{\mathcal{R}}\sum_{m\geq 1}^{\mathcal{R}} f(m,n)=\sum_{m\geq 1}^{\mathcal{R}}\sum_{n\geq 1}^{\mathcal{R}} f(m,n)$.
□

Examples
1) We have by the shift property

$$\begin{aligned}\sum_{m\geq 1}^{\mathcal{R}}\sum_{n\geq 1}^{\mathcal{R}}\frac{1}{n+m} &= \sum_{m\geq 1}^{\mathcal{R}}\gamma-\sum_{m\geq 1}^{\mathcal{R}} H_m+\sum_{m\geq 1}^{\mathcal{R}} Log(m+1)\\ &= \frac{1}{2}\gamma-\frac{3}{2}\gamma-\frac{1}{2}+Log(\sqrt{2\pi})+Log(\sqrt{2\pi})-1+2Log2-1\end{aligned}$$

This gives

$$\boxed{\sum_{m\geq 1}^{\mathcal{R}}\sum_{n\geq 1}^{\mathcal{R}}\frac{1}{n+m}=-\gamma+3Log(2)+Log(\pi)-\frac{5}{2}}$$

2) We have $\sum_{n=1}^{+\infty} \frac{1}{(n+m)^2} = \zeta(2) - H_m^{(2)}$ thus

$$\begin{aligned}
\sum_{m\geq 1}^{\mathcal{R}} \sum_{n\geq 1}^{\mathcal{R}} \frac{1}{(n+m)^2} &= \sum_{m\geq 1}^{\mathcal{R}} \sum_{n=1}^{+\infty} \frac{1}{(n+m)^2} - \sum_{m\geq 1}^{\mathcal{R}} \int_1^{+\infty} \frac{1}{(x+m)^2} dx \\
&= \sum_{m\geq 1}^{\mathcal{R}} (\zeta(2) - H_m^{(2)}) - \sum_{m\geq 1}^{\mathcal{R}} \frac{1}{m+1}
\end{aligned}$$

Thus

$$\boxed{\sum_{m\geq 1}^{\mathcal{R}} \sum_{n\geq 1}^{\mathcal{R}} \frac{1}{(n+m)^2} = 3 - \zeta(2) - \gamma - Log(2)}$$

Remark
By the same method we get more generally

$$\sum_{m\geq 1}^{\mathcal{R}} \sum_{n\geq 1}^{\mathcal{R}} f(n+m) = \sum_{n\geq 1}^{\mathcal{R}} (n-1) f(n) + 2 \sum_{n\geq 1}^{\mathcal{R}} F(n) + \int_1^2 F(y) dy$$

This is obtained by

$$\begin{aligned}
\sum_{m\geq 1}^{\mathcal{R}} \sum_{n\geq 1}^{\mathcal{R}} f(n+m) &= \sum_{m\geq 1}^{\mathcal{R}} (\sum_{n\geq 1}^{\mathcal{R}} f(n) - S_m(f) + \int_1^{m+1} f(x)dx) \\
&= \frac{1}{2} \sum_{n\geq 1}^{\mathcal{R}} f(n) - \sum_{m\geq 1}^{\mathcal{R}} S_m(f) + \sum_{m\geq 1}^{\mathcal{R}} \int_1^{m+1} f(x)dx \\
&= \frac{1}{2} \sum_{n\geq 1}^{\mathcal{R}} f(n) - (\frac{3}{2} \sum_{n\geq 1}^{\mathcal{R}} f(n) - \sum_{n\geq 1}^{\mathcal{R}} n f(n) - \sum_{n\geq 1}^{\mathcal{R}} \int_1^n f(x)dx) + \sum_{m\geq 1}^{\mathcal{R}} \int_1^{m+1} f(x)dx \\
&= -\sum_{n\geq 1}^{\mathcal{R}} f(n) + \sum_{n\geq 1}^{\mathcal{R}} n f(n) + \sum_{n\geq 1}^{\mathcal{R}} \int_1^n f(x)dx + \sum_{m\geq 1}^{\mathcal{R}} \int_1^{m+1} f(x)dx \\
&= \sum_{n\geq 1}^{\mathcal{R}} (n-1) f(n) + 2 \sum_{n\geq 1}^{\mathcal{R}} \int_1^n f(x)dx + \int_1^2 \int_1^y f(x)dxdy \\
&= \sum_{n\geq 1}^{\mathcal{R}} (n-1) f(n) + 2 \sum_{n\geq 1}^{\mathcal{R}} F(n) + \int_1^2 F(y)dy
\end{aligned}$$

□

Example

$$\sum_{m\geq 1}^{\mathcal{R}} \sum_{n\geq 1}^{\mathcal{R}} Log(n+m) = \sum_{n\geq 1}^{\mathcal{R}} (n-1) Log(n) + 2 \sum_{n\geq 1}^{\mathcal{R}} n Log(n) - 1 + \int_1^2 (yLog(y) - y)dy$$

2.10.2 The case of convergence

Like in one dimension we have a formula that link $\sum_{n\geq 1}^{\mathcal{R}} \sum_{m\geq 1}^{\mathcal{R}}$ to $\sum_{n=1}^{+\infty} \sum_{m=1}^{+\infty}$.
We have

$$\boxed{\sum_{n\geq 1}^{\mathcal{R}} \sum_{m\geq 1}^{\mathcal{R}} f(m,n) = \sum_{n=1}^{+\infty} \sum_{m=1}^{+\infty} f(m,n) + \int_1^{+\infty} \int_1^{+\infty} f(x,y)dxdy - \sum_{n=1}^{+\infty} \int_1^{+\infty} f(x,n)dx - \sum_{n=1}^{+\infty} \int_1^{+\infty} f(m,y)dy}$$

Examples

1) For $Re(s) > 2$ we have

$$\boxed{\sum_{m\geq 1}^{\mathcal{R}}\sum_{n\geq 1}^{\mathcal{R}}\frac{1}{(n+m)^s}=\frac{s-3}{s-1}\zeta(s-1)+\frac{2^{2-s}}{(s-1)(s-2)}+\frac{1}{s-1}-(\zeta(s)-\frac{1}{s-1})}$$

Note that

$$\begin{aligned}\sum_{m=1}^{+\infty}\sum_{n=1}^{+\infty}\frac{1}{(n+m)^s} &= \sum_{m\geq 1}^{\mathcal{R}}\sum_{n=1}^{+\infty}\frac{1}{(n+m)^s}+\int_1^{+\infty}\sum_{n=1}^{+\infty}\frac{1}{(n+x)^s}dx\\ &= \sum_{m\geq 1}^{\mathcal{R}}(\zeta(s)-H_m^{(s)})+\sum_{n=1}^{+\infty}\frac{1}{s-1}\frac{1}{(n+1)^{s-1}}\end{aligned}$$

This gives the relation

$$\sum_{m=1}^{+\infty}\sum_{n=1}^{+\infty}\frac{1}{(n+m)^s}=\zeta(s-1)-\zeta(s)$$

Independently this relation is easily deduced from the summation by packets

$$\sum_{m=1}^{+\infty}\sum_{n=1}^{+\infty}\frac{1}{(n+m)^s}=\sum_{k=1}^{+\infty}\sum_{m+n=k+1}\frac{1}{(n+m)^s}$$

This type of summation cannot be applied to $\sum_{m\geq 1}^{\mathcal{R}}\sum_{n\geq 1}^{\mathcal{R}}\frac{1}{(n+m)^s}$

2) With $f(x,y)=\frac{1}{xy(x+y)}$ we know that

$$\sum_{n=1}^{+\infty}\sum_{m=1}^{+\infty}f(m,n)=2\zeta(3)$$

thus we get

$$\sum_{n\geq 1}^{\mathcal{R}}\sum_{m\geq 1}^{\mathcal{R}}\frac{1}{mn(m+n)}=2\zeta(3)+2Log(2)-2\sum_{n=1}^{+\infty}\frac{Log(n+1)}{n^2}$$

But

$$\frac{1}{mn(m+n)}=\frac{1}{n^2m}-\frac{1}{n^2(m+n)}$$

and by the shift property

$$\sum_{m\geq 1}^{\mathcal{R}}\frac{1}{m+n}=\gamma-H_n+Log(n+1)$$

we get

$$\sum_{n\geq 1}^{\mathcal{R}}\sum_{m\geq 1}^{\mathcal{R}}\frac{1}{mn(m+n)}=\gamma\sum_{n\geq 1}^{\mathcal{R}}\frac{1}{n^2}-\sum_{n\geq 1}^{\mathcal{R}}\frac{1}{n^2}(\gamma-H_n+Log(n+1))$$

After some simplifications this gives

$$\int_1^{+\infty}\frac{\Psi(x+1)+\gamma}{x^2}dx=\sum_{n\geq 1}^{+\infty}\frac{Log(n+1)}{n^2}$$

2.10.3 The sums $\sum_{n\geq 1}^{\mathcal{R}} \frac{H_n}{n^p}$

We have by the shift property

$$\sum_{n\geq 1}^{\mathcal{R}} \frac{1}{m+n} = \gamma - H_m + Log(m+1)$$

this gives

$$\sum_{m\geq 1}^{\mathcal{R}}\sum_{n\geq 1}^{\mathcal{R}} \frac{1}{m}\frac{1}{m+n} = \sum_{m\geq 1}^{\mathcal{R}} \frac{1}{m}\sum_{n\geq 1}^{\mathcal{R}} \frac{1}{m+n} = \sum_{m\geq 1}^{\mathcal{R}} \frac{1}{m}(\gamma - H_m + Log(m+1))$$

thus

$$\sum_{m\geq 1}^{\mathcal{R}}\sum_{n\geq 1}^{\mathcal{R}} \frac{1}{m}\frac{1}{m+n} = \gamma^2 - \sum_{m\geq 1}^{\mathcal{R}} \frac{H_m}{m} + \sum_{m\geq 1}^{\mathcal{R}} \frac{Log(m+1)}{m}$$

On the other side

$$\sum_{n\geq 1}^{\mathcal{R}}\sum_{m\geq 1}^{\mathcal{R}} \frac{1}{m}\frac{1}{m+n} = \sum_{n\geq 1}^{\mathcal{R}} \frac{1}{n}(\sum_{m\geq 1}^{\mathcal{R}}(\frac{1}{m} - \frac{1}{m+n})) = \sum_{n\geq 1}^{\mathcal{R}} \frac{1}{n}(H_n - Log(n+1))$$

thus

$$\sum_{n\geq 1}^{\mathcal{R}}\sum_{m\geq 1}^{\mathcal{R}} \frac{1}{m}\frac{1}{m+n} = \sum_{m\geq 1}^{\mathcal{R}} \frac{H_m}{m} - \sum_{m\geq 1}^{\mathcal{R}} \frac{Log(m+1)}{m}$$

By the preceding theorem we get

$$\boxed{\sum_{m\geq 1}^{\mathcal{R}} \frac{Log(m+1)}{m} = \sum_{m\geq 1}^{\mathcal{R}} \frac{H_m}{m} - \frac{\gamma^2}{2}}$$

The sum $\sum_{m\geq 1}^{\mathcal{R}} \frac{Log(m+1)}{m}$ is simply related to $\sum_{m\geq 1}^{\mathcal{R}} \frac{Log(m)}{m}$:

$$\begin{aligned}
\sum_{m\geq 1}^{\mathcal{R}} \frac{Log(m+1)}{m} &= \sum_{m\geq 1}^{\mathcal{R}} \frac{Log(m+1)}{m+1} + \sum_{m\geq 1}^{\mathcal{R}} \frac{Log(m+1)}{m+1}\frac{1}{m} \\
&= \sum_{m\geq 1}^{\mathcal{R}} \frac{Log(m)}{m} + \int_1^2 \frac{Log(x)}{x}dx + \sum_{m\geq 1}^{\mathcal{R}} \frac{Log(m+1)}{m+1}\frac{1}{m} \\
&= \sum_{m\geq 1}^{\mathcal{R}} \frac{Log(m)}{m} + \frac{1}{2}Log^2(2) + \sum_{m\geq 1}^{\mathcal{R}} \frac{Log(m+1)}{m+1}\frac{1}{m}
\end{aligned}$$

But the last series $\sum_{m\geq 1} \frac{Log(m+1)}{m+1}\frac{1}{m}$ is convergent and thus we have

$$\begin{aligned}
\sum_{m\geq 1}^{\mathcal{R}} \frac{Log(m+1)}{m+1}\frac{1}{m} &= \sum_{m=1}^{+\infty} \frac{Log(m+1)}{m(m+1)} - \int_1^{+\infty} \frac{Log(x+1)}{x(x+1)}dx \\
&= \sum_{m=1}^{+\infty} \frac{Log(m+1)}{m(m+1)} - \frac{\pi^2}{12} - \frac{1}{2}Log^2(2)
\end{aligned}$$

Finally we get

$$\boxed{\sum_{m\geq 1}^{\mathcal{R}} \frac{Log(m)}{m} - \frac{\pi^2}{12} + \sum_{m=1}^{+\infty} \frac{Log(m+1)}{m(m+1)} = \sum_{m\geq 1}^{\mathcal{R}} \frac{H_m}{m} - \frac{\gamma^2}{2}}$$

thus

$$\boxed{\sum_{m\geq 1}^{\mathcal{R}} \frac{H_m}{m} = \gamma_1 - \frac{\pi^2}{12} + \frac{\gamma^2}{2} + \sum_{m=1}^{+\infty} \frac{Log(m+1)}{m(m+1)}}$$

Remark

We can obtain directly the sums $\sum_{m\geq 1}^{\mathcal{R}} \frac{Log(m)}{m}$ by

$$\sum_{n\geq 1}^{\mathcal{R}} \frac{1}{m+n-1} = \gamma - H_m + \frac{1}{m} + Log(m)$$

this gives

$$\boxed{\sum_{m\geq 1}^{\mathcal{R}} \sum_{n\geq 1}^{\mathcal{R}} \frac{1}{m} \frac{1}{m+n-1} = \gamma^2 - \sum_{m\geq 1}^{\mathcal{R}} \frac{H_m}{m} + \zeta(2) - 1 + \sum_{m\geq 1}^{\mathcal{R}} \frac{Log(m)}{m}}$$

Now consider the sum $\sum_{m\geq 1}^{\mathcal{R}} \frac{1}{m} \frac{1}{m+n-1}$. We have if $n=1$

$$\sum_{m\geq 1}^{\mathcal{R}} \frac{1}{m} \frac{1}{m+n-1} = \sum_{m\geq 1}^{\mathcal{R}} \frac{1}{m^2}$$

and if $n > 1$

$$\begin{aligned} \sum_{m\geq 1}^{\mathcal{R}} \frac{1}{m} \frac{1}{m+n-1} &= \sum_{m=1}^{+\infty} \frac{1}{m(m+n-1)} - \int_1^{+\infty} \frac{1}{x(x+n-1)} dx \\ &= \frac{H_{n-1}}{n-1} - \frac{Log(n)}{n-1} \end{aligned}$$

Thus the sum $\sum_{n\geq 1}^{\mathcal{R}} \sum_{m\geq 1}^{\mathcal{R}} \frac{1}{m} \frac{1}{m+n-1}$ is $\sum_{n\geq 1}^{\mathcal{R}} f(n)$ where

$$\begin{aligned} f(n) &= \sum_{m\geq 1}^{\mathcal{R}} \frac{1}{m^2} \text{ if } n=1 \\ &= \frac{H_{n-1}}{n-1} - \frac{Log(n)}{n-1} \text{ if } n>1 \end{aligned}$$

The function f is given by $f(x) = \frac{1}{x-1}(\gamma + \psi(x) - Log(x))$ and we have

$$\sum_{n\geq 1}^{\mathcal{R}} f(n) = \sum_{n\geq 1}^{\mathcal{R}} f(n+1) + f(1) - \int_1^2 f(x) dx$$

thus

$$\sum_{n\geq 1}^{\mathcal{R}} \sum_{m\geq 1}^{\mathcal{R}} \frac{1}{m} \frac{1}{m+n-1} = \sum_{n\geq 1}^{\mathcal{R}} \frac{H_n}{n} - \sum_{n\geq 1}^{\mathcal{R}} \frac{Log(n+1)}{n} + \sum_{m\geq 1}^{\mathcal{R}} \frac{1}{m^2} - \int_1^2 \frac{1}{x-1} (\gamma + \Psi(x) - Log(x)) dx$$

With the preceding results, after some simplifications we get

$$\int_0^1 \frac{\gamma + \Psi(x+1)}{x} dx = \sum_{m=1}^{+\infty} \frac{Log(m+1)}{m(m+1)}$$

We have for an integer $q > 1$

$$\sum_{k=1}^{2q-1} (-1)^{k-1} \frac{1}{n^k m^{2q-k}} = \frac{1}{m^{2q-1}(m+n)} + \frac{1}{n^{2q-1}(m+n)}$$

thus if $\zeta^{\mathcal{R}}(k)=\sum_{m\geq 1}^{\mathcal{R}}\frac{1}{m^k}$ then

$$\sum_{k=1}^{2q-1}\zeta^{\mathcal{R}}(k)\zeta^{\mathcal{R}}(2q-k)(-1)^{k-1}=2\sum_{n\geq 1}^{\mathcal{R}}\sum_{m\geq 1}^{\mathcal{R}}\frac{1}{m^{2q-1}(m+n)}$$

With

$$\sum_{n\geq 1}^{\mathcal{R}}\frac{1}{m+n}=\gamma-H_m+Log(m+1)$$

we get

$$2\sum_{m\geq 1}^{\mathcal{R}}\sum_{n\geq 1}^{\mathcal{R}}\frac{1}{m^{2q-1}(m+n)}=2\gamma\zeta^{\mathcal{R}}(2q-1)+2\sum_{m\geq 1}^{\mathcal{R}}\frac{Log(m+1)-H_m}{m^{2q-1}}$$

thus

$$\boxed{\sum_{m\geq 1}^{\mathcal{R}}\frac{Log(m+1)}{m^{2q-1}}-\sum_{m\geq 1}^{\mathcal{R}}\frac{H_m}{m^{2q-1}}=\frac{1}{2}\sum_{k=2}^{2q-2}\zeta^{\mathcal{R}}(k)\zeta^{\mathcal{R}}(2q-k)(-1)^{k-1}}$$

For $q=2$ we get

$$\boxed{\sum_{m\geq 1}^{\mathcal{R}}\frac{Log(m+1)}{m^3}-\sum_{m\geq 1}^{\mathcal{R}}\frac{H_m}{m^3}=-\frac{1}{2}(\zeta^{\mathcal{R}}(2))^2}$$

2.10.4 The sums $\sum_{n\geq 1}^{\mathcal{R}}\sum_{m\geq 1}^{\mathcal{R}}\frac{1}{m^p(m+n)^q}$

We begin with the sum $\sum_{m\geq 1}^{\mathcal{R}}\sum_{n\geq 1}^{\mathcal{R}}\frac{m}{m+n}$. We have

$$\sum_{m\geq 1}^{\mathcal{R}}\sum_{n\geq 1}^{\mathcal{R}}\frac{m}{m+n}=\sum_{m\geq 1}^{\mathcal{R}}\sum_{n\geq 1}^{\mathcal{R}}\frac{m+n}{m+n}-\sum_{m\geq 1}^{\mathcal{R}}\sum_{n\geq 1}^{\mathcal{R}}\frac{n}{m+n}=\frac{1}{4}-\sum_{m\geq 1}^{\mathcal{R}}\sum_{n\geq 1}^{\mathcal{R}}\frac{n}{m+n}$$

thus we get

$$\sum_{m\geq 1}^{\mathcal{R}}\sum_{n\geq 1}^{\mathcal{R}}\frac{m}{m+n}=\frac{1}{8}$$

Remark
Since $\sum_{n\geq 1}^{\mathcal{R}}\frac{m}{m+n}=m(\gamma-H_m+Log(m+1))$, we get

$$\sum_{m\geq 1}^{\mathcal{R}}\sum_{n\geq 1}^{\mathcal{R}}\frac{m}{m+n}=\sum_{m\geq 1}^{\mathcal{R}}m(\gamma-H_m+Log(m+1))=\frac{5}{12}\gamma-\sum_{m\geq 1}^{\mathcal{R}}mH_m+\sum_{m\geq 1}^{\mathcal{R}}mLog(m+1)$$

thus we find

$$\sum_{m\geq 1}^{\mathcal{R}}mH_m=\frac{5}{12}\gamma+\sum_{m\geq 1}^{\mathcal{R}}mLog(m+1)-\frac{1}{8}$$

Since

$$\begin{aligned}\sum_{m\geq 1}^{\mathcal{R}}mLog(m+1)&=\sum_{m\geq 1}^{\mathcal{R}}(m+1)Log(m+1)-\sum_{m\geq 1}^{\mathcal{R}}Log(m+1)\\&=-\zeta'(-1)-Log(\sqrt{2\pi})+1\end{aligned}$$

we have another proof of the relation

$$\sum_{m\geq 1}^{\mathcal{R}}mH_m=\frac{5}{12}\gamma-\zeta'(-1)-Log(\sqrt{2\pi})+\frac{7}{8}$$

A general relation

Let an integer $p \geq 2$ and

$$\begin{aligned} V_p(X,m,n) &= \sum_{k=1}^{p-1}\left(\frac{1}{m^k n^{p-k}} + \frac{1}{n^k m^{p-k}}\right)X^{p-k-1} \\ W_p(X,m,n) &= \sum_{k=1}^{p-1}\left(\frac{1}{m^k (m+n)^{p-k}} + \frac{1}{n^k (m+n)^{p-k}}\right)X^{p-k-1} \end{aligned}$$

The partial fraction decomposition of $\frac{1}{m^k(m+n)^{p-k}} + \frac{1}{n^k(m+n)^{p-k}}$ gives the relation

$$W_p(1+X,m,n) + X^{p-2}W_p(1+1/X,m,n) = V_p(X,m,n)$$

With $\mathcal{R}-$ summation on m and n, we get

$$\sum_{n\geq 1}^{\mathcal{R}}\sum_{m\geq 1}^{\mathcal{R}} W_p(1+X,m,n) + X^{p-2}\sum_{n\geq 1}^{\mathcal{R}}\sum_{m\geq 1}^{\mathcal{R}} W_p(1+1/X,m,n) = \sum_{n\geq 1}^{\mathcal{R}}\sum_{m\geq 1}^{\mathcal{R}} V_p(X,m,n)$$

If we set

$$\zeta^{\mathcal{R}}(k) = \sum_{m\geq 1}^{\mathcal{R}} \frac{1}{m^k}$$

and

$$\sum_{n\geq 1}^{\mathcal{R}}\sum_{m\geq 1}^{\mathcal{R}} \frac{1}{m^k(m+n)^{p-k}} = \zeta^{\mathcal{R}}(k,p-k)$$

then the polynomials

$$S_p(X) = \sum_{k=1}^{p-1} \zeta^{\mathcal{R}}(k,p-k)X^{p-k-1}$$

satisfies the relation (cf. Zagier)

$$S_p(1+X) + X^{p-2}S_p(1+1/X) = \sum_{k=1}^{p-1} \zeta^{\mathcal{R}}(k)\zeta^{\mathcal{R}}(p-k)X^{p-k-1}$$

Chapter 3

Ramanujan summation of alternating series.

3.1 Definition

The sum $\sum_{n\geq 1}^{\mathcal{R}}(-1)^n$ is not defined in the preceding sections. This is a consequence of the fact that for the sequence $n \mapsto (-1)^n$ there is no interpolation function $f \in \mathcal{O}^\pi$.

To define the summation of alternating series we begin to use the Euler-Boole summation formula

$$\begin{aligned} f(1) - f(2) + ... + (-1)^{n-1} f(n) &= \frac{1}{2}\sum_{k=0}^{m} \partial^k f(1) \frac{E_k}{k!} \\ &+ \frac{(-1)^{n-1}}{2} \sum_{k=0}^{m} \partial^k f(n+1) \frac{E_k}{k!} \\ &+ \frac{1}{2}\int_1^{n+1} \frac{1}{m!} e_m(t)\, \partial^{m+1} f(t)\, dt \end{aligned}$$

which we can write on the form

$$\begin{aligned} f(1) - ... + (-1)^{n-1} f(n) &= \frac{1}{2}\sum_{k=0}^{m} \partial^k f(1) \frac{E_k}{k!} + \frac{1}{2}\int_1^{+\infty} \frac{1}{m!} e_m(t)\, \partial^{m+1} f(t)\, dt \\ &+ \frac{(-1)^{n-1}}{2} \sum_{k=0}^{m} \partial^k f(n+1) \frac{E_k}{k!} - \frac{1}{2}\int_{n+1}^{\infty} \frac{1}{m!} e_m(t)\, \partial^{m+1} f(t)\, dt \end{aligned}$$

We see that we have the same structure like in the Euler-MacLaurin formula. By integration by parts we verify that the constant term

$$\tilde{C}(f) = \frac{1}{2}\sum_{k=0}^{m} \partial^k f(1) \frac{E_k}{k!} + \frac{1}{2}\int_1^{+\infty} \frac{1}{m!} e_m(t)\, \partial^{m+1} f(t)\, dt$$

is independent on m for $m \geq M$.

We can define a summation for an alternating series $\sum_{k\geq 1}(-1)^{k-1} f(k)$, like the preceding Ramanujan summation, by defining

$$\sum_{k\geq 1}^{\mathcal{R}} (-1)^{k-1} f(k) = \tilde{C}(f) = \frac{1}{2}\sum_{k=0}^{m} \partial^k f(1) \frac{E_k}{k!} + \frac{1}{2}\int_1^{+\infty} \frac{1}{m!} e_m(t)\, \partial^{m+1} f(t)\, dt$$

We get for example:
$\sum_{k\geq 1}^{\mathcal{R}}(-1)^{k-1} = \frac{1}{2}E_0 = 1/2$ and $\sum_{k\geq 1}^{\mathcal{R}}(-1)^{k-1}k = \frac{1}{2}E_0 + \frac{1}{2}\frac{E_1}{1!} = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$.

To obtain a definition not directly dependent on the Euler-Boole summation formula, we note

$$T_f(n) = \frac{(-1)^n}{2}\sum_{k=0}^{m}\partial^k f(n)\frac{E_k}{k!} - \frac{1}{2}\int_n^{\infty}\frac{1}{m!}e_m(t)\,\partial^{m+1}f(t)\,dt$$

then we have

$$f(1) - ... + (-1)^{n-1}f(n) = \tilde{C}(f) + T_f(n+1)$$

thus

$$(-1)^{n-1}f(n) = T_f(n+1) - T_f(n)$$

If we define $A_f(n) = (-1)^n T_f(n)$ we get

$$A_f(n) + A_f(n+1) = f(n)$$

and

$$\sum_{n\geq 1}^{\mathcal{R}}(-1)^{n-1}f(n) = \tilde{C}(f) = -T_f(1) = A_f(1)$$

The equation $A_f(x) + A_f(x+1) = f(x)$ does not specify an unique function A_f we must avoid the the solutions of $A(x) + A(x+1) = 0$, that are combinations of the functions $e^{(2k+1)i\pi x}$.

Lemma 2
If $f \in \mathcal{O}^{\pi}$ then there exist a unique solution $A_f \in \mathcal{O}^{\pi}$ of

$$A_f(x) + A_f(x+1) = f(x)$$

We have

$$A_f(x) = R_{f(2x)}(\frac{x}{2}) - R_{f(2x)}(\frac{x+1}{2}). \tag{3.1}$$

Proof
The function $x \mapsto f(2x)$ is in $\mathcal{O}^{2\pi}$ and by the theorem 1 there is a function $R \in \mathcal{O}^{2\pi}$ which is solution of $R(x) - R(x+1) = f(2x)$ with $\int_1^2 R(x)dx = 0$. And let

$$A(x) = R(\frac{x}{2}) - R(\frac{x+1}{2})$$

then we have

$$\begin{aligned} A(x) + A(x+1) &= -R(\frac{x}{2}+1) + R(\frac{x+1}{2}) - R(\frac{x+1}{2}) + R(\frac{x}{2}) \\ &= -R(\frac{x}{2}+1) + R(\frac{x}{2}) \\ &= f(x). \end{aligned}$$

Unicity of the solution: if a function $A \in \mathcal{O}^{\pi}$ is a solution of $A(x) + A(x+1) = 0$ then the function $R(x) = A(x)e^{i\pi x}$ is a solution of $R(x) - R(x+1) = 0$ of exponential type $< 2\pi$, thus by the Lemma 1 the function R is a constant C and we have $A(x) = Ce^{i\pi x}$, and $A \in \mathcal{O}^{\pi}$ implies $C = 0$.
□

Definition
If $f \in \mathcal{O}^{\pi}$ there exist a unique function $A_f \in \mathcal{O}^{\pi}$ solution of $A_f(x) + A_f(x+1) = f(x)$ and we define

$$\sum_{n\geq 1}^{\mathcal{R}}(-1)^{n-1}f(n) = A_f(1)$$

Remarks

1) This definition does not contradict the preceding definition, because if $f \in \mathcal{O}^\pi$ there is no function $g \in \mathcal{O}^\pi$ such that $g(n) = (-1)^n f(n)$. The condition $g \in \mathcal{O}^\pi$ is essential since in the case $f = 1$ if we take $g(x) = e^{i\pi x}$ then $R_g(x) = \frac{e^{i\pi x}}{2} - \frac{1}{i\pi}$ thus

$$\sum_{n\geq 1}^{\mathcal{R}} e^{i\pi n} = -\frac{1}{2} - \frac{1}{i\pi}$$

And if we take $g(x) = e^{-i\pi x}$ then $R_g(x) = \frac{e^{-i\pi x}}{2} + \frac{1}{i\pi}$ thus

$$\sum_{n\geq 1}^{\mathcal{R}} e^{-i\pi n} = -\frac{1}{2} + \frac{1}{i\pi}$$

2) If $f \in \mathcal{O}^\pi$ then $x \mapsto f(x)e^{i\pi x}$ is in $\mathcal{O}^{2\pi}$ and by theorem 1 there is a unique function $R_{f(x)e^{i\pi x}}$ solution of $R(x) - R(x+1) = f(x)e^{i\pi x}$. Then the function $A(x) = e^{-i\pi x}R_{f(x)e^{i\pi x}}$ is solution of $A(x) + A(x+1) = f(x)$, but A is not A_f since it is not of exponential type $< \pi$.

3) We have $\int_x^{x+1} A_f(x)dx + \int_{x+1}^{x+2} A_f(x)dx = \int_x^{x+1} f(x)dx$ thus by the preceding definition

$$\sum_{n\geq 1}^{\mathcal{R}} (-1)^{n-1} \int_n^{n+1} f(x)dx = \int_1^2 A_f(x)dx$$

Examples

1) The Euler polynomials $E_k(x)$ given by

$$\sum_{k\geq 0} \frac{E_k(x)}{k!} z^k = \frac{2e^{xz}}{e^z + 1}$$

are solution of

$$E_k(x) + E_k(x+1) = 2x^k$$

thus $A_{x^k} = \frac{1}{2}E_k(x)$ By Lemma 2 we have

$$A_{x^k} = \frac{2^k}{k+1}(B_{k+1}(\frac{x+1}{2}) - B_{k+1}(\frac{x}{2}))$$

By the properties of Bernoulli polynomials we deduce that for k integer > 0

$$\sum_{n\geq 1}^{\mathcal{R}} (-1)^{n-1} n^{2k} = 0$$

thus

$$\sum_{n\geq 1}^{\mathcal{R}} (-1)^{n-1} n^{2k-1} = \frac{2^{2k}}{4k}(B_{2k}(1) - B_{2k}(\frac{1}{2}))$$

this gives

$$\boxed{\sum_{n\geq 1}^{\mathcal{R}} (-1)^{n-1} n^{2k-1} = \frac{2^{2k}}{4k}(B_{2k} - B_{2k}(2^{1-4k} - 1))}$$

2) If $f(x) = Log(x)$ then $R_f(x) = -Log(\Gamma(x)) + Log(\sqrt{2\pi}) - 1$ thus

$$R_{Log(2x)} = R_{Log(2)} - Log(\Gamma(x)) + Log(\sqrt{2\pi}) - 1 = (\frac{3}{2} - x)Log(2) - Log(\Gamma(x)) + Log(\sqrt{2\pi}) - 1$$

By Lemma 2 we get

$$A_f(x) = \frac{1}{2}Log(2) - Log(\Gamma(x/2)) + Log(\Gamma((x+1)/2))$$

and $A_f(1) = \frac{1}{2}Log(2) - Log(\Gamma(1/2))$ thus

$$\boxed{\sum_{n\geq 1}^{\mathcal{R}} (-1)^{n-1} Log(n) = \frac{1}{2}Log(2) - \frac{1}{2}Log(\pi)}$$

3) We have for $|z| < \pi$

$$e^{ixz} + e^{i(x+1)z} = e^{ixz}(1 + e^{iz})$$

thus $A_{e^{ixz}} = \frac{e^{ixz}}{1+e^{iz}}$ and

$$\sum_{n\geq 1}^{\mathcal{R}} (-1)^{n-1} e^{inz} = \frac{e^{iz}}{1+e^{iz}}$$

We deduce that for $-\pi < t < \pi$

$$\sum_{n\geq 1}^{\mathcal{R}} (-1)^{n-1} \cos(nt) = \frac{1}{2}$$

$$\sum_{n\geq 1}^{\mathcal{R}} (-1)^{n-1} \sin(nt) = \frac{1}{2}\tan(\frac{t}{2})$$

4) The function $\Psi = \Gamma'/\Gamma$ satisfies

$$\Psi(x) + \Psi(x+1) = 2\Psi(x) + \frac{1}{x}$$

thus

$$2\sum_{n\geq 1}^{\mathcal{R}} (-1)^{n-1}\Psi(n) + \sum_{n\geq 1}^{\mathcal{R}} \frac{(-1)^{n-1}}{n} = \Psi(1)$$

We know that $\sum_{n\geq 1}^{\mathcal{R}} \frac{(-1)^{n-1}}{n} = \ln(2)$, this gives

$$2\sum_{n\geq 1}^{\mathcal{R}} (-1)^{n-1}\Psi(n) + \ln(2) = -\gamma$$

With $\Psi(n) + \frac{1}{n} + \gamma = H_n$ we find that

$$\sum_{n\geq 1}^{\mathcal{R}} (-1)^{n-1} H_n = \frac{1}{2}\ln 2$$

In the same manner

$$x\Psi(x) + (x+1)\Psi(x+1) = 2x\Psi(x) + \Psi(x) + 1 + \frac{1}{x}$$

gives

$$\sum_{n\geq 1}^{\mathcal{R}} (-1)^{n-1} nH_n = \frac{1}{4} - \frac{1}{4}\ln 2$$

3.2 Relation to usual summation.

Let $f \in \mathcal{O}^{\pi}$ from the equation $A_f(x) + A_f(x+1) = f(x)$ we get

$$A_f(1) + (-1)^n A_f(n) = \sum_{k=1}^{n-1} (-1)^{k-1} f(k)$$

If the alternating series $\sum_{n\geq 1}(-1)^{n-1}f(n)$ is convergent then $(-1)^n A_f(n)$ must have a limit when $n\to\infty$ but with the oscillation of $(-1)^n$ and the fact that $A_f\in\mathcal{O}^\pi$ then it seems that this limit is 0. Thus in the case the series $\sum_{n\geq 1}(-1)^{n-1}f(n)$ is a convergent series we expect that

$$\sum_{n\geq 1}^{\mathcal{R}}(-1)^{n-1}f(n)=\sum_{n=1}^{\infty}(-1)^{n-1}f(n)$$

To prove this we note that if the series $\sum_{n\geq 0}(-1)^n f(x+n)$ is convergent for all $Re(x)>0$ and if the function

$$x\mapsto\sum_{n=0}^{\infty}(-1)^n f(x+n)$$

is in $\mathcal{O}^\pi$, this sum is the unique solution of $A(x)+A(x+1)=f(x)$. Thus we get

$$A_f(x)=\sum_{n=0}^{\infty}(-1)^n f(x+n)$$

and

$$\sum_{n\geq 1}^{\mathcal{R}}(-1)^{n-1}f(n)=\sum_{n=0}^{\infty}(-1)^n f(n+1)$$

Remark

The fact that for alternating series there is not a corrective integral term is also clear if we note that for f sufficiently decreasing when $Re(x)\to+\infty$ we have

$$\begin{aligned}
A_f(x) &= R_{f(2x)}(\frac{x}{2})-R_{f(2x)}(\frac{x+1}{2})\\
&= \sum_{n\geq 0}^{\infty}f(x+2n)-\int_1^{+\infty}f(2x)dx-\sum_{n\geq 0}^{\infty}f(x+n+1)+\int_1^{+\infty}f(2x)dx\\
&= \sum_{n\geq 0}^{\infty}f(x+2n)-\sum_{n\geq 0}^{\infty}f(x+n+1)\\
&= \sum_{n=0}^{\infty}(-1)^n f(x+n)
\end{aligned}$$

3.3 Properties of the summation

We have immediately the property of linearity

$$\sum_{n\geq 1}^{\mathcal{R}}(-1)^{n-1}(af(n)+bg(n))=a\sum_{n\geq 1}^{\mathcal{R}}(-1)^{n-1}f(n)+b\sum_{n\geq 1}^{\mathcal{R}}(-1)^{n-1}g(n)$$

3.3.1 The shift property

Let $f\in\mathcal{O}^\pi$ we have for any integer $p\geq 1$

$$A_f(x+p)+A_f(x+p+1)=f(x+p)$$

thus if we note $f(+p):x\mapsto f(x+p)$ then $A_{f(+p)}(x)=A_f(x+p)$ and

$$\sum_{n\geq 1}^{\mathcal{R}}(-1)^{n-1}f(n+p)=A_f(p+1)=(-1)^p A_f(1)+(-1)^{p+1}\sum_{k=1}^{p}(-1)^{k-1}f(k)$$

This gives the usual property for the shift

$$\sum_{n\geq 1}^{\mathcal{R}}(-1)^{n-1}f(n+p) = f(p) - f(p-1) + ... + (-1)^{p-1}f(1) + (-1)^p \sum_{n\geq 1}^{\mathcal{R}}(-1)^{n-1}f(n)$$

thus for the special case $p = 1$ we get

$$\sum_{n\geq 1}^{\mathcal{R}}(-1)^n f(n+1) = -f(1) + \sum_{n\geq 1}^{\mathcal{R}}(-1)^{n-1}f(n)$$

Remark
Let $f \in \mathcal{O}^\pi$ and $F(x) = \int_1^x f(t)dt$ as a consequence of the shift property we have

$$\sum_{n\geq 1}^{\mathcal{R}}(-1)^{n-1}F(n) = -\frac{1}{2}\int_1^2 A_f(x)dx \tag{3.2}$$

Proof
Let $f \in \mathcal{O}^\pi$ we have

$$\int_x^{x+1} A_f(x)dx + \int_{x+1}^{x+2} A_f(x)dx = \int_x^{x+1} f(x)dx$$

thus by the preceding definition

$$\sum_{n\geq 1}^{\mathcal{R}}(-1)^{n-1}\int_n^{n+1} f(x)dx = \int_1^2 A_f(x)dx$$

With $F(x) = \int_1^x f(t)dt$ this gives

$$\int_1^2 A_f(x)dx = \sum_{n\geq 1}^{\mathcal{R}}(-1)^{n-1}(F(n+1) - F(n)) = -\sum_{n\geq 1}^{\mathcal{R}}(-1)^n F(n+1) - \sum_{n\geq 1}^{\mathcal{R}}(-1)^{n-1}F(n)$$

and by the shift property this gives $\int_1^2 A_f(x)dx = -2\sum_{n\geq 1}^{\mathcal{R}}(-1)^{n-1}F(n)$.
□

3.3.2 Summation of even and odd terms.

The classical properties

$$\sum_{n=1}^{+\infty} f(2n-1) + \sum_{n=1}^{+\infty} f(2n) = \sum_{n=1}^{+\infty} f(n)$$

and

$$\sum_{n=1}^{+\infty}(-1)^{n-1}f(n) = \sum_{n=1}^{+\infty}(f(2n-1) - f(2n))$$

are not satisfied by Ramanujan sums.

Theorem 12 *Let $f \in \mathcal{O}^\pi$ we have*

$$\begin{aligned}
\sum_{n\geq 1}^{\mathcal{R}} f(2n-1) &= \frac{1}{2}\sum_{n\geq 1}^{\mathcal{R}} f(n) + \frac{1}{2}\sum_{n\geq 1}^{\mathcal{R}}(-1)^{n-1}f(n) \\
\sum_{n\geq 1}^{\mathcal{R}} f(2n) &= \frac{1}{2}\sum_{n\geq 1}^{\mathcal{R}} f(n) - \frac{1}{2}\sum_{n\geq 1}^{\mathcal{R}}(-1)^{n-1}f(n) + \frac{1}{2}\int_1^2 f(t)dt
\end{aligned}$$

Proof
It is equivalent to prove the following assertions

$$\sum_{n\geq 1}^{\mathcal{R}} f(2n-1) + \sum_{n\geq 1}^{\mathcal{R}} f(2n) = \sum_{n\geq 1}^{\mathcal{R}} f(n) + \frac{1}{2}\int_1^2 f(t)dt$$

and

$$\sum_{n\geq 1}^{\mathcal{R}} f(2n-1) - \sum_{n\geq 1}^{\mathcal{R}} f(2n) = \sum_{n\geq 1}^{\mathcal{R}} (-1)^{n-1} f(n) - \frac{1}{2}\int_1^2 f(t)dt$$

The first assertion is simply (2.7). For the second assertion let $g(x) = f(2x)$ thus

$$A_f(x) = R_g(\frac{x}{2}) - R_g(\frac{x+1}{2})$$

and

$$\sum_{n\geq 1}^{\mathcal{R}} (-1)^{n-1} f(n) = R_g(\frac{1}{2}) - \sum_{n\geq 1}^{\mathcal{R}} g(n)$$

But we know by eq(10) that

$$R_g(\frac{1}{2}) = \sum_{n\geq 1}^{\mathcal{R}} g(n-\frac{1}{2}) + \int_{1/2}^1 g(t)dt$$

Then

$$\sum_{n\geq 1}^{\mathcal{R}} (-1)^{n-1} f(n) = \sum_{n\geq 1}^{\mathcal{R}} g(n-\frac{1}{2}) - \sum_{n\geq 1}^{\mathcal{R}} g(n) + \int_{1/2}^1 g(t)dt$$

thus we obtain

$$\sum_{n\geq 1}^{\mathcal{R}} (-1)^{n-1} f(n) = \sum_{n\geq 1}^{\mathcal{R}} f(2n-1) - \sum_{n\geq 1}^{\mathcal{R}} f(2n) + \frac{1}{2}\int_1^2 f(t)dt$$

□

Remark
From the preceding theorem we have

$$\sum_{n\geq 1}^{\mathcal{R}} f(n) = \sum_{n\geq 1}^{\mathcal{R}} f(2n-1) + \sum_{n\geq 1}^{\mathcal{R}} f(2n) - \frac{1}{2}\int_1^2 f(t)dt$$

$$\sum_{n\geq 1}^{\mathcal{R}} (-1)^{n-1} f(n) = \sum_{n\geq 1}^{\mathcal{R}} f(2n-1) - \sum_{n\geq 1}^{\mathcal{R}} f(2n) + \frac{1}{2}\int_1^2 f(t)dt$$

Examples
1) Let $f(x) = \frac{1}{x}$

$$\sum_{n\geq 1}^{\mathcal{R}} \frac{1}{2n-1} = \frac{1}{2}\sum_{n\geq 1}^{\mathcal{R}} \frac{1}{n} + \frac{1}{2}\sum_{n\geq 1}^{\mathcal{R}} (-1)^{n-1}\frac{1}{n}$$

thus

$$\boxed{\sum_{n\geq 1}^{\mathcal{R}} \frac{1}{2n-1} = \frac{1}{2}\gamma + \frac{1}{2}\ln 2}$$

2) Let $f(x) = \ln(x)$ then

$$\sum_{n\geq 1}^{\mathcal{R}}(-1)^{n-1}Log(n)=\sum_{n\geq 1}^{\mathcal{R}}Log(n)-2\sum_{n\geq 1}^{\mathcal{R}}Log(2n)+\int_1^2 Log(t)dt$$

With

$$\sum_{n\geq 1}^{\mathcal{R}}Log(2n)=\sum_{n\geq 1}^{\mathcal{R}}Log(2)+\sum_{n\geq 1}^{\mathcal{R}}\ln(n)=\frac{1}{2}Log(2)+Log(\sqrt{2\pi})-1$$

we get

$$\boxed{\sum_{n\geq 1}^{\mathcal{R}}(-1)^{n-1}Log(n)=\frac{1}{2}Log(\frac{2}{\pi})}$$

And we have also

$$\sum_{n\geq 1}^{\mathcal{R}}Log(2n-1)=\frac{1}{2}\sum_{n\geq 1}^{\mathcal{R}}Log(n)+\frac{1}{2}\sum_{n\geq 1}^{\mathcal{R}}(-1)^{n-1}Log(n)=\frac{1}{2}(Log(\sqrt{2\pi})-1)+\frac{1}{4}Log(\frac{2}{\pi})$$

thus

$$\boxed{\sum_{n\geq 1}^{\mathcal{R}}Log(2n-1)=\frac{1}{2}(Log(2)-1)}$$

and by the shift property

$$\boxed{\sum_{n\geq 1}^{\mathcal{R}}Log(2n+1)=\frac{1}{2}(Log(2)+3Log(3))-1}$$

3) Let $f(x)=\frac{Log(x)}{x}$ then

$$\sum_{n\geq 1}^{\mathcal{R}}\frac{Log(2n)}{2n}=\frac{1}{2}\sum_{n\geq 1}^{\mathcal{R}}\frac{Log(n)}{n}-\frac{1}{2}\sum_{n\geq 1}^{\mathcal{R}}\frac{(-1)^{n-1}Log(n)}{n}+\frac{1}{2}\int_1^2\frac{Log(x)}{x}dx$$

but

$$\sum_{n\geq 1}^{\mathcal{R}}\frac{Log(2n)}{2n}=\sum_{n\geq 1}^{\mathcal{R}}\frac{Log(2)}{2n}+\sum_{n\geq 1}^{\mathcal{R}}\frac{Log(n)}{2n}=\frac{Log(2)}{2}\gamma+\frac{1}{2}\sum_{n\geq 1}^{\mathcal{R}}\frac{Log(n)}{n}$$

thus

$$\frac{Log(2)}{2}\gamma=-\frac{1}{2}\sum_{n\geq 1}^{\mathcal{R}}\frac{(-1)^{n-1}Log(n)}{n}+\frac{Log^2(2)}{4}$$

this gives

$$\boxed{\sum_{n\geq 1}^{\mathcal{R}}\frac{(-1)^{n-1}Log(n)}{n}=\frac{Log^2(2)}{2}-\gamma Log(2)}$$

3.3.3 Derivation and integration

Let $f\in\mathcal{O}^{\pi}$ we have by theorem 8

$$\sum_{n\geq 1}^{\mathcal{R}}(-1)^{n-1}f(n)=\sum_{n\geq 1}^{\mathcal{R}}(2f(2n-1)-f(n))=\sum_{n\geq 1}^{\mathcal{R}}(-1)^{n-1}f(n)=\sum_{n\geq 1}^{\mathcal{R}}(f(n)-2f(2n))+\int_1^2 f(t)dt$$

Thus if f depend on an extra parameter z or t then the theorems of analyticity en integration of chapter 2 remains valid.

Examples

1) We know that for $Re(z) < 0$

$$\sum_{n\geq 1}^{\mathcal{R}}(-1)^{n-1}e^{zn}H_n = \frac{\ln(e^z+1)}{e^z+1}$$

Thus expanding in powers of z we obtain

$$\sum_{n\geq 1}^{\mathcal{R}}(-1)^{n-1}H_n = \frac{\ln(2)}{2}$$

and

$$\sum_{n\geq 1}^{\mathcal{R}}(-1)^{n-1}nH_n = \frac{1}{4} - \frac{\ln(2)}{4}$$

$$\sum_{n\geq 1}^{\mathcal{R}}(-1)^{n-1}n^2H_n = -\frac{1}{16}$$

2) For $Re(s) > 0$

$$\sum_{n\geq 1}^{\mathcal{R}}(-1)^{n-1}\frac{1}{n^s} = (1-2^{1-s})\zeta(s)$$

thus

$$\sum_{n\geq 1}^{\mathcal{R}}(-1)^{n-1} = (1-2^1)\zeta(0) \Rightarrow \zeta(0) = -\frac{1}{2}$$

By derivation, for $s \neq 1$ we have

$$\sum_{n\geq 1}^{\mathcal{R}}(-1)^{n-1}\frac{\ln(n)}{n^s} = -2^{1-s}(\ln 2)\,\zeta(s) - \left(1-2^{1-s}\right)\zeta'(s)$$

thus

$$\begin{aligned}\sum_{n\geq 1}^{\mathcal{R}}(-1)^{n-1}\ln(n) &= -2(\ln 2)\,\zeta(0) + \zeta'(0)\\ &= \ln 2 + \zeta'(0)\end{aligned}$$

$$\sum_{n\geq 1}^{\mathcal{R}}(-1)^{n-1}n^2\ln(n) = -\left(1-2^3\right)\zeta'(-2) = -7\zeta(3)/(2\pi)^2$$

$$\sum_{n\geq 1}^{\mathcal{R}}(-1)^{n-1}n^k\ln(n) = -2^{k+1}(\ln 2)\,\zeta(-k) - \left(1-2^{k+1}\right)\zeta'(-k)$$

3) We have for $-\pi < t < \pi$

$$\sum_{n\geq 1}^{\mathcal{R}}(-1)^{n-1}e^{int} = \frac{e^{it}}{1+e^{it}} = \frac{e^{it/2}}{e^{-it/2}+e^{it/2}} = \frac{1}{2} + i\frac{1}{2}\mathrm{tg}(\frac{t}{2})$$

Thus for $-\pi < t < \pi$

$$\sum_{n\geq 1}^{\mathcal{R}}(-1)^{n-1}\cos(nt) = \frac{1}{2}$$

and

$$\sum_{n\geq 1}^{\mathcal{R}}(-1)^{n-1}\sin(nt) = \frac{1}{2}\mathrm{tg}(\frac{t}{2})$$

3.4 Application: Expression of the Stieltjes constants

Theorem 13 *All the Stieltjes constants* $\gamma_m = \sum_{n\geq 1}^{\mathcal{R}} \frac{Log^m(n)}{n}$ *can be expressed in terms linear combinations of the constants* $\tilde{\gamma}_k$ *given by the convergent series*

$$\tilde{\gamma}_k = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1} Log^k(n)}{n}$$

More precisely we have

$$\gamma_m = -m! \sum_{\substack{k+l=m \\ l\geq -1, k\geq 0}} \frac{B_{l+1} Log^l(2)}{(l+1)!} \frac{\tilde{\gamma}_k}{k!}$$

Proof

By the preceding theorem we have

$$\sum_{m\geq 1}^{\mathcal{R}} \frac{1}{(2n)^{z+1}} = \frac{1}{2} \sum_{m\geq 1}^{\mathcal{R}} \frac{1}{n^{z+1}} - \frac{1}{2} \sum_{m\geq 1}^{\mathcal{R}} \frac{(-1)^{n-1}}{n^{z+1}} + \frac{1}{2} \int_1^2 \frac{1}{x^{z+1}} dx$$

thus

$$\sum_{m\geq 1}^{\mathcal{R}} \frac{1}{n^{z+1}} = \frac{1}{1-2^{-z}} \sum_{m\geq 1}^{\mathcal{R}} \frac{(-1)^{n-1}}{n^{z+1}} - \frac{1}{z}$$

But we can expand all the terms

$$\sum_{m\geq 1}^{\mathcal{R}} \frac{1}{n^{z+1}} = \sum_{m\geq 1}^{\mathcal{R}} \frac{e^{-zLog(n)}}{n} = \sum_{k=0}^{+\infty} \frac{(-1)^k}{k!} z^k \sum_{m\geq 1}^{\mathcal{R}} \frac{Log^k(n)}{n}$$

$$\sum_{m\geq 1}^{\mathcal{R}} \frac{(-1)^{n-1}}{n^{z+1}} = \sum_{m\geq 1}^{\mathcal{R}} \frac{(-1)^{n-1} e^{-zLog(n)}}{n} = \sum_{k=0}^{+\infty} \frac{(-1)^k}{k!} z^k \sum_{m\geq 1}^{\mathcal{R}} \frac{(-1)^{n-1} Log^k(n)}{n}$$

$$\frac{1}{1-2^{-z}} = \frac{1}{zLog(2)} \frac{-zLog(2)}{e^{-zLog(2)}-1} = \sum_{k=0}^{+\infty} \frac{(-1)^k B_k}{k!} Log^{k-1}(2) z^{k-1}$$

This gives

$$\sum_{m=0}^{+\infty} \frac{(-1)^m}{m!} z^m \sum_{n\geq 1}^{\mathcal{R}} \frac{Log^m(n)}{n}$$

$$= \left[\sum_{l=-1}^{+\infty} \frac{(-1)^{l+1} B_{l+1}}{(l+1)!} Log^l(2) z^l\right] \left[\sum_{k=0}^{+\infty} \frac{(-1)^k}{k!} z^k \sum_{n\geq 1}^{\mathcal{R}} \frac{(-1)^{n-1} Log^k(n)}{n}\right] - \frac{1}{z}$$

$$= \sum_{m=-1}^{+\infty} z^m \sum_{k+l=m} \frac{(-1)^{l+1} B_{l+1}}{(l+1)!} Log^l(2) \frac{(-1)^k}{k!} \sum_{n\geq 1}^{\mathcal{R}} \frac{(-1)^{n-1} Log^k(n)}{n} - \frac{1}{z}$$

the coefficient of z^{-1} in the sum is $Log^{-1}(2) \sum_{n\geq 1}^{\mathcal{R}} \frac{(-1)^{n-1}}{n} = 1$ thus we have

$$\sum_{m=0}^{+\infty} \frac{(-1)^m}{m!} z^m \sum_{n\geq 1}^{\mathcal{R}} \frac{Log^m(n)}{n} = \sum_{m=0}^{+\infty} z^m \sum_{k+l=m} \frac{(-1)^{l+1} B_{l+1}}{(l+1)!} Log^l(2) \frac{(-1)^k}{k!} \sum_{n\geq 1}^{\mathcal{R}} \frac{(-1)^{n-1} Log^k(n)}{n}$$

By identification we get

$$\frac{1}{m!} \sum_{n\geq 1}^{\mathcal{R}} \frac{Log^m(n)}{n} = - \sum_{\substack{k+l=m \\ l\geq -1, k\geq 0}} \frac{B_{l+1} Log^l(2)}{(l+1)!} \left[\frac{1}{k!} \sum_{n\geq 1}^{\mathcal{R}} \frac{(-1)^{n-1} Log^k(n)}{n}\right]$$

□

Examples
For $m = 0$

$$\sum_{n\geq 1}^{\mathcal{R}} \frac{1}{n} = -\frac{1}{Log(2)}\left[\sum_{n\geq 1}^{\mathcal{R}} \frac{(-1)^{n-1}Log(n)}{n}\right] - B_1\left[\sum_{n\geq 1}^{\mathcal{R}} \frac{(-1)^{n-1}}{n}\right]$$

thus

$$\sum_{n=1}^{+\infty} \frac{(-1)^{n-1}Log(n)}{n} = -\gamma Log(2) + \frac{1}{2}Log^2(2)$$

For $m = 1$

$$\begin{aligned}\sum_{n\geq 1}^{\mathcal{R}} \frac{Log(n)}{n} &= -\frac{1}{Log(2)}\left[\frac{1}{2}\sum_{n\geq 1}^{\mathcal{R}} \frac{(-1)^{n-1}Log^2(n)}{n}\right] \\ &\quad -B_1\left[\sum_{n\geq 1}^{\mathcal{R}} \frac{(-1)^{n-1}Log(n)}{n}\right] \\ &\quad -\frac{B_2 Log(2)}{2!}\left[\sum_{n\geq 1}^{\mathcal{R}} \frac{(-1)^{n-1}}{n}\right]\end{aligned}$$

thus

$$\boxed{\sum_{n\geq 1}^{\mathcal{R}} \frac{Log(n)}{n} = -\frac{1}{Log(2)}\left[\frac{1}{2}\sum_{n=1}^{+\infty} \frac{(-1)^{n-1}Log^2(n)}{n}\right] + \frac{1}{2}[-\gamma Log(2)] + \frac{1}{6}Log^2(2)}$$

For $m = 2$

$$\begin{aligned}\frac{1}{2}\sum_{n\geq 1}^{\mathcal{R}} \frac{Log^2(n)}{n} &= -\frac{1}{Log(2)}\left[\frac{1}{3!}\sum_{n\geq 1}^{\mathcal{R}} \frac{(-1)^{n-1}Log^3(n)}{n}\right] \\ &\quad -B_1\left[\frac{1}{2!}\sum_{n\geq 1}^{\mathcal{R}} \frac{(-1)^{n-1}Log^2(n)}{n}\right] \\ &\quad -\frac{B_2 Log(2)}{2!}\left[\sum_{n\geq 1}^{\mathcal{R}} \frac{(-1)^{n-1}Log(n)}{n}\right]\end{aligned}$$

3.5 Other alternate sums

Theorem 14 *Let $f \in \mathcal{O}^\pi$ then*

$$\sum_{n\geq 1}^{\mathcal{R}} (-1)^{n-1}\sum_{k=1}^{n} f(k) = \frac{1}{2}\sum_{n\geq 1}^{\mathcal{R}} (-1)^{n-1}f(n)$$

Proof
We write

$$(-1)^{n-1}R_f(n) - (-1)^{n-1}R_f(n+1) = (-1)^{n-1}f(n)$$

then

$$\sum_{n\geq 1}^{\mathcal{R}}(-1)^{n-1}R_f(n)+\sum_{n\geq 1}^{\mathcal{R}}(-1)^{n}R_f(n+1)=\sum_{n\geq 1}^{\mathcal{R}}(-1)^{n-1}f(n)$$

but

$$\sum_{n\geq 1}^{\mathcal{R}}(-1)^{n}R_f(n+1)=\sum_{n\geq 1}^{\mathcal{R}}(-1)^{n-1}R_f(n)+R_f(1)$$

thus

$$2\sum_{n\geq 1}^{\mathcal{R}}(-1)^{n-1}R_f(n)=\sum_{n\geq 1}^{\mathcal{R}}(-1)^{n-1}f(n)+\sum_{n\geq 1}^{\mathcal{R}}f(n)$$

□

Example
We have

$$-\sum_{n\geq 1}^{\mathcal{R}}(-1)^{n-1}Log(n)=2\sum_{n\geq 1}^{\mathcal{R}}(-1)^{n-1}Log(\Gamma(n+1))$$

thus

$$\boxed{\sum_{n\geq 1}^{\mathcal{R}}(-1)^{n-1}Log(n!)=\frac{1}{4}Log(\frac{2}{\pi})}$$

Theorem 15 *Let We now study series of type*

$$S_n^A(f)=\sum_{k=1}^{n}(-1)^{k-1}f(k)$$

If f is of moderate growth then

$$\sum_{n\geq 1}^{\mathcal{R}}S_n^A(f)=\frac{3}{2}\sum_{n\geq 1}^{\mathcal{R}}(-1)^{n-1}f(n)-\sum_{n\geq 1}^{\mathcal{R}}(-1)^{n-1}nf(n)$$

$$\sum_{n\geq 1}^{\mathcal{R}}(-1)^{n-1}S_n^A(f)=\frac{1}{2}\sum_{n\geq 1}^{\mathcal{R}}(-1)^{n-1}f(n)-\sum_{n\geq 1}^{\mathcal{R}}(-1)^{n-1}F(n)\ \textit{with}\ F(x)=\int_1^x f(u)du$$

Proof
We have $S_n^A(f)=A_f(1)+(-1)^nA_f(n)+(-1)^{n-1}f(n)$ thus the sums

$$\sum_{n\geq 1}^{\mathcal{R}}S_n^A(f)\ \text{and}\ \sum_{n\geq 1}^{\mathcal{R}}(-1)^{n-1}S_n^A(f)$$

are well defined if f is of moderate growth and it is equivalent to prove that

$$\sum_{n\geq 1}^{\mathcal{R}}(-1)^{n-1}A_f(n)=\sum_{n\geq 1}^{\mathcal{R}}(-1)^{n-1}nf(n)$$

$$\sum_{n\geq 1}^{\mathcal{R}}A_f(n)=\frac{1}{2}\sum_{n\geq 1}^{\mathcal{R}}(-1)^{n-1}f(n)+\frac{1}{2}\sum_{n\geq 1}^{\mathcal{R}}f(n)-\frac{1}{2}\int_1^2 A_f(x)dx$$

We have

$$xA_f(x) + (x+1)A_f(x+1) = A_f(x+1) + xf(x)$$

thus

$$A_f(1) = \sum_{n\geq 1}^{\mathcal{R}}(-1)^{n-1}A_f(n+1) + \sum_{n\geq 1}^{\mathcal{R}}(-1)^{n-1}nf(n)$$

but $A_f(1) + (-1)^{n-1}A_f(n+1) = S_n^A(f)$ this gives the first assertion.

For the second assertion we use

$$A_f(x) - A_f(x+1) = 2A_f(x) - f(x)$$

and the function $R(x) = A_f(x) - \int_1^2 A_f(x)dx$ is solution of

$$\begin{aligned} R(x) - R(x+1) &= 2A_f(x) - f(x) \\ \int_1^2 R(x)dx &= 0 \end{aligned}$$

thus

$$A_f(1) - \int_1^2 A_f(x)dx = 2\sum_{n\geq 1}^{\mathcal{R}} A_f(n) - \sum_{n\geq 1}^{\mathcal{R}} f(n)$$

i.e.

$$\sum_{n\geq 1}^{\mathcal{R}}(-1)^{n-1}f(n) - \int_1^2 A_f(x)dx = 2\sum_{n\geq 1}^{\mathcal{R}} A_f(n) - \sum_{n\geq 1}^{\mathcal{R}} f(n)$$

We have $\int_1^2 A_f(x)dx = \sum_{n\geq 1}^{\mathcal{R}}(-1)^{n-1}(F(n+1) - F(n))$ with $F(x) = \int_1^x f(u)du$

$$\int_1^2 A_f(x)dx = -2\sum_{n\geq 1}^{\mathcal{R}}(-1)^{n-1}F(n)$$

Finaly we note that

$$(-1)^{n-1}A_f(n) = A_f(1) - S_n^A(f) + (-1)^{n-1}f(n)$$

thus

$$A_f(n) = A_f(1) - 1)^{n-1} - (1)^{n-1}S_n^A(f) + f(n)$$

□

Example
For $f(x) = 1/x$ let $H_n^A = \sum_{k=1}^{n}(-1)^{k-1}\frac{1}{k}$ then

$$\boxed{\sum_{n\geq 1}^{\mathcal{R}} H_n^A = \frac{3}{2}\ln(2) - \frac{1}{2}}$$

Theorem 16 *If f and g are of modeate growth then*

$$\sum_{n\geq 1}^{\mathcal{R}} f(n)S_n^A(g) + \sum_{n\geq 1}^{\mathcal{R}}(-1)^{n-1}g(n)S_n(f) = \sum_{n\geq 1}^{\mathcal{R}}(-1)^{n-1}f(n)g(n) + \sum_{n\geq 1}^{\mathcal{R}} f(n)\sum_{n\geq 1}^{\mathcal{R}}(-1)^{n-1}g(n)$$

Proof
Multiplying by $R_f(x)$ the equation $A_g(x) + A_g(x+1) = g(x)$ we get

$$A_g(x)R_f(x) + A_g(x+1)R_f(x) = g(x)R_f(x)$$

or

$$A_g(x)R_f(x) + A_g(x+1)(R_f(x+1) + f(x)) = g(x)R_f(x)$$

This gives

$$A_g(x)R_f(x) + A_g(x+1)R_f(x+1) = g(x)R_f(x) - f(x)A_g(x+1)$$

We obtain

$$\begin{aligned}
A_g(1)R_f(1) &= \sum_{n\geq 1}^{\mathcal{R}}(-1)^{n-1}(g(n)R_f(n) - f(n)A_g(n+1)) \\
&= \sum_{n\geq 1}^{\mathcal{R}}(-1)^{n-1}g(n)R_f(n) - \sum_{n\geq 1}^{\mathcal{R}}(-1)^{n-1}f(n)A_g(n+1) \\
&= \sum_{n\geq 1}^{\mathcal{R}}(-1)^{n-1}g(n)(\sum_{n\geq 1}^{\mathcal{R}} f(n) + f(n) - S_n(f)) \\
&\quad - \sum_{n\geq 1}^{\mathcal{R}}(-1)^{n-1}f(n)((-1)^{n-1}S_n^A(g) - (-1)^{n-1}\sum_{n\geq 1}^{\mathcal{R}}(-1)^{n-1}g(n))
\end{aligned}$$

Finaly

$$0 = \sum_{n\geq 1}^{\mathcal{R}}(-1)^{n-1}f(n)g(n) - \sum_{n\geq 1}^{\mathcal{R}}(-1)^{n-1}g(n)S_n(f) - \sum_{n\geq 1}^{\mathcal{R}} f(n)S_n^A(g) + \sum_{n\geq 1}^{\mathcal{R}} f(n)\sum_{n\geq 1}^{\mathcal{R}}(-1)^{n-1}g(n)$$

□

Example
With $f = g$ the preceding theorem gives

$$\sum_{n\geq 1}^{\mathcal{R}} f(n)S_n^A(f) + \sum_{n\geq 1}^{\mathcal{R}}(-1)^{n-1}f(n)S_n(f) = \sum_{n\geq 1}^{\mathcal{R}}(-1)^{n-1}[f(n)]^2 + \sum_{n\geq 1}^{\mathcal{R}} f(n)\sum_{n\geq 1}^{\mathcal{R}}(-1)^{n-1}f(n)$$

For $f(x) = 1/x$ we have

$$\sum_{n\geq 1}^{\mathcal{R}}(-1)^{n-1}\frac{H_n}{n} + \sum_{n\geq 1}^{\mathcal{R}}\frac{H_n^A}{n} = \gamma\ln 2 + \sum_{n\geq 1}^{\mathcal{R}}(-1)^{n-1}\frac{1}{n^2} = \gamma\ln 2 + \frac{1}{2}\zeta(2)$$

We have (cf. Srivastava and Choi p.357 (40))

$$\sum_{n\geq 1}^{\mathcal{R}}(-1)^{n-1}\frac{H_n}{n} = \frac{1}{2}\zeta(2) - \frac{1}{2}(Log(2))^2$$

We get a formula that generalize a formula of Sitaramachandrarao ([4] A formula of Ramanujan, th 3.5).

$$\boxed{\sum_{n\geq 1}^{\mathcal{R}}\frac{H_n^A}{n} = \gamma\ln 2 + \frac{1}{2}(\ln 2)^2}$$

where $H_n^A = \sum_{k=1}^{n}(-1)^{k-1}\frac{1}{k}$.

3.6 Generalization

Let N integer > 1 and ω a root of unity $\omega = e^{2i\pi m/N}$, $m = 1, ..., N-1$, we can define the Ramanujan summation of the series

$$\sum_{n\geq 1} \omega^{n-1} f(n)$$

Let N integer > 1 we note simply $f(Nx)$ the function

$$: x \mapsto f(Nx)$$

Lemma
Let ω a root of unity $\omega = e^{2i\pi m/N}$, $m = 1, ..., N-1$. Let $f \in \mathcal{O}^{2\pi/N}$ then the equation

$$R(x) - \omega R(x+1) = f(x)$$

has a unique solution $R_f^\omega \in \mathcal{O}^{2\pi/N}$. We have

$$R_f^\omega(x) = \sum_{k=0}^{N-1} \omega^k R_{f(Nx)}(\frac{x+k}{N}) \quad (3.3)$$

Proof
The function R_f^ω defined by (5.3) verify

$$R_f^\omega(x) - R_f^\omega(x+1) = R_{f(Nx)}(\frac{x}{N}) - R_{f(Nx)}(\frac{x}{N}+1) = f(x)$$

For the unicity we note that if $R \in \mathcal{O}^{2\pi/N}$ is solution of

$$R(x) - \omega R(x+1) = 0$$

then

$$\omega^x R(x) - \omega^{x+1} R(x+1) = 0$$

and $T : x \mapsto \omega^x R(x)$ is in $\mathcal{O}^{2\pi}$ and is solution of $T(x) - T(x+1) = 0$. Thus T is constant and

$$R(x) = C\omega^{-x}$$

With $R \in \mathcal{O}^{2\pi/N}$ this implies $C = 0$.
□

Definition
Let ω a root of unity $\omega = e^{2i\pi m/N}$, $m = 1, ..., N-1$. Let $f \in \mathcal{O}^{2\pi/N}$ we define

$$\sum_{n\geq 1}^{\mathcal{R}} \omega^{n-1} f(n) = R_f^\omega(1)$$

where $R_f^\omega \in \mathcal{O}^{2\pi/N}$ is the unique solution of

$$R(x) - \omega R(x+1) = f(x)$$

Theorem 17 *We have for* $\omega = e^{2i\pi m/N}$, $m = 1, ..., N-1$

$$\sum_{n\geq 1}^{\mathcal{R}} \omega^{n-1} f(n) = \sum_{k=0}^{N-1} \omega^k \sum_{n\geq 1}^{\mathcal{R}} f(Nn+k+1-N) + \frac{1}{N}\sum_{k=0}^{N-1} \omega^k \int_{k+1}^{N} f(x)dx$$

For $k = 1, ..., N$

$$\sum_{n\geq 1}^{\mathcal{R}} f(Nn+k-N) = \frac{1}{N}\sum_{n\geq 1}^{\mathcal{R}} f(n) + \frac{1}{N}\sum_{m=1}^{N-1} e^{-\frac{2i\pi m}{N}k} \sum_{n\geq 1}^{\mathcal{R}} e^{\frac{2i\pi m}{N}n} f(n) + \frac{1}{N}\int_1^k f(x)dx$$

Proof
We have

$$\sum_{n\geq 1}^{\mathcal{R}} \omega^{n-1} f(n) = \sum_{k=0}^{N-1} \omega^k R_{f(Nx)}(\frac{k+1}{N})$$

which is true for $\omega = e^{2i\pi m/N}$, $m = 1, ..., N-1$.
If $\omega = 1$ the function R given by

$$R(x) = \sum_{k=0}^{N-1} R_{f(Nx)}(\frac{x+k}{N})$$

is solution of

$$R(x) - R(x+1) = f(x)$$

Thus

$$R_f(x) = \sum_{k=0}^{N-1} R_{f(Nx)}(\frac{x+k}{N}) - \int_1^2 \sum_{k=0}^{N-1} R_{f(Nx)}(\frac{x+k}{N})dx$$

$$\int_1^2 \sum_{k=0}^{N-1} R_{f(Nx)}(\frac{x+k}{N})dx = N\int_{1/N}^{1+1/N} R_{f(Nx)}(x)dx = N\int_{1/N}^{1} f(Nx)dx = \int_1^N f(x)dx$$

and

$$R_f(x) = \sum_{k=0}^{N-1} R_{f(Nx)}(\frac{x+k}{N}) - \int_1^N f(x)dx$$

$$\sum_{n\geq 1}^{\mathcal{R}} f(n) = \sum_{k=0}^{N-1} R_{f(Nx)}(\frac{k+1}{N}) - \int_1^N f(x)dx$$

The system of equations

$$\begin{aligned}\sum_{k=0}^{N-1} R_{f(Nx)}(\frac{k+1}{N}) &= \sum_{n\geq 1}^{\mathcal{R}} f(n) + \int_1^N f(x)dx \\ \sum_{k=0}^{N-1} e^{\frac{2i\pi m}{N}k} R_{f(Nx)}(\frac{k+1}{N}) &= e^{-\frac{2i\pi m}{N}} \sum_{n\geq 1}^{\mathcal{R}} e^{\frac{2i\pi m}{N}n} f(n)\ , m = 1, ..., N-1\end{aligned}$$

is of type $\sum_{k=0}^{N-1} b_k e^{\frac{2i\pi m}{N}k} = a_m$ and can be solved by and $b_k = \frac{1}{N}\sum_{m=0}^{N-1} e^{-\frac{2i\pi m}{N}k} a_m$ thus

$$\begin{aligned}R_{f(Nx)}(\frac{k+1}{N}) &= \frac{1}{N}(\sum_{n\geq 1}^{\mathcal{R}} f(n) + \int_1^N f(x)dx) \\ &+ \frac{1}{N}\sum_{m=1}^{N-1} e^{-\frac{2i\pi m}{N}k} e^{-\frac{2i\pi m}{N}} \sum_{n\geq 1}^{\mathcal{R}} e^{\frac{2i\pi m}{N}n} f(n)\end{aligned}$$

With

$$\begin{aligned}R_{f(Nx)}(\frac{k+1}{N}) &= \sum_{n\geq 1}^{\mathcal{R}} f(Nn - N + 1 + k) - \int_1^{\frac{1+k}{N}} f(Nx)dx \\ &= \sum_{n\geq 1}^{\mathcal{R}} f(Nn + k + 1 - N) + \frac{1}{N}\int_{k+1}^N f(x)dx\end{aligned}$$

we get for $k = 0, ..., N-1$

$$\begin{aligned}\sum_{n\geq 1}^{\mathcal{R}} f(Nn + k + 1 - N) &= \frac{1}{N}\sum_{n\geq 1}^{\mathcal{R}} f(n) + \frac{1}{N}\sum_{m=1}^{N-1} e^{-\frac{2i\pi m}{N}(k+1)} \sum_{n\geq 1}^{\mathcal{R}} e^{\frac{2i\pi m}{N}n} f(n) \\ &+\frac{1}{N}\int_1^{k+1} f(x)dx\end{aligned}$$

□

Remark. Relation with the usual summation
Let ω a root of unity $\omega = e^{2i\pi m/N}$, $m = 1, ..., N-1$.
Let $f \in \mathcal{O}^{2\pi/N}$. If the series $\sum_{n\geq 1} \omega^{n-1} f(n+x-1)$ is convergent for $Re(x) > 0$ then the function

$$R(x) = \sum_{n\geq 1}^{+\infty} \omega^{n-1} f(n+x-1)$$

is solution of

$$R(x) - \omega R(x+1) = f(x)$$

If $R \in \mathcal{O}^{2\pi/N}$ then by unicity of the solution we obtain

$$R_f^{\omega}(x) = \sum_{n\geq 1}^{+\infty} \omega^{n-1} f(n+x-1)$$

thus

$$\sum_{n\geq 1}^{\mathcal{R}} f(n) = \sum_{n\geq 1}^{+\infty} \omega^{n-1} f(n)$$

Examples
1) $N = 2$

$$\begin{aligned}\sum_{n\geq 1}^{\mathcal{R}} f(2n-1) &= \frac{1}{2}\sum_{n\geq 1}^{\mathcal{R}} f(n) - \frac{1}{2}\sum_{m=1}^{1}\sum_{n\geq 1}^{\mathcal{R}} e^{\frac{2i\pi m}{2}n} f(n)\\ \sum_{n\geq 1}^{\mathcal{R}} f(2n) &= \frac{1}{2}\sum_{n\geq 1}^{\mathcal{R}} f(n) + \frac{1}{2}\sum_{n\geq 1}^{\mathcal{R}} (-1)^n f(n) + \frac{1}{2}\int_1^2 f(x)dx\end{aligned}$$

2) $N = 3$

$$\sum_{n\geq 1}^{\mathcal{R}} f(3n-2) = \frac{1}{3}\sum_{n\geq 1}^{\mathcal{R}} f(n) + \frac{1}{3}e^{-\frac{2i\pi}{3}}\sum_{n\geq 1}^{\mathcal{R}} e^{\frac{2i\pi}{3}n} f(n) + \frac{1}{3}e^{-\frac{4i\pi}{3}}\sum_{n\geq 1}^{\mathcal{R}} e^{\frac{4i\pi}{3}n} f(n)$$

$$\begin{aligned}\sum_{n\geq 1}^{\mathcal{R}} f(3n-1) &= \frac{1}{3}\sum_{n\geq 1}^{\mathcal{R}} f(n) + \frac{1}{3}e^{-\frac{4i\pi}{3}}\sum_{n\geq 1}^{\mathcal{R}} e^{\frac{2i\pi}{3}n} f(n) + \frac{1}{3}e^{-\frac{2i\pi}{3}}\sum_{n\geq 1}^{\mathcal{R}} e^{\frac{4i\pi}{3}n} f(n)\\ &\quad + \frac{1}{3}\int_1^2 f(x)dx\end{aligned}$$

$$\begin{aligned}\sum_{n\geq 1}^{\mathcal{R}} f(3n) &= \frac{1}{3}\sum_{n\geq 1}^{\mathcal{R}} f(n) + \frac{1}{3}\sum_{n\geq 1}^{\mathcal{R}} e^{\frac{2i\pi}{3}n} f(n) + \frac{1}{3}\sum_{n\geq 1}^{\mathcal{R}} e^{\frac{4i\pi}{3}n} f(n)\\ &\quad + \frac{1}{3}\int_1^3 f(x)dx\end{aligned}$$

Chapter 4

Formal transforms and numerical evaluations

4.1 Formal transforms

We know that the Ramanujan summation is related to the series $-\sum_{k\geq 1}\frac{B_k}{k!}\partial^{k-1}f(1)$ by the formulas

$$\sum_{n\geq 1}^{\mathcal{R}} f(n) = -\sum_{k=1}^{m}\frac{B_k}{k!}\partial^{k-1}f(1) + (1)^{m+1}\int_0^1 R_{\partial^m f}(t+1)\frac{B_m(t)}{m!}dt$$

or

$$\sum_{n\geq 1}^{\mathcal{R}} f(n) = -\sum_{k=1}^{m}\frac{B_k}{k!}\partial^{k-1}f(1) + \int_1^{+\infty}\frac{b_{m+1}(x)}{(m+1)!}\partial^{m+1}f(x)dx$$

The relation with the series $-\sum_{k\geq 1}\frac{B_k}{k!}\partial^{k-1}f(1)$ can also be viewed in an operator setting.
Let E the shift operator defined by

$$Eg(x) = g(x+1)$$

by Taylor formula we have formally $E = e^{\partial}$, and the equation $R(x) - R(x+1) = f(x)$ is

$$(I - e^{\partial})R = f$$

and we have

$$R = \frac{I}{I-E}f = -\frac{\partial}{e^{\partial}-I}\partial^{-1}f = -\partial^{-1}f - \sum_{k\geq 1}\frac{B_k}{k!}\partial^{k-1}f$$

With $\partial^{-1}f(x) = \int_1^x f(t)dt$ we get formally

$$\boxed{\sum_{n\geq 1}^{\mathcal{R}} f(n) = -\sum_{k\geq 0}\frac{B_{k+1}}{(k+1)!}\partial^k f(1)}$$

Unfortunately this last series is often divergent for the usual Cauchy summation. A more useful formula is obtained if we work wiyh the difference operator

$$\Delta g(x) = g(x+1) - g(x)$$

Thus we write $\Delta = E - I$, and translate the equation $R(x) - R(x+1) = f(x)$ in the form

$$-\Delta R = f$$

To get an expansion of $\frac{I}{\Delta}$ we use that $\frac{I}{\log(I+\Delta)} - \frac{I}{\Delta}$ can be expanded in powers of Δ and that formally we have

$$\frac{I}{\log(I+\Delta)} = \frac{I}{\log(e^{\partial})} = \partial^{-1}$$

Thus define the *modified Bernoulli numbers* β_n by

$$\frac{t}{\log(1+t)} = \sum_{n\geq 0} \frac{\beta_n}{n!} t^n = 1 + \sum_{n\geq 0} \frac{\beta_{n+1}}{(n+1)!} t^{n+1}$$

We get

$$\frac{I}{\log(I+\Delta)} - \frac{I}{\Delta} = \sum_{n\geq 0} \frac{\beta_{n+1}}{(n+1)!} \Delta^n$$

Thus

$$R = -\frac{I}{\Delta} f = -\partial^{-1} f + \sum_{n\geq 0} \frac{\beta_{n+1}}{(n+1)!} \Delta^n f$$

This gives formally

$$\boxed{\sum_{n\geq 1}^{\mathcal{R}} f(n) = \sum_{n\geq 0} \frac{\beta_{n+1}}{(n+1)!} (\Delta^n f)(1)} \tag{4.1}$$

Remark
The modified Bernoulli numbers β_{n+1} are given by

$$\frac{t}{\log(1+t)} = \sum_{n\geq 0} \frac{\beta_n}{n!} t^n$$

Thus they are given by $\beta_0 = 1$ and the relation

$$\sum_{k=0}^{n} \frac{\beta_n}{n!} \frac{(-1)^k}{n-k+1} = 0$$

this gives

$$\beta_1 = \frac{1}{2}, \beta_2 = -\frac{1}{12}, \beta_3 = \frac{1}{24}, \beta_4 = -\frac{19}{720}, \beta_5 = \frac{3}{160}, \beta_6 = -\frac{863}{60480}, \ldots$$

We can give an integral expression of these numbers if we write

$$\frac{t}{\log(1+t)} = \int_0^1 e^{xLog(1+t)} dx = \int_0^1 (1+t)^x dx = \sum_{n\geq 0} t^n \int_0^1 \frac{x(x-1)\ldots(x-n+1)}{n!} dx$$

thus we get

$$\beta_n = \int_0^1 x(x-1)\ldots(x-n+1) dx$$

This formula and the sum $\sum_{n\geq 0} \frac{\beta_{n+1}}{(n+1)!} (\Delta^n f)(1)$ in (4.1) show that Ramanujan summation can be related to Newton interpolation series.

4.2 Newton interpolation series

The Newton interpolation series are series of type

$$\sum_{n\geq 0} a_n \frac{(z-1)\ldots(z-n)}{n!}$$

with the convention $\frac{(z-1)\ldots(z-n)}{n!} = 1$ si $n = 0$. They have the following property:

If for $x_0 \in \mathbb{R}\backslash\mathbb{N}$ the series $\sum_{n\geq 0} a_n \frac{(x_0-1)\ldots(x_0-n)}{n!}$ is convergent then the series $\sum_{n\geq 0} a_n \frac{(z-1)\ldots(z-n)}{n!}$ is uniformly convergent on every compact of the half plane $\{Re(z) > x_0\}$ thus defining an analytic function

$$f(z) = \sum_{n=0}^{+\infty} a_n \frac{(z-1)\ldots(z-n)}{n!} \text{ for } z \in \{Re(z) > x_0\} \tag{4.2}$$

If for $x_0 < 1$ the series $\sum_{n\geq 0} a_n \frac{(x_0-1)...(x_0-n)}{n!}$ is convergent then the coefficients a_n are related to the values $f(1), f(2), ...$ of the fonction f by

$$f(k+1) = \sum_{n=0}^{k} a_n \frac{k(k-1)...(k-n+1)}{n!} = \sum_{n=0}^{k} a_n C_k^n$$

a relation that we can invert to get an expression of the coefficients a_n

$$a_n = \Delta^n f(1) = \sum_{k=0}^{n} f(k+1) C_n^k (-1)^{n-k}$$

Thus for $z \in \{Re(z) > x_0\}$ we get the *Newton interpolation formula*

$$\boxed{f(z) = \sum_{n=0}^{+\infty} \Delta^n f(1) \frac{(z-1)...(z-n)}{n!}}$$

Remark: Ramanujan interpolation formula

To get the expansion of a function f in Newton series we have to evaluate the terms $\Delta^n f(1)$, this can be done by the generating function

$$\sum_{n\geq 0} \Delta^n f(1) \frac{t^n}{n!} = \sum_{n\geq 0} \sum_{k=0}^{n} f(k+1) \frac{t^n}{k!(n-k)!} (-1)^{n-k} = \sum_{k\geq 0} f(k+1) \frac{t^k}{k!} \sum_{l\geq 0} \frac{(-1)^l t^l}{l!}$$

thus

$$\boxed{\sum_{n\geq 0} \Delta^n f(1) \frac{t^n}{n!} = e^{-t} \sum_{k\geq 0} f(k+1) \frac{t^k}{k!}}$$

Now if we write

$$(z-1)...(z-k) = (-1)^k \frac{1}{\Gamma(-z+1)} \int_0^{+\infty} e^{-t} t^{k-z} dt$$

then we have

$$\sum_{k\geq 0} \frac{(\Delta^k f)(1)}{k!} (z-1)...(z-k) = \frac{1}{\Gamma(-z+1)} \sum_{k\geq 0} \frac{(-1)^k (\Delta^k f)(1)}{k!} \int_0^{+\infty} e^{-t} t^{k-z} dt$$

and interchanging $\sum_{k\geq 0}$ and $\int_0^{+\infty}$ we get

$$\sum_{k\geq 0} \frac{(\Delta^k f)(1)}{k!} (z-1)...(z-k) = \frac{1}{\Gamma(-z+1)} \int_0^{+\infty} t^{-z} e^{-t} \sum_{k\geq 0} \frac{(-1)^k (\Delta^k f)(1)}{k!} t^k dt$$

But

$$e^{-t} \sum_{k\geq 0} \frac{(-1)^k (\Delta^k f)(1)}{k!} t^k = \sum_{k\geq 0} \frac{(-1)^k}{k!} f(k+1) t^k$$

thus the Newton interpolation formula become the *Ramanujan interpolation formula*

$$f(z) = \frac{1}{\Gamma(-z+1)} \int_0^{+\infty} t^{-z} \sum_{k\geq 0} \frac{(-1)^k}{k!} f(k+1) t^k dt$$

4.3 Another formula for Ramanujan summation

Let f be given by the Newton interpolation formula, then for any integer $n \geq 1$

$$f(n) = \sum_{k=0}^{+\infty} \frac{(\Delta^k f)(1)}{k!}(n-1)...(n-k)$$

To evaluate the sum $\sum_{n\geq 1}^{\mathcal{R}} f(n)$ we thus evaluate the sums $\sum_{n\geq 1}^{\mathcal{R}}(n-1)...(n-k)$ and try to prove that

$$\sum_{n\geq 1}^{\mathcal{R}} f(n) = \sum_{k=0}^{+\infty} \frac{(\Delta^k f)(1)}{k!} \sum_{n\geq 1}^{\mathcal{R}}(n-1)...(n-k)$$

We first note that

$$(x-1)...(x-(k+1)) - x(x-1)...(x-k) = -(k+1)(x-1)...(x-k)$$

thus

$$R_{(x-1)...(x-k)} = -\frac{1}{k+1}(x-1)...(x-(k+1)) + \frac{1}{k+1}\int_1^2 (x-1)...(x-(k+1))dx$$

and we get

$$\sum_{n\geq 1}^{\mathcal{R}}(n-1)...(n-k) = \frac{1}{k+1}\int_1^2 (x-1)...(x-(k+1))dx$$

We verify that

$$\frac{1}{k+1}\int_1^2 (x-1)...(x-(k+1))dx = \frac{1}{k+1}\int_0^1 x(x-1)...(x-k)dx = \frac{\beta_{k+1}}{k+1}$$

Thus

$$\sum_{n\geq 1}^{\mathcal{R}}(n-1)...(n-k) = \frac{\beta_{k+1}}{k+1}$$

and the formula (4.1) is simply

$$\sum_{n\geq 1}^{\mathcal{R}} f(n) = \sum_{k\geq 0} \frac{(\Delta^k f)(1)}{k!} \sum_{n\geq 1}^{\mathcal{R}}(n-1)...(n-k)$$

Theorem 18 *Let f analytic for $Re(z) > x_0$ with*

$$|f(z)| \leq Ce^{|z|Log(2)}$$

Let

$$\beta_{k+1} = \int_0^1 x(x-1)...(x-k)dx$$

and

$$\Delta^k f(1) = \sum_{j=0}^{k} f(j+1)C_k^j(-1)^{k-j}$$

then the series $\sum_{k\geq 0} \frac{\beta_{k+1}}{(k+1)!}(\Delta^k f)(1)$ is convergent and

$$\sum_{n\geq 1}^{\mathcal{R}} f(n) = \sum_{k=0}^{\infty} \frac{\beta_{k+1}}{(k+1)!}(\Delta^k f)(1)$$

Proof

This is a consequence of the following theorems of Nörlund (Sur les series d'interpolation, Gauthier-Villars, 1926):

Theorem 1 of Nörlund

Let $x_0 < 1$.

If the series $\sum_{n\geq 0} a_n \frac{(x_0-1)...(x_0-n)}{n!}$ is convergent then the function

$$f(z) = \sum_{n=0}^{+\infty} a_n \frac{(z-1)...(z-n)}{n!}$$

is analytic for $Re(z) > x_0$ and

$$|f(z)| \leq Ce^{|z|\frac{\pi}{2}} |z|^{x_0+\frac{1}{2}}$$

Theorem 2 of Nörlund

If a function f is analytic for $Re(z) > x_0$ and verify

$$|f(z)| \leq Ce^{|z| Log(2)}$$

then for $Re(z) > \sup(x_0, 1/2)$ we have

$$f(z) = \sum_{n=0}^{+\infty} \Delta^n f(1) \frac{(z-1)...(z-n)}{n!}$$

this expansion is uniformly convergent for $Re(z) > \sup(x_0, 1/2) + \varepsilon$

By the Theorem 2 of Nörlund we have

$$f(x) = \sum_{k\geq 0} \frac{(\Delta^k f)(1)}{k!} (x-1)...(x-k)$$

this expansion being uniformly convergent in every compact of $Re(z) > \sup(x_0, 1/2)$. The series

$$\sum_{k\geq 0} \frac{(\Delta^k f)(1)}{(k+1)!} (x-1)...(x-(k+1))$$

is also convergent for $Re(x) > \sup(x_0, 1/2)$. This can be proved by

$$\frac{(\Delta^k f)(1)}{(k+1)!} (x-1)...(x-(k+1)) = \frac{(\Delta^k f)(1)}{k!} (x-1)...(x-k) \frac{x-(k+1)}{k+1}$$

and applying the classical summation by part. Thus this series define an analytic function

$$R(x) = -\sum_{k=0}^{+\infty} \frac{(\Delta^k f)(1)}{(k+1)!} (x-1)...(x-(k+1))$$

for $Re(x) > \alpha = \sup(x_0, 1/2)$. This function verify

$$R(x) - R(x+1) = f(x)$$

and by the Theorem 1 of Nörlund we have

$$|R(x)| \leq Ce^{|x|\frac{\pi}{2}} |x|^{\alpha+\frac{1}{2}}$$

Thus R is in $\mathcal{O}^{\pi}$, it suffices now to define R_f by

$$R_f(x) = R(x) - \int_1^2 R(t)dt$$

By uniform convergence of the series defining R on the interval $[1,2]$ we get the result.
□

Example
If $f(x) = \frac{1}{x}$ then

$$\sum_{n\geq 0} \Delta^n f(1) \frac{t^n}{n!} = e^{-t} \sum_{k\geq 0} \frac{t^k}{(k+1)!} = \frac{e^{-t}}{t}(e^t - 1) = \frac{1-e^{-t}}{t}$$

thus

$$\Delta^n f(1) = \frac{(-1)^n}{n+1}$$

and

$$\gamma = \sum_{n\geq 1}^{\mathcal{R}} \frac{1}{n} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} \frac{\beta_{k+1}}{(k+1)!}$$

Remark
Note that for any integer $m \geq 1$ we have if $g(x) = f(x+m)$

$$\Delta^n g(1) = \sum_{k=0}^{n} f(k+m+1) C_n^k (-1)^{n-k} = \Delta^n f(m+1)$$

and by the shift property

$$\sum_{n\geq 1}^{\mathcal{R}} f(n) = \sum_{n\geq 1}^{\mathcal{R}} g(n) + \sum_{n=1}^{m} f(n) - \int_1^{m+1} f(x)dx$$

thus for any integer $m \geq 1$ we get

$$\boxed{\sum_{n\geq 1}^{\mathcal{R}} f(n) = \sum_{k=0}^{\infty} \frac{\beta_{k+1}}{(k+1)!} (\Delta^k f)(m+1) + \sum_{n=1}^{m} f(n) - \int_1^{m+1} f(x)dx}$$

This can be used in some cases to get numerical evaluations of the Ramanujan sums, for example if we set $f(x) = \frac{1}{(x+1)Log(x+1)}$ we get with $m = 20$

$$\sum_{n\geq 1}^{\mathcal{R}} \frac{1}{(n+1)Log(n+1)} = 0.42816572487123...$$

The case of alternating series
For the use of Δ in the case of summation of alternating series we write the equation $A(x) + A(x+1) = f(x)$ in the form

$$(2I + \Delta)A = f$$

Thus

$$A = \frac{1}{2} \frac{I}{I + \frac{1}{2}\Delta} f$$

This gives formally

$$\sum_{n\geq 1}^{\mathcal{R}} (-1)^{n-1} f(n) = \sum_{n\geq 0} \frac{(-1)^n}{2^{n+1}} (\Delta^n f)(1)$$

and we see that the Ramanujan summation of alternating series is simply the classical Euler summation which is defined by

$$\sum_{n\geq 0}^{Euler} v_n = \sum_{n=0}^{+\infty} \frac{1}{2^{n+1}} \sum_{k=0}^{n} C_n^k v_k$$

Chapter 5

A general algebraic view on summation of series

5.1 Introduction

The Ramanujan summation differs of the classical summations methods by the fact that for convergent series the Ramanujan summation does not give the usual sum. And also there is the shift property which seems very strange for a summation procedure. Thus it is necessary to give a general algebraic formalism to unify Ramanujan summation and the classical methods of summation of series.

To introduce this formalism we begin with the analysis of the example of Borel summation.

The Borel summation is formally given by the interversion formula

$$\sum_{n\geq 0} a_n = \sum_{n=0}^{+\infty} a_n \int_0^{+\infty} e^{-t}\frac{t^n}{n!}\, dt = \int_0^{+\infty} e^{-t}(\sum_{n=0}^{+\infty} a_n \frac{t^n}{n!})\, dt$$

More precisely let a complex sequence (a_n), then if the series

$$f(x) = \sum_{n=0}^{+\infty} a_n \frac{x^n}{n!}$$

is convergent for x near 0 then the function f is analytic near 0 and such that for all $n \geq 0$

$$a_n = \partial^n f(0)$$

And formally we get

$$\sum_{n\geq 0} a_n = \sum_{n\geq 0} \partial^n f(0) = (\sum_{n\geq 0} \partial^n f)(0) = ((I-\partial)^{-1} f)(0)$$

Thus $\sum_{n\geq 0} a_n = R(0)$ with R solution of the equation

$$(I-\partial)R = f$$

Assume that f has an analytic continuation near $[0,+\infty[$ and take the solution

$$R(x) = e^x \int_x^{+\infty} e^{-t} f(t)dt$$

if this integral is convergent, thus we get formally

$$\sum_{n\geq 0} a_n = R(0) = \int_0^{+\infty} e^{-t} f(t)dt$$

The problem is that the differential equation $(I-\partial)R = f$ has an infinity of solutions thus we must introduce a condition to get a unique solution.

Let E the space of complex analytic functions f near $[0, +\infty[$ such that $e^{-x}f(x)$ has a finite limit when $x \to +\infty$.

And let

$$\begin{aligned} D(f)(x) &= \frac{df}{dx}(x) = \partial f(x) \\ v_0(f) &= f(0) \\ v_\infty(f) &= \lim_{x \to +\infty} e^{-x} f(x) \end{aligned}$$

A sequence (a_n) has the generating function $f \in E$ if

$$a_n = \partial^n f(0)$$

Since f is analytic near 0 then in a small disk $D(0, \rho)$ we have

$$f(x) = \sum_{n=0}^{+\infty} \partial^n f(0) \frac{x^n}{n!}$$

The differential equation

$$R - \partial R = f$$

gives the general solution

$$R(x) = -e^x \int_0^x e^{-t} f(t)dt + Ke^x$$

The condition $R \in E$ is equivalent to the convergence of the integral $\int_\alpha^{+\infty} e^{-t} f(t)dt$ and the condition $v_\infty(R) = 0$ gives

$$K = \int_\alpha^{+\infty} e^{-t} f(t)dt$$

Finally we see that the equation $R - D(R) = f$ with the condition $v_\infty(R_f) = 0$ gives the unique solution

$$R(x) = e^x \int_x^{+\infty} e^{-t} f(t)dt$$

Thus the series $\sum_{n \geq 0} a_n$ is Borel-summable if the the series $\sum_{n \geq 0} a_n \frac{x^n}{n!}$ is convergent for x near 0 and define by analytic continuation a function $f \in E$ then

$$\sum_{n \geq 0}^{\mathcal{B}} a_n = v_0(R) = \int_0^{+\infty} e^{-t} f(t)dt$$

5.2 An algebraic formalism

Let a $\mathbb{C}$-vector space E with a linear operator $D : E \to E$ and two linear operators $v_0, v_\infty : E \to \mathbb{C}$ such that :

(*) The solutions of $Dg = g$ form a one dimensional subspace of E generated by $\alpha \in E$ with

$$v_0(\alpha) = v_\infty(\alpha) = 1$$

(**) If $v_0(D^n g) = 0$ for all $n \geq 0$ then $g = 0$

Remark

By the property (*) we have

$$\text{If } Dg = g \text{ and } v_\infty(g) = 0 \text{ then } g = 0$$

Definition
If (a_n) is a complex sequence then we say that (a_n) is generated by $f \in E$ if

$$a_n = v_0(D^n f) \text{ for all } n \geq 0$$

then by (**) this element f is unique.

Formally the sum $\sum_{n\geq 0} a_n$ is defined by

$$\sum_{n\geq 0} a_n = \sum_{n\geq 0} v_0(D^n f) = v_0(\sum_{n\geq 0} D^n f) = v_0((I-D)^{-1} f)$$

thus

$$\sum_{n\geq 0} a_n = v_0(R) \text{ with } (I-D)R = f$$

If R_1 and R_2 are solutions of the equation $R - DR = f$ then $g = R_1 - R_2$ is solution of $Dg = g$ thus to get unicity we use the preceeding remark and we add to the equation $R - DR = f$ the condition $v_\infty(R) = 0$.

Definition
Let $\mathcal{T} = (E, D, v_0, v_\infty)$ as above. Let (a_n) a complex sequence generated by $f \in E$ and assume that there is $R_f \in E$ solution of

$$R_f - DR_f = f \text{ with } v_\infty(R_f) = 0$$

then this R_f is unique and we define

$$\sum_{n\geq 0}^{\mathcal{T}} a_n = v_0(R_f)$$

Remark
Since $R_f - DR_f = f$ we have for any positive integer k

$$D^k R_f - D^{k+1} R_f = D^k f$$

thus we get for any integer $N \geq 1$

$$R_f - D^N R_f = \sum_{k=0}^{N-1} D^k f$$

This gives

$$\sum_{n\geq 0}^{\mathcal{T}} a_n = \sum_{k=0}^{N-1} a_k + v_0(D^N R_f)$$

and also

$$v_\infty(D^N R_f) = -\sum_{k=0}^{N-1} v_\infty(D^k f)$$

Examples
1) The usual Cauchy summation
Let E the vector space of convergent complex sequences $u = (u_n)_{n\geq 0}$. Let the operators

$$\begin{aligned} D &: (u_n) \mapsto (u_1, u_2, u_3, ...) \\ v_0 &: (u_n) \mapsto u_0 \\ v_\infty &: (u_n) \mapsto \lim_{n\to+\infty} u_n \end{aligned}$$

Since $v_0(D^n f) = f_n$ a complex sequence $(a_n) \in E$ has the generating element

$$f = (a_n)$$

The equation $R - D(R) = f$ is

$$(R_0, R_1, R_2, ...) - (R_1, R_2, R_3, ...) = (a_0, a_1, a_2, ...)$$

this gives $R_n - R_{n+1} = a_n$ thus

$$R_{n+1} = R_0 - \sum_{k=0}^{n} a_k$$

and the condition $v_\infty(R) = 0$ gives $R_{n+1} \to 0$ thus

$$\sum_{k=0}^{n} a_k \to R_0 = v_0(R)$$

We see that the series $\sum_{n\geq 0} a_n$ is Cauchy-summable if $\sum_{k=0}^{n} a_k$ has a finite limit when $n \to +\infty$ and we write

$$\sum_{n=0}^{+\infty} a_n = v_0(R) = \lim_{n\to+\infty} \sum_{k=0}^{n} a_k$$

we said simply that the series $\sum_{n\geq 0} a_n$ is convergent.

2) The Ramanujan summation.

We have defined the Ramanujan summation for series $\sum_{n\geq 1}^{\mathcal{R}} a_n$, indexed by $n \geq 1$, these can be seen series indexed by $n \geq 0$ if we let

$$\sum_{n\geq 1}^{\mathcal{R}} a_n = \sum_{n\geq 0}^{\mathcal{R}} b_n \text{ with } b_n = a_{n+1}$$

Let the space $E = \mathcal{O}^\pi$ and the operators

$$\begin{aligned} Df(x) &= f(x+1) \\ v_0(f) &= f(1) \\ v_\infty(f) &= \int_1^2 f(t)dt \end{aligned}$$

The condition (*) is simply that a function $f \in \mathcal{O}^\pi$ which is 1-periodic is constant and we get $\alpha = 1$. The condition (**) is a consequence of Carlson theorem's.

A complex sequence $(a_n)_{n\geq 1} = (b_n)_{n\geq 0}$ has the generating element $f \in \mathcal{O}^\pi$ if for all integer $n \geq 0$ we have

$$b_n = v_0(D^n f) = f(n+1)$$

that is for all integer $n \geq 1$ we have

$$a_n = f(n)$$

The equation

$$R_f - DR_f = f$$

is simply our difference equation

$$R_f(x) - R_f(x+1) = f(x)$$

and the condition $v_\infty(R_f) = 0$ is simply the condition $\int_1^2 R_f(t)dt = 0$ that we have in the Ramanujan summation.

We write as usual

$$\sum_{n\geq 1}^{\mathcal{R}} a_n = v_0(R_f) = R_f(1)$$

Properties

1) The linearity property
We have clearly for $C \in \mathbb{C}$

$$\boxed{\sum_{n\geq 0}^{\mathcal{T}} a_n + Cb_n = \sum_{n\geq 0}^{\mathcal{T}} a_n + C\sum_{n\geq 0}^{\mathcal{T}} b_n}$$

2) The shift property
If (a_n) is generated by f, for any integer $N \geq 1$ we have the shift property

$$\boxed{\sum_{n\geq 0}^{\mathcal{T}} a_{n+N} = \sum_{n\geq 0}^{\mathcal{T}} a_n - \sum_{n=0}^{N-1} a_n + \sum_{k=0}^{N-1} v_\infty(D^k f)}$$

Proof
If (a_n) is generated by $f \in E$ then for any integer $N \geq 1$

$$a_{n+N} = v_0(D^{n+N} f) = v_0(D^n(D^N f))$$

thus the sequence (a_{n+N}) is generated by $D^N f \in E$.
The equation $R_f - DR_f = f$ gives

$$D^N R_f - D(D^N R_f) = D^N f$$

but generally we don't have $v_\infty(D^N R_f) = 0$.
If we add $-v_\infty(D^N R_f)\alpha$ to $D^N R_f$ we get

$$[D^N R_f - v_\infty(D^N R_f)\alpha] - D[D^N(R_f) - v_\infty(D^N R_f)\alpha] = D^N f$$
$$v_\infty[D^N R_f + -v_\infty(D^N R_f)\alpha] = 0$$

Thus

$$\sum_{n\geq 0}^{\mathcal{T}} a_{n+N} = v_0[D^N(R_f) - v_\infty(D^N R_f)\alpha] = v_0(D^N(R_f)) - v_\infty(D^N R_f)$$

Since and by the preceeding remark we have

$$-v_\infty(D^N R_f) = \sum_{k=0}^{N-1} v_\infty(D^k f)$$

this gives the shift property

$$\sum_{n\geq 0}^{\mathcal{T}} a_{n+N} = \sum_{n\geq 0}^{\mathcal{T}} a_n - \sum_{n=0}^{N-1} a_n + \sum_{k=0}^{N-1} v_\infty(D^k f)$$

□

In the special case $N = 1$ we get

$$\boxed{\sum_{n\geq 0}^{\mathcal{T}} a_{n+1} = \sum_{n\geq 0}^{\mathcal{T}} a_n - a_0 + v_\infty(f)}$$

Note that if we have the additional property:

(***) If $v_\infty(g) = 0$ then $v_\infty(Dg) = 0$

then $v_\infty(R_f)=0$ gives $v_\infty(D^N R_f)=0$ for all positive integer thus the shift property is the usual property

$$\sum_{n\geq 0}^{\mathcal{T}} a_{n+N} = \sum_{n\geq 0}^{\mathcal{T}} a_n - \sum_{n=0}^{N-1} a_n$$

This is the case for most summations but not for the Ramanujan summation.

Remark: Generalized limit
If we define the generalized limit of a sequence (a_n) by

$$\sum_{n\geq 0}^{\mathcal{T}} (a_n - a_{n+1}) = a_0 - \lim_{\mathcal{T}} a_n$$

then the shift property gives

$$\lim_{\mathcal{T}} a_n = v_\infty(f)$$

Thus the generalized limit associated to the summation is nothing else than v_∞.
To see that the summation is related to the partial sums by this generalized limit let the sequence (r_n) defined by

$$\begin{aligned} r_0 &= \sum_{n\geq 0}^{\mathcal{T}} a_n \\ r_n &= r_0 - \sum_{k=0}^{n-1} a_k \text{ for } n \geq 1 \end{aligned}$$

Since we have $r_0 = v_0(R_f)$ and (by the preceding remark) for any integer $n \geq 1$

$$r_n = v_0(D^n R_f)$$

we see that the sequence $(r)_n$ is generated by R_f thus

$$\lim_{\mathcal{T}} r_n = v_\infty(R_f) = 0$$

and finally we get

$$\lim_{\mathcal{T}} (r_0 - \sum_{k=0}^{n-1} a_k) = 0$$

5.3 Examples

1) Cesaro summation
Let E the vector space of complex sequences $u=(u_n)_{n\geq 0}$ such that

$$\lim_{n\to+\infty} \frac{u_0+...+u_{n-1}}{n} \text{ is finite}$$

And let the operators

$$\begin{aligned} D &: (u_n) \mapsto (u_1, u_2, u_3, ...) \\ v_0 &: (u_n) \mapsto u_0 \\ v_\infty &: (u_n) \mapsto \lim_{n\to+\infty} \frac{u_0+...+u_{n-1}}{n} \end{aligned}$$

A sequence $(a_n) \in E$ is generated by

$$f = (a_n)$$

The equation $R - DR = f$ is $R_k - R_{k+1} = a_k$ thus

$$R_0 - R_n = \sum_{k=0}^{n-1} a_k = S_{n-1}$$

this gives

$$\begin{aligned} R_0 - R_1 &= a_0 = S_0 \\ R_0 - R_2 &= a_0 + a_1 = S_1 \\ & \dots \\ R_0 - R_n &= \sum_{k=0}^{n-1} a_k = S_{n-1} \end{aligned}$$

thus

$$R_0 - \frac{R_1 + \dots + R_n}{n} = \frac{S_0 + \dots + S_{n-1}}{n}$$

We see that $R = (R_n) \in E$ if and only if $\frac{S_0+\dots+S_{n-1}}{n}$ has a finite limit when $n \to +\infty$. Since

$$\frac{S_0 + \dots + S_{n-1}}{n} = \frac{S_0 + \dots + S_{n-2}}{n-1}\frac{n-1}{n} + \frac{S_{n-1}}{n}$$

this implies that $\frac{S_{n-1}}{n}$ has a finite limit when $n \to +\infty$. Thus automatically we get $f \in E$.

The condition $v_\infty(R) = 0$ is $\lim_{n\to+\infty} \frac{R_1+\dots+R_n}{n} = 0$ this gives

$$R_0 = \lim_{n\to+\infty} \frac{S_0 + \dots + S_{n-1}}{n}$$

Thus series $\sum_{n\geq 0} a_n$ is Cesaro-summable if the sequence $(\frac{S_0+\dots+S_{n-1}}{n})$ has a finite limit when $n \to +\infty$ and we write

$$\sum_{n=0}^{\mathcal{C}} a_n = v_0(R) = R_0 = \lim_{n\to+\infty} \frac{S_0 + \dots + S_{n-1}}{n}$$

Remark

This can be generalized to Toeplitz summations in this case we let

$$v_\infty(u_n) = \lim_{t\to+\infty} \sum_{n=0}^{+\infty} a_{t,n} u_n$$

where $(a_{t,n})_{n\in\mathbb{N}}$ a family of sequences indexed by $t \in \mathbb{N}$ such that:

a) For all $t \in \mathbb{N}$ the series $\sum_{n\geq 0} |a_{t,n}|$ is convergent. There is $M > 0$ such that $\sum_{n=0}^{+\infty} |a_{t,n}| \leq M$ for all $t \in \mathbb{N}$.

b) $\lim_{t\to\alpha} \sum_{n=0}^{+\infty} a_{t,n} = 1$.

c) $\lim_{t\to\alpha} a_{t,n} = 0$ for all n.

2) Euler summation

Let E the vector space of complex sequences $u = (u_n)$ with $\lim_{n\to+\infty} \frac{u_n}{2^n}$ is finite. And let the operators

$$\begin{aligned} D &: (u_n) \mapsto (u_{n+1} - u_n) \\ v_0 &: (u_n) \mapsto u_0 \\ v_\infty &: \lim_{n\to+\infty} \frac{u_n}{2^n} \end{aligned}$$

We have for all $n \geq 0$

$$v_0(D^n f) = \sum_{k=0}^{n} C_n^k f_k (-1)^{n-k}$$

thus we get

$$f_n = \sum_{k=0}^{n} v_0(D^k f)C_n^k$$

Let a complex sequence (a_n) and the sequence $f = (f_n)$ defined by

$$f_n = \sum_{k=0}^{n} a_k C_n^k$$

Assume that $f \in E$, we have $v_0(D^n f) = a_n$, thus the sequence (a_n) is generated by f.
The equation $R - DR = f$ is

$$2R_k - R_{k+1} = f_k$$

this gives

$$\begin{aligned} R_0 &= \frac{1}{2}R_1 + \frac{1}{2}f_0 \\ \frac{1}{2}R_1 &= \frac{1}{2^2}R_2 + \frac{1}{2^2}f_1 \\ \frac{1}{2^2}R_2 &= \frac{1}{2^3}R_3 + \frac{1}{2^3}f_2 \\ &\ \ \cdots \end{aligned}$$

we get

$$R_0 = \frac{1}{2}f_0 + \frac{1}{2^2}f_1 + ... + \frac{1}{2^{n+1}}f_n - \frac{1}{2^n}R_n$$

We have $R \in E$ if and only if the sequence $(\frac{R_n}{2^n})$ has a finite limit, in this case the series $\sum_{n\geq 0} \frac{1}{2^{n+1}} f_n$ is convergent and this implies that $\lim_{n\to+\infty} \frac{1}{2^n} f_n = 0$ thus we have automaticaly $f \in E$.
The condition

$$0 = v_\infty(R) = \lim_{n\to+\infty} \frac{1}{2^n}R_n$$

gives

$$R_0 = \lim_{n\to+\infty} \frac{1}{2}f_0 + \frac{1}{2^2}f_1 + ... + \frac{1}{2^{n+1}}f_n$$

Finally the series $\sum_{n\geq 0} a_n$ is Euler-summable if the series

$$\sum_{k\geq 0} \frac{1}{2^{n+1}} (\sum_{k=0}^{n} a_k C_n^k)$$

is convergent and we write

$$\sum_{n=0}^{\mathcal{E}} a_n = v_0(R) = R_0 = \sum_{k=0}^{+\infty} \frac{1}{2^{n+1}} (\sum_{k=0}^{n} a_k C_n^k)$$

3) Abel summation
Let E the vector space of analytic functions on $]-1,1[$ such that

$$\lim_{x\to 1} (1-x)f(x) \text{ is finite}$$

Let the operators

$$\begin{aligned} Df(x) &= \frac{f(x) - f(0)}{x} \text{ if } x \neq 0 \text{ and } Df(0) = f'(0) \\ v_0(f) &= f(0) \\ v_\infty(f) &= \lim_{x\to 1} (1-x)f(x) \end{aligned}$$

Since $f \in E$ is analytic on $]-1,1[$ we can write

$$f(x) = \sum_{m=0}^{+\infty} \alpha_m x^m \text{ with } \alpha_m = \frac{\partial^m f(0)}{m!}$$

and we have

$$Df(x) = \sum_{m=0}^{+\infty} \alpha_{m+1} x^m$$

thus for all $n \geq 0$

$$D^n f(x) = \sum_{m=0}^{+\infty} \alpha_{m+n} x^m$$

and we get

$$v_0(D^n f) = \alpha_n = \frac{\partial^n f(0)}{n!}$$

Let a sequence (a_n) and assume that the series

$$f(x) = \sum_{n=0}^{+\infty} a_n x^n$$

is convergent for $x \in]-1,1[$ and define a function $f \in E$.
Then

$$a_n = \frac{\partial^n f(0)}{n!} = v_0(D^n f)$$

and the sequence (a_n) is generated by f.
The equation $R - DR = f$ gives

$$R(x) = \frac{1}{1-x}(R(0) - xf(x))$$

Thus R is analytic on $]-1,1[$ and $R \in E$ if and only if $\lim_{x \to 1} f(x)$ is finite then automatically $f \in E$.
The condition $v_\infty(R) = 0$ gives

$$R(0) = \lim_{x \to 1} f(x)$$

Finally if the series $\sum_{n \geq 0} a_n x^n$ is convergent for all $x \in [-1,1[$ and if $\lim_{x \to 1} \sum_{n=0}^{+\infty} a_n x^n$ is finite, then $f \in E$ and $\sum_{n \geq 0} a_n$ is Abel-summable

$$\sum_{n=0}^{\mathcal{A}} a_n = v_0(R) = R(0) = \lim_{x \to 1} \sum_{n=0}^{+\infty} a_n x^n$$

Chapter 6

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Chapter 7

Appendix

7.1 Euler-MacLaurin and Euler-Boole formulas

7.1.1 A Taylor formula

The classical Taylor formula

$$f(x) = \sum_{k=0}^{m} \partial^k f(0) \frac{x^k}{k!} + \int_0^x \frac{(x-t)^m}{m!} \partial^{m+1} f(t) dt$$

can be generalized if we replace the polynomial $\frac{x^k}{k!}$ by other polynomials.

Définition
Let μ a linear form on $C^0(\mathbb{R})$ such that $\mu(1) = 1$, we define the polynomials (P_n) by:

$$\begin{aligned} P_0 &= 1 \\ \partial P_n &= P_{n-1} \text{ , } \mu(P_n) = 0 \text{ for } n \geq 1 \end{aligned}$$

Generating function for the P_n
We have formaly

$$\partial_x(\sum_{k\geq 0} P_k(x)z^k) = \sum_{k\geq 1} P_{k-1}(x)z^k = z\sum_{k\geq 0} P_k(x)z^k$$

thus

$$\sum_{k\geq 0} P_k(x)z^k = Ce^{xz}$$

and

$$\begin{aligned} \mu_x(\sum_{k\geq 0} P_k(x)z^k) &= \sum_{k\geq 0} \mu_x(P_k(x))z^k = 1 \\ \mu_x(\sum_{k\geq 0} P_k(x)z^k) &= \mu_x(Ce^{xz}) = C\mu_x(e^{xz}) \end{aligned}$$

this gives $C = \frac{1}{\mu_x(e^{xz})}$. Thus the generating function of the sequence (P_n) is

$$\sum_n P_n(x)z^n = e^{xz}/M_\mu(z)$$

where the function M_μ is defined by $M_\mu(z) = \mu_x(e^{xz})$.

Examples
1) $\mu(f) = f(0)$, $P_n(x) = \frac{x^n}{n!}$, $M_\mu(z) = 1$, $\sum P_n(x)z^n = e^{xz}$

2) $\mu(f)=\int_0^1 f(t)dt$, $P_n(x)=\frac{B_n(x)}{n!}$, $M_\mu(z)=\sum\int_0^1\frac{t^n}{n!}z^n dt=\sum\frac{z^n}{(n+1)!}=\frac{1}{z}(e^z-1)$

$$\sum_n \frac{B_n(x)}{n!}z^n=\frac{ze^{xz}}{e^z-1}$$

The $B_n(x)$ are the Bernoulli polynomials and the $B_n=B_n(0)$ the Bernoulli numbers. With the generating function we verify that $B_0=1, B_1=-1/2, B_{2n+1}=0$ if $n\geq 1$, $B_n(1-x)=(-1)^n B_n(x)$.

3) $\mu(f)=\frac{1}{2}(f(0)+f(1))$, $P_n(x)=\frac{E_n(x)}{n!}$

$$\sum_n \frac{E_n(x)}{n!}z^n=\frac{2e^{xz}}{e^z+1}$$

The $E_n(x)$ are the Euler polynomials and we call $E_n=E_n(0)$ the Euler numbers.
With the generating function we verify that $E_0=1, E_1=-1/2$ if $n\geq 1$, $E_n(1-x)=-1)^n E_n(x)$.

Taylor formula
Let f be a $C^\infty(\mathbb{R})$ function, then we have

$$f(x)=f(y)+\int_y^x \partial P_1(x+y-t)\partial f(t)dt$$

and by integration by parts we get for every $m\geq 1$

$$f(x)=f(y)+\sum_{k=1}^m(P_k(x)\partial^k f(y)-P_k(y)\partial^k f(x))+\int_y^x P_m(x+y-t)\partial^{m+1}f(t)dt$$

Applying μ at this function as a function of y this gives a **general Taylor formula**: for every $m\geq 0$

$$f(x)=\sum_{k=0}^m \mu_y(\partial^k f(y))P_k(x)+\mu_y(\int_y^x P_m(x+y-t)\partial^{m+1}f(t)dt)$$

7.1.2 Euler-MacLaurin formula

We can transform the Taylor formula to get a summation formula. Taking $x=0$ we get

$$f(0)=\sum_{k=0}^m \mu_y(\partial^k f(y))P_k(0)-\mu_y(\int_0^y P_m(y-t)\partial^{m+1}f(t)dt)$$

In the case of $\mu: f\mapsto\int_0^1 f(t)dt$ we have

$$f(0)=\sum_{k=0}^m\frac{B_k}{k!}\partial^{k-1}f]_0^1-\int_0^1(\int_0^y\frac{B_m(y-t)}{m!}\partial^{m+1}f(t)dt)dy$$

Replacing m by $2m$ and with $B_1=-1/2$ and $B_{2k+1}=0$, we get

$$f(0)=\int_0^1 f(t)dt+\frac{1}{2}(f(0)-f(1))+\sum_{k=1}^m\frac{B_{2k}}{(2k)!}\partial^{2k-1}f]_0^1-\int_0^1(\int_0^y\frac{B_{2m}(y-t)}{(2m)!}\partial^{2m+1}f(t)dt)dy$$

The last integral can easily be evaluated by Fubini, we obtain

$$f(0)=\int_0^1 f(t)dt+\frac{1}{2}(f(0)-f(1))+\sum_{k=1}^m\frac{B_{2k}}{(2k)!}\partial^{2k-1}f]_0^1+\int_0^1\frac{B_{2m+1}(t)}{(2m+1)!}\partial^{2m+1}f(t)dt \qquad (7.1)$$

Let j be a positive integer, by replacing f by $x\mapsto f(j+x)$ in the last formula we obtain

$$f(j)=\int_j^{j+1}f(t)dt+\frac{1}{2}(f(j)-f(j+1))+\sum_{k=1}^m\frac{B_{2k}}{(2k)!}\partial^{2k-1}f]_j^{j+1}+\int_j^{j+1}\frac{b_{2m+1}(t)}{(2m+1)!}\partial^{2m+1}f(t)dt$$

where $b_{2m+1}(t) = B_{2m+1}(t-[t])$.

Summing these relations on j from 1 to $n-1$, we get for $f \in C^\infty(]0,\infty[)$ the **Euler-MacLaurin formula**

$$f(1)+...+f(n) = \int_1^n f(x)dx + \frac{f(1)+f(n)}{2} \tag{7.2}$$

$$+\sum_{k=1}^{m} \frac{B_{2k}}{(2k)!}[\partial^{2k-1}f]_1^n \tag{7.3}$$

$$+\int_1^n \frac{b_{2m+1}(x)}{(2m+1)!}\partial^{2m+1}f(x)dx \tag{7.4}$$

7.1.3 Euler-Boole formula

In the case of the Euler polynomials, the formula

$$f(0) = \sum_{k=0}^{m} \mu_y(\partial^k f(y))P_k(0) - \mu_y(\int_0^y P_m(y-t)\partial^{m+1}f(t)dt)$$

gives

$$f(0) = \sum_{k=0}^{m} \frac{1}{2}(\partial^k f(0) + \partial^k f(1))\frac{E_k}{k!} - \frac{1}{2}\int_0^1 \frac{(-1)^m E_m(t)}{m!}\partial^{m+1}f(t)dt$$

Let j be a positive integer, by replacing f by $x \mapsto f(j+x)$ in the last formula we obtain

$$\begin{aligned} f(j) &= \sum_{k=0}^{m} \frac{1}{2}(\partial^k f(j) + \partial^k f(j+1))\frac{E_k}{k!} \\ &\quad -\frac{1}{2}\int_j^{j+1} \frac{(-1)^m E_m}{m!}(t-j)\,\partial^{m+1}f(t)\,dt \end{aligned}$$

Define

$$e_m(t) = (-1)^{[t]}(-1)^m E_m(t-[t])$$

we obtain by summation on j, the **Euler-Boole summation formula**

$$\begin{aligned} f(1)-f(2)+...+(-1)^{n-1}f(n) &= \frac{1}{2}\sum_{k=0}^{m} \partial^k f(1)\frac{E_k}{k!} \\ &\quad +\frac{(-1)^{n-1}}{2}\sum_{k=0}^{m} \partial^k f(n+1)\frac{E_k}{k!} \\ &\quad +\frac{1}{2}\int_1^{n+1} \frac{1}{m!}e_m(t)\,\partial^{m+1}f(t)\,dt \end{aligned}$$

Chapter 8

The chapter VI of the second Ramanujan Notebook

In the chapter VI of his second notebook Ramanujan gives the defintion of the constant of a series. We give here an exact copy of this chapter.

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CHAPTER VI

Let $f(1)+f(2)+f(3)+f(4)+.....+f(x)=\phi(x)$, *then*

$\phi(x)=c+\int f(x)dx+\frac{1}{2}f(x)+\frac{B_2}{\lfloor 2}f'(x)-\frac{B_4}{\lfloor 4}f'''(x)+$

$\frac{B_6}{\lfloor 6}f^V(x)+\frac{B_8}{\lfloor 8}f^{VII}(x)+\&c$

sol. $\phi(x)-\phi(x-1)=f(x)$; *apply V 1.* (*)

N.B. *By giving any value to* x, c *can be found. R.S. is not a terminating series except in some special cases. Consequently no constant can be found in* $\frac{1}{2}f(x)+\frac{B_2}{\lfloor 2}f'(x)-\frac{B_4}{\lfloor 4}f'''(x)+\&c$ *except in those special cases. If R.S. be a terminating series, it must be some integral function of* x. *In this case there is no possibility of a constant (according to the ordinary sense) in* $\phi(x)$ *; for*

$\phi(1)=f(1)+\phi(0)$: *But* $\phi(1)=f(1)\ \therefore\ \phi(0)$ *is always* 0

whether $\phi(x)$ *is rational or irrational.* $\therefore$ *When* $\phi(x)$ *is a rational integral function of* (x) *it must be divisible and hence no constant but* 0 *can exist. The algebraic constant of a se -ries is the constant obtained by completing the remaining part in the above theorem. We can substitute this constant which is like the cen -tre of gravity of a body instead of its di- vergent infinite series.*

(*) V 1.*If* $f(x+h)-f(x)=h\phi'(x)$, *then*
$f(x)=\phi(x)-\frac{h}{2}\phi'(x)+\frac{B_2}{\lfloor 2}h^2\phi''(x)-\frac{B_4}{\lfloor 4}h^4\phi^{IV}(x)+\&c$
If $f(x+h)+f(x)=h\phi'(x)$, *then*
$f(x)=\frac{h}{2}\phi'(x)-(2^2-1)\frac{B_2}{\lfloor 2}h^2\phi''(x)+(2^4-1)\frac{B_4}{\lfloor 4}h^4\phi^{IV}(x)-\&c$
Sol. If we write e^x *for* $\phi(x)$, *we see that the coeff*^{ts} *in R.S. are the same as those in the expansion of* $\frac{h}{e^h-1}$ *and* $\frac{h}{e^h+1}$ *respectively.*

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E.G. The constant of the series $1+1+1+\&c = -\frac{1}{2}$; *for the sum to* x *terms* $= x = c + \int 1dx + \frac{1}{2}$ $\therefore c = -\frac{1}{2}$

We may also find the constant thus
$c = 1+2+3+4+\&c$
$\therefore 4c = 4+8+\&c$
$\therefore -3c = 1-2+3-4+\&c = \frac{1}{(1+1)^2} = \frac{1}{4}$
$\therefore c = -\frac{1}{12}$

2. $\phi(x) + \sum_{n=0}^{n=\infty} \frac{B_n}{\lfloor n} f^{n-1}(x) \cos\frac{\pi n}{2} = 0$

Sol. Let $\frac{B_n}{\lfloor n}\psi(n)$ *be the coeff*^{t}. *of* $f^{n-1}(x)$ *then we*

see $\psi(0) = 1$, $\psi(2) = -1$, $\psi(4) = 1$, $\psi(6) = -1$ *&c*
$\psi(3) = 0$, $\psi(5) = 0$, $\psi(7) = 0$, $\frac{B_1}{\lfloor 1}\psi(1) = \frac{1}{2}$; *but* $B_1 = \infty$
$\therefore \psi(1) = 0$. *Again by V 26 cor 2. (*) we have*
$\pi(n-1)B_n = 1$ *when* $n = 1$ $\therefore \frac{B_n\psi(n)}{\lfloor n} = \frac{\pi(n-1)B_n}{\lfloor n} . \frac{\psi(n)}{\pi(n-1)}$
$= \frac{1}{2}$ *when* $n = 1$, *i.e* $\frac{\psi(n)}{\pi(n-1)} = \frac{1}{2}$ *when* $n = 1$ $\therefore$
$\therefore \psi(n) = -\cos\frac{\pi n}{2}$.

3. The sum to a negative number of terms is the sum with the sign changed, calculated backwards from the term previous to the first to the given number of terms with positive sign instead of negative.

Sol. $\phi(x) = f(1) + f(2) + ... + f(n+x)$
$-f(1+x) - f(2+x) - ... - f(n+x)$

(*) *V 26 cor 2.* $\pi n B_{n+1} = 1$ *when* $n = 0$
Sol. $nS_{n+1} = \frac{(2\pi)^n}{\lfloor n+1}\pi n B_{n+1} = 1$ *when* $n = 0$
i.e. $\pi n B_{n+1} = 1$ *when* n *approaches* 0.

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change x *to* $-x$ *and put* $n = x$, *then we have*

$\phi(-x) = \phi(0) - \{f(0) + f(-1) + f(-2) + ... + f(-x+1)\};$

but $\phi(0) = 0.$
E.G. $1 + 2 + 3 + \&c$ *to* -5 *terms*
$= -(0 - 1 - 2 - 3 - 4) = \underline{10}$

4.i. For finding the sum to a fractional num-ber of terms assume the sum to be true al-ways and if there is any difficulty in find-ing $\phi(x)$, *take* n *any integer you choose, find* $\phi(n+x)$ *and then subtract* $\{f(1+x)+ f(2+x) + f(3+x) + ... + f(n+x)\}$ *from the result.*

ii. $\phi(h) = \phi(n) - \{f(1+h) + f(2+h) + ... + f(n+h)\}$
$+hf(n) + \frac{\Sigma h}{\lfloor 1} f'(n) + \frac{\Sigma h^2}{\lfloor 2} f''(n) + \&c$ *where* n
is any integer or infinity.

E.G.1. $1 + \frac{1}{2} + \frac{1}{3} + ... \frac{1}{h}$
$= (1 + \frac{1}{2} + ... + \frac{1}{n}) - (\frac{1}{1+h} + \frac{1}{2+h} + ... + \frac{1}{n+h})$ *when* $n = \infty$
$= c_0 + log_e n - (\frac{1}{1+h} + \frac{1}{2+h} + ... + \frac{1}{n+h})$ *when* $n = \infty$
where c_0 *is the constant of* $\Sigma \frac{1}{n}$

2. $\lfloor h = \frac{n^h}{(1+\frac{h}{1})(1+\frac{h}{2})...(1+\frac{h}{n})}$ *when* $n = \infty.$

sol. $\lfloor h = \frac{\lfloor n+h}{\lfloor n} . \frac{\lfloor n \lfloor h}{\lfloor n+h} = \frac{n^h(1+\frac{1}{n})(1+\frac{2}{n})...(1+\frac{h}{n})}{(1+\frac{h}{1})(1+\frac{h}{2})...(1+\frac{h}{n})}$

$\therefore \lfloor h \div (1 + \frac{1}{n})(1 + \frac{2}{n})...(1 + \frac{h}{n}) = \frac{n^h}{(1+\frac{h}{1})(1+\frac{h}{2})...(1+\frac{h}{n})}$

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iii. $\phi(h) = xf(1) - x^{1+h}f(1+h) + x^2 f(2) - x^2 f(2+h) + \&c$

5. *Def. A series is said to be corrected when its constant is subtracted from it.*
The differential coeff^t. of a series is a corrected series.

i.e $\frac{d\{\phi(1)+\phi(2)+...+\phi(x)\}}{dx} = \phi'(1) + \phi'(2) + ...$
$+\phi'(x) - c'$ *where* c' *is the constant of* $\phi'(1) + \phi'(2)$
$+\phi'(3) + ... + \phi'(x)$.

Sol. In the diff^l. coeff^t. of $\phi(1) + \phi(2) + ... + \phi(x)$ *there can't be any constant. Therefore it should be corrected.*

N.B. If $f(1) + f(2) + ... + f(x)$ *be a convergent series then its constant is the sum of the series*

E.G.1. $\frac{d(1+\frac{1}{2}+...+\frac{1}{x})}{dx} = \frac{1}{(x+1)^2} + \frac{1}{(x+2)^2} + \frac{1}{(x+3)^2} + \&c$

Sol. $\frac{d\Sigma\frac{1}{x}}{dx} = -\frac{1}{1^2} - \frac{1}{2^2} - ... - \frac{1}{x^2} - c$
$= \frac{1}{(x+1)^2} + \frac{1}{(x+2)^2} + \&c$

2. *If* c_0 *be the constant of* $\Sigma\frac{1}{x}$, *then*
$\frac{d\lfloor x}{dx} = \lfloor x(\Sigma\frac{1}{x} - c_0)$

Sol. $\frac{d\lfloor x}{dx} = \lfloor x\frac{d\log_e\lfloor x}{dx} = \lfloor x(\Sigma\frac{1}{x} - c_0)$

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3. $\int_0^x \Sigma\frac{1}{x}dx = \log_e\lfloor x + xc_0.$

4. $\int_0^x (1^{13} + 2^{13} + ... + x^{13})dx = \frac{1}{14}(1^{14} + 2^{14} + ... + x^{14}) - \frac{x}{12}.$

5. $\frac{d(1^{10}+2^{10}+...+x^{10})}{dx} = 10(1^9 + 2^9 + ... + x^9) + \frac{10}{132}.$

6. $\int_0^x (\sqrt{1} + \sqrt{2} + ... + \sqrt{x})dx = \frac{2}{3}(1\sqrt{1} + 2\sqrt{2} + ... + x\sqrt{x})$
$-\frac{x}{4\pi}(\frac{1}{1\sqrt{1}} + \frac{1}{2\sqrt{2}} + \&c).$

6. *If* $f^n(x)$ *stands for the* n *th derivative of* $f(x)$
and c_n *be the constant of* $\{f^n(1) + f^n(2) + ... + f^n(x)\}$
then $\phi(x) = -c_1x - c_2\frac{x^2}{\lfloor 2} - c_3\frac{x^3}{\lfloor 3} - c_4\frac{x^4}{\lfloor 4} - \&c$

Sol. $\phi(x) = \phi(0) + \frac{x}{\lfloor 1}\phi'(0) + \frac{x^2}{\lfloor 2}\phi''(0) + \&c$
But from VI 5 we have $\phi(0) = 0$, $\phi'(0) = -c_1$, $\phi''(0) = -c_2$ $\&c$

E.g. 1. $log_e\lfloor x = -S_1x + \frac{S_2}{2}x^2 - \frac{S_3}{3}x^3 + \&c$ *where* S_n
is the constant of $(\frac{1}{1^n} + \frac{1}{2^n} + \frac{1}{3^n} + \&c).$
2. $\Sigma\frac{1}{x} = S_2x - S_3x^2 + S_4x^3 - \&c$ *where* $S_n = \frac{1}{1^n} + \frac{1}{2^n} + \&c$
N.B. This is very useful in finding $\phi(x)$ *for fractional values of* x.

7. If c'_n *be the constant of*
$f(\frac{1}{n}) + f(\frac{2}{n}) + f(\frac{3}{n}) + ... + f(\frac{x}{n})$, *then*

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$\phi(\frac{x}{n}) + \phi(\frac{x-1}{n}) + \phi(\frac{x-2}{n}) + ... + \phi(\frac{x-n+1}{n}) - nc$
$= f(\frac{1}{n}) + f(\frac{2}{n}) + ... + f(\frac{x}{n}) - c'_n$

Sol. Let $\psi(x) = \phi(\frac{x}{n}) + \phi(\frac{x-1}{n}) + ... + \phi(\frac{x-n+1}{n})$ *then*
$\psi(x) - \psi(x-1) = \phi(\frac{x}{n}) - \phi(\frac{x-n}{n}) = f(\frac{x}{n})$
$\therefore \psi(x)$ *&* $f(\frac{x}{n}) + f(\frac{x-1}{n}) + ... + f(\frac{1}{n})$ *differ only by some constant; hence if these be corrected they must be equal.* $\psi(x)$ *contains* n *terms each each of which is of the form* $\phi(y)$ *whose constant is* c*. The constant of* $\psi(x)$ *is* nc *& the constant of* $f(\frac{1}{n}) + f(\frac{2}{n}) + ... + f(\frac{x}{n})$ *is* c'_n *by our supposition.*

Cor.1. $\phi(-\frac{1}{n}) + \phi(-\frac{2}{n}) + ... + \phi(-\frac{n-1}{n}) = nc - c'_n$
Sol. Put $x = 0$ *in the above theorem.*
2.i. $\phi(-\frac{1}{2}) = 2c - c'_2$.
ii. $c = c_0 = c'_1$.
iii. $\phi(-\frac{1}{3}) + \phi(-\frac{2}{3}) = 3c - c'_3$
iv. $\phi(-\frac{1}{4}) + \phi(-\frac{3}{4}) = 2c + c'_2 - c'_4$.
v. $\phi(-\frac{1}{6}) + \phi(-\frac{5}{6}) = c + c'_2 + c'_3 - c'_4$.

8. $\phi(x - \frac{1}{2}) = c + \int f(x)dx - (1 - \frac{1}{2})\frac{B_2}{\lfloor 2}f'(x) + (1 - \frac{1}{2^3})\frac{B_4}{\lfloor 4}f'''(x)$
$-\&c = \sum_{n=0}^{n=\infty} \left\{(1 - \frac{1}{2^{n-1}})\frac{B_n}{\lfloor n}f^{n-1}(x)\cos\frac{\pi n}{2}\right\}$

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Sol. Put $n = 2$, *change* x *to* $2x$ *and apply VI 1.*

9.i. $S(a_1 + a_2 + a_3 + \&c)$ *means that the series is a conver gent series and its sum to infinity is required*
ii. $C(a_1 + a_2 + a_3 + \&c)$ *means that the series is a divergent series and its constant is* req^d.
iii. $G(a_1 + a_2 + a_3 + \&c)$ *means that the series is oscillating or divergent and the value of its generating function is required.*

N.B. Hereafter the series will only be given omit -ting S, C *or* G *and from the nature of the series we should infer whether* C, S *or* G *is* req^d; *more over if a series appear to be equal to a finite qua -ntity we must select* S, C *or* G *from the nature of the series.*

10.i. The value of an oscillating series is only true when the series is deduced from a regular series. For example the series $1 - 1 + 1 - 1 + \&c = \frac{1}{2}$ *only when it is deduced from a regular series of the form* $\phi(1) - \phi(2) + \phi(3) - \&c$. *Again if we take an irregular series* $a^r - b^r + c^r - d^r + \&c$ *we get the same series* $1 - 1 + 1 - 1 + \&c$ *when* r *becomes* 0 *; yet its value is not* $\frac{1}{2}$ *in this case*

ii. $a_1 - a_2 + a_3 - a_4 + \&c$ *is not equal to the series* $(a_1 - a_2) + (a_3 - a_4) + (a_5 - a_6) + \&c$ *or to the series*

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$a_1 - (a_2 - a_3) - (a_4 - a_5) - (a_6 - a_7) - \&c$ *; but to the series* $a_1 - (a_2 - a_3 + a_4 - \&c)$
e.g. $1 - 2 + 3 - 4 + \&c$ *is not equal to* $(1 - 2) + (3 - 4) + (5 - 6) + \&c$ *or to* $1 - (2 - 3) - (4 - 5) - \&c$

iii. $(a_1 - a_2 + a_3 - \&c) \pm (b_1 - b_2 + b_3 - \&c)$
$= (a_1 \pm b_1) - (a_2 \pm b_2) + (a_3 \pm b_3) - \&c$

Ex.i.shew that $(a_1 - a_2 + a_3 - \&c) + (b_1 - b_2 + \&c)$
$= a_1 + (b_1 - a_2) - (b_2 - a_3) + (b_3 - a_4) - \&c$

Sol. $L.S = a_1 + (b_1 - b_2 + b_3 - \&c) - (a_2 - a_3 + \&c)$
$= a_1 + (b_1 - a_2) - (b_2 - a_3) + \&c$

2. $a_1 - a_2 + a_3 - a_4 + \&c = \frac{a_1}{2} + \frac{1}{2}\{(a_1 - a_2) - (a_2 - a_3) + \&c\}$

3. $= \frac{3a_1 - a_2}{4} + \frac{1}{4}\{(a_1 - 2a_2 + a_3) - (a_2 - 2a_3 + a_4) + \&c\}$

4. $= \frac{7a_1 - 4a_2 + a_3}{8} + \frac{1}{8}\{(a_1 - 3a_2 + 3a_3 - a_4) - (a_2 - 3a_3 + 3a_4 - a_5) + (a_3 - 3a_4 + 3a_5 - a_6) - \&c\}$

ii. $a_1 - a_2 + a_3 - a_4 + \&c$
$= \frac{a_1}{2} + \frac{a_1 - a_2}{4} + \frac{a_1 - 2a_2 + a_3}{8} + \&c$
$= xa_1 - x^2a_2 + x^3a_3 - x^4a_4 + \&c$
$= x\frac{a_1}{2} - x^2\frac{a_1 - a_2}{4} + x^3\frac{a_1 - 2a_2 + a_3}{8} + \&c$
when x *approaches unity.*

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12. *If* $\frac{a_2}{a_3}$ *lies between* $\frac{a_1}{a_2}$ & $\frac{a_3}{a_4}$, *then*
$a_1 - a_2 + a_3 - a_4 + \&c$ *lies between* $\frac{a_1^2}{a_1+a_2}$ & $a_1 - \frac{a_2^2}{a_2+a_3}$

e.g. $1 - 2 + 3 - 4 + \&c$ *lies between* $\frac{1}{3}$ & $\frac{1}{5}$ *and its values is* $\frac{1}{4}$. $\lfloor 0 - \lfloor 1 + \lfloor 2 - \lfloor 3 + \&c$ *lies between* $\frac{1}{2}$ & $\frac{2}{3}$ *; its value is* $\frac{3}{5}$ *very nearly.*
But $2 - 2\frac{1}{2} + 3\frac{1}{3} - 4\frac{1}{4} + 5\frac{1}{5} - \&c$ *cannot lie between* $\frac{2^2}{2+2\frac{1}{2}}$ & $2 - \frac{(2\frac{1}{2})^2}{2\frac{1}{2}+3\frac{1}{3}}$ *as* $\frac{2\frac{1}{2}}{3\frac{1}{3}}$ *is not lying between* $\frac{2}{2\frac{1}{2}}$ & $\frac{3\frac{1}{3}}{4\frac{1}{4}}$ *.i.e it cannot lie between .889 & 929 as its value is 1.193*

13. $\phi_1(x) + \phi_2(x) + \phi_3(x) + \&c$ *can be expanded in ascending powers of* x *, say* $A_0 + A_1x + A_2x^2 + \&c$ *where each of* $A_1, A_2, \&c$ *is a series.*

Case I When A_n *is a convergent series*
(1) If $A_0 + A_1x + A_2x^2 + \&c$ *be a rapidly conver -gent series what is required is got.*
(2) But if it is a slowly convergent or an oscillating series, convergent or divergent (at least for some values of x *)*
(a).Change x *into a suitable function of* y *so that the new series in ascending powers*

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of y may be a rapidly convergent series;
e.g. let $\frac{x}{1+\frac{x}{2}} = y$, *then* $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \&c$
$= y - \frac{y^3}{12} + \frac{y^5}{80} - \frac{y^7}{448} + \&c$
(b) or convert it into a continued fraction
e.g. $x - \frac{x^2}{3} + \frac{2}{15}x^3 - \frac{17}{315}x^4 + \&c = \frac{x}{1+}\frac{x}{3+}\frac{x}{5+\&c}$

$$\frac{1}{x} - \frac{\lfloor 1}{x^2} + \frac{\lfloor 2}{x^3} - \frac{\lfloor 3}{x^4} + \&c = \cfrac{x}{x+1-\cfrac{1^2}{x+3-\cfrac{2^2}{x+5-\&c}}}$$

(c) or transform it into another series by applying III 8; e.g. $\frac{1}{x} - \frac{2}{x^2} + \frac{5}{x^3} - \frac{15}{x^4} + \&c$
$= \frac{1}{x+1} - \frac{1}{(x+1)(x+2)} + \frac{1}{(x+1)(x+2)(x+3)} - \&c$

(d) or take the reciprocal of the series and try to make it a rapidly convergent series in anyway

Case II When A_n *is an oscillating (convergent or divergent) or a pure divergent series*

(1) Let C_n *be the constant or the value of its generating function. Then the given series*
$= \Psi(x) + c_0 + c_1x + c_2x^2 + c_3x^3 + \&c$ *where* $\Psi(x)$ *can be found in special cases.*

(2) But if $c_0 + c_1x + c_2x^2 + \&c$ *be a divergent series find some function of* n *(say* P_n*) such that the value of* $P_0 + P_1x + P_2x^2 + \&c$ *may be easily*

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found and $c_n - P_n$ *may be rapidly diminish as* n *increases. Then the given series* $=$
$F(x) + (c_0 - P_0) + (c_1 - P_1)x + (c_2 - P_2)x^2 + \&c$

e.g.1. $\frac{1}{x+1} - \frac{1}{x+2} + \frac{1}{x+3} - \&c = \frac{1}{x}(1 - 1 + 1 - \&c)$
$-\frac{1}{x^2}(1 - 2 + 3 - \&c) = \frac{1}{2x} - \frac{1}{4x^2} + \&c$

2. $\frac{1}{1^2-x^2} + \frac{1}{2^2-x^2} + \frac{1}{3^2-x^2} + \&c = -\frac{1}{x^2}(1 + 1 + 1 + \&c)$
$-\frac{1}{x^4}(1^2 + 2^2 + 3^2 + \&c) - \frac{1}{x^4}(1^4 + 2^4 + 3^4 + \&c) = \Psi(x)$
$+\frac{1}{2x^2} = \frac{1}{2x^2} - \frac{\pi\, cotg(\pi x)}{2x}$

3. $\frac{1}{1^x} + \frac{1}{2^x} + \frac{1}{3^x} + \&c = (1 + 1 + 1 + \&c)$
$-x(\log 1 + \log 2 + \&c = -\frac{1}{2} - x\log\sqrt{2\pi} - \&c$
$= \frac{1}{x-1} + 1 + x + x^2 + \&c - \frac{1}{2} - x\log\sqrt{2\pi} - \&c$
$= \frac{1}{x-1} + \frac{1}{2} + (1 - .91894)x - \&c$
$= \frac{1}{x-1} + \frac{1}{2} + .8106x - \&c$

14.i $\frac{x}{e^x+1} + \frac{x}{e^{2x}+1} + \frac{x}{e^{3x}+1} + \&c$
$= \log 2 - \frac{x}{4} + (B_2)^2 \frac{x^2(2^2-1)}{2\lfloor 2} + (B_4)^2 \frac{x^4(2^4-1)}{4\lfloor 4} +$
$(B_6)^2 \frac{x^6(2^6-1)}{6\lfloor 6} + \&c$

Sol. $\frac{x}{e^x+1} + \frac{x}{e^{2x}+1} + \frac{x}{e^{3x}+1} + \frac{x}{e^{4x}+1} + \&c$
$= \frac{x}{2}(1 + 1 + 1 + \&c) - B_2 \frac{x^2(2^2-1)}{\lfloor 2}(1 + 2 + 3 + \&c)$
$+B_4 \frac{x^4(2^4-1)}{\lfloor 4}(1^3 + 2^3 + 3^3 + \&c) - \&c.$

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$= \Psi(x) - \frac{x}{4} + (B_2)^2 \frac{x^2(2^2-1)}{2\lfloor 2} + (B_4)^2 \frac{x^4(2^4-1)}{4\lfloor 4} + \&c$

Now it is req^d to find $\Psi(x)$
The given series $= \frac{x}{e^x-1} - \frac{x}{e^{2x}-1} + \frac{x}{e^{3x}-1} - \&c$
$=$ log2+ *terms involving* x *& higher powers of* x. $\therefore \Psi(x) = \log_e 2$.

ii. $\frac{x}{e^x-1} + \frac{x}{e^{2x}-1} + \frac{x}{e^{3x}-1} + \frac{x}{e^{4x}-1} + \&c$
$= C - \log_e x + \frac{x}{4} - (B_2)^2 \frac{x^2}{2\lfloor 2} - B_4^2 \frac{x^4}{4\lfloor 4} - B_6^2 \frac{x^6}{6\lfloor 6} - \&c$

Sol. Proceedind as in the previous theorem we have the series $= \Psi(x) + C + \frac{x}{4}$
$-{B_2}^2 \frac{x^2}{2\lfloor 2} - {B_4}^2 \frac{x^4}{4\lfloor 4} - \&c$
But we know $\frac{x}{e^x+1} + \frac{x}{e^{2x}+1} + \frac{x}{e^{3x}+1} + \&c$
$= (\frac{x}{e^x-1} + \frac{x}{e^{2x}-1} + \&c) - (\frac{2x}{e^{2x}-1} + \frac{2x}{e^{4x}-1} + \&c)$
$\therefore \Psi(x) - \Psi(2x) = \log 2$; *hence* $\Psi(x) = -\log_e x$.

Ex.1.shew that the constant in the series
$\sqrt[100]{1} + \sqrt[100]{2} + \sqrt[100]{3} + \sqrt[100]{4} + ... + \sqrt[100]{x}$
is $-.4969100$

2. $\frac{1}{2+1} + \frac{1}{2^2+1} + \frac{1}{2^3+1} + \&c = \frac{3}{4} + \frac{\log_e 2}{48}$ *nearly*

3. $\frac{1}{1+\frac{10}{9}} + \frac{1}{1+(\frac{10}{9})^2} + \frac{1}{1+(\frac{10}{9})^3} + \&c = 6.331009$.

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4. $\frac{1}{\frac{10}{9}-1}+\frac{1}{(\frac{10}{9})^2-1}+\frac{1}{(\frac{10}{9})^3-1}+\&c = 27$ *nearly*

15. i. $\frac{1}{x-1}+\frac{1}{x^2-1}+\frac{1}{x^3-1}+\&c$
$=\frac{1}{x}\frac{x+1}{x-1}+\frac{1}{x^4}\frac{x^2+1}{x^2-1}+\frac{1}{x^9}\frac{x^3+1}{x^3-1}+\&c$
ii. $\frac{1}{x-1}-\frac{1}{x^2-1}+\frac{1}{x^3-1}-\frac{1}{x^4-1}+\&c$
$=\frac{1}{x}\frac{x^2+1}{x^2-1}-\frac{1}{x^4}\frac{x^4+1}{x^4-1}+\frac{1}{x^9}\frac{x^6+1}{x^6-1}-\&c$

sol. $\frac{1}{x-1}=\frac{1}{x-1}$
$\pm\frac{1}{x^2-1}=\pm\left\{\frac{1}{x^2}+\frac{1}{x^2(x^2-1)}\right\}$
$\frac{1}{x^3-1}=\frac{1}{x^3}+\frac{1}{x^6}+\frac{1}{x^6(x^3-1)}$
$\pm\frac{1}{x^4-1}=\pm\left\{\frac{1}{x^4}+\frac{1}{x^8}+\frac{1}{x^{12}}+\frac{1}{x^{12}(x^4-1)}\right\}$
$\&c\&c\&c$

Adding up all the terms we can get the results.

16. $\frac{r}{1-ax}+\frac{r^2}{1-ax^2}+\frac{r^3}{1-ax^3}+\&c$ *to n terms*
$=\frac{arx}{1-ax}+\frac{(arx^2)^2}{1-ax^2}+\frac{(arx^3)^3}{1-ax^3}+\&c$ *to n terms*
$+\frac{r-r^{n+1}}{1-r}+a\frac{(rx)^2-(rx)^{n+1}}{1-rx}+a^2\frac{(rx^2)^3-(rx^2)^{n+1}}{1-rx^2}+\&c$
to n terms.

sol. $\frac{r}{1-ax}=\frac{arx}{1-ax}+r.$
$\frac{r^2}{1-ax^2}=\frac{(arx^2)^2}{1-ax^2}+r^2+ar^2x^2.$

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$\frac{r^3}{1-ax^3} = \frac{(arx^3)^3}{1-ax^3} + r^3 + ar^3x^3 + a^2r^3x^6.$
&c&c&c

Adding up all the terms in the n rows we can get the results.
Cor. $\frac{r}{1-ax} + \frac{r^2}{1-ax^2} + \frac{r^3}{1-ax^3} + \&c$
$= \frac{arx}{1-ax} + \frac{(arx^2)^2}{1-ax^2} + \frac{(arx^3)^3}{1-ax^3} + \&c$
$+\frac{r}{1-r} + \frac{a(rx)^2}{1-rx} + \frac{a^2(rx^2)^3}{1-rx^2} + \frac{a^3(rx^3)^4}{1-rx^3} + \&c$

17. $\frac{a}{1-m} + \frac{(a+h)n}{1-mx} + \frac{(a+2h)n^2}{1-mx^2} + \frac{(a+3h)n^3}{1-mx^3} + \&c$
$= .a.\frac{1-mn}{(1-m)(1-n)} + (a+b)\frac{1-mnx^2}{(1-mx)(1-nx)}(mnx)$
$+(a+2b)\frac{1-mnx^4}{(1-mx^2)(1-nx^2)}(mnx^2)^2 + (a+3b)\frac{1-mnx^6}{(1-mx^3)(1-nx^3)}$
$+\&c + \frac{b}{m}\{\frac{mx}{(1-x)^2} + \frac{(mnx)^2}{(1-nx)^2} + \frac{(mnx^2)^3}{(1-nx^2)^2} + \&c$

Cor 1. $\frac{a}{1-n} + \frac{(a+b)n}{1-nx} + \frac{(a+2b)n^2}{1-nx^2} + \&c$
$= a.\frac{1+n}{1-n} + (a+b)\frac{1+nx}{1-nx}(n^2x) + (a+2b)\frac{1+nx^2}{1-nx^2}(n^2x^2)^2$
$+b\{\frac{x}{(1-n)^2} + \frac{n^3x^2}{(1-nx)^2} + \frac{n^5x^6}{(1-nx^2)^2} + \frac{n^7x^{12}}{(1-nx^3)^2} + \&c\}$

2. *If* A_n *denotes the no. of factores in* n *including* 1 & n *then* $\frac{A_1}{x} + \frac{A_2}{x^2} + \frac{A_3}{x^3} + \&c = \frac{1}{x-1} + \frac{1}{x^2-1} + \&c$
and hence deduce VI 15 i