Inverse binomial series and values of Arakawa-Kaneko zeta functions

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Article submitted to the *Journal of Number Theory*

Abstract

In this article, we present a variety of evaluations of series of polylogarithmic nature. More precisely, we express the special values at positive integers of two families of zeta functions of Arakawa-Kaneko-type by means of inverse binomial series involving harmonic sums which appeared fifteen years ago in physics in relation with the Feynman diagrams. In certain cases, these series may be explicitly evaluated in terms of zeta values and other related numbers. Incidentally, this connection allows us to deduce new identities for the constant $C = \sum_{n\geq 1} \frac{1}{(2n)^3}(1+\frac{1}{3}+\cdots+\frac{1}{2n-1})$ considered by S. Ramanujan in his notebooks.

Mathematical Subject Classification (2010): 11M06, 11M41, 11Y60, 33B30.

Keywords: Inverse binomial series, Harmonic sums, Binomial transformations, Polylogarithms, Log-sine integrals, Mellin transforms, Zeta values.

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1 Introduction

The function β defined for $\Re(s) > 0$ by the Dirichlet series

$$\beta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^s}$$

has the integral representation

$$\beta(s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} \frac{e^{-t}}{1+e^{-2t}} t^{s-1}\, dt = \frac{1}{\Gamma(s)} \int_0^{+\infty} \frac{e^{-t}}{1-e^{-2t}} \operatorname{Li}_0\left(\frac{1-e^{-2t}}{2}\right) t^{s-1}\, dt$$

where Li_k denotes the classical polylogarithm $\operatorname{Li}_k(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^k}$. One may also observe that

$$\frac{1}{\Gamma(s)} \int_0^{+\infty} \frac{e^{-t}}{1-e^{-2t}} \operatorname{Li}_1(1-e^{-2t})\, t^{s-1}\, dt = (2-2^{-s}) s\, \zeta(s+1)\,.$$

We define two families of functions α_k and β_k by the Mellin transforms

$$\alpha_k(s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} \frac{e^{-t}}{1-e^{-2t}} \operatorname{Li}_k\left(1-e^{-2t}\right) t^{s-1}\, dt \qquad \text{for } \Re(s) > 0 \text{ and } k \geq 1\,,$$

$$\beta_k(s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} \frac{e^{-t}}{1-e^{-2t}} \operatorname{Li}_k\left(\frac{1-e^{-2t}}{2}\right) t^{s-1}\, dt \qquad \text{for } \Re(s) > 0 \text{ and } k \geq 0\,,$$

so that

$$\alpha_1(s) = (2-2^{-s}) s\zeta(s+1), \qquad \text{and} \qquad \beta_0(s) = \beta(s)\,.$$

We notice a complete analogy between the couple of functions (α_k, β_k) and the couple (ξ_k, η_k) defined by

$$\xi_k(s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} \frac{e^{-t}}{1-e^{-t}} \operatorname{Li}_k(1-e^{-t})\, t^{s-1}\, dt \qquad \text{for } \Re(s) > 0, \quad k \geq 1\,,$$

$$\eta_k(s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} \frac{e^{-t}}{1-e^{-t}} \operatorname{Li}_k(\frac{1-e^{-t}}{2})\, t^{s-1}\, dt \qquad \text{for } \Re(s) > 0, \quad k \geq 0\,,$$

where the function ξ_k was introduced by Arakawa and Kaneko in 1999 (cf. [1]), and formed the subject of recent works and further generalizations (cf. [3], [5], [10]). One can easily verify that

$$\xi_1(s) = s\, \zeta(s+1) = (2-2^{-s})^{-1} \alpha_1(s), \text{ and } \eta_0(s) = (1-2^{1-s})\zeta(s)\,.$$

In the case when s is a positive integer, the values $\alpha_k(s)$ et $\beta_k(s)$ can be expressed by means of the inverse binomial series studied by Kalmykov and Davydychev in relation with the Feynman diagrams (cf. [6]). More precisely, we obtain the following identities:

$$\alpha_k(m+1) = \sum_{n=1}^{\infty} \frac{2^{2n-1}}{\binom{2n}{n} n^{k+1}} P_m(O_n^{(1)}, \ldots, O_n^{(j)}, \ldots, O_n^{(m)}) \quad (\text{for } k \geq 1)\,,$$
$$\beta_k(m+1) = \sum_{n=1}^{\infty} \frac{2^{n-1}}{\binom{2n}{n} n^{k+1}} P_m(O_n^{(1)}, \ldots, O_n^{(j)}, \ldots, O_n^{(m)}) \quad (\text{for } k \geq 0)\,,$$

where P_m is the modified Bell polynomial of order m and $O_n^{(j)} = \sum_{k=1}^{n} \frac{1}{(2k-1)^j}$ is the "odd" harmonic number of order j. For small values of k ans s, these series may be explicitly evaluated in terms of zeta values ans related numbers. For instance, we show that

$$\alpha_1(2) = \sum_{n=1}^{\infty} \frac{2^{2n-1}}{\binom{2n}{n}} \frac{O_n}{n^2} = \frac{7}{2}\zeta(3)\,,$$
$$\beta_1(2) = \sum_{n=1}^{\infty} \frac{2^{n-1}}{\binom{2n}{n}} \frac{O_n}{n^2} = \frac{7}{4}\zeta(3) - \frac{\pi}{2}G\,,$$
$$\alpha_2(2) = \sum_{n=1}^{\infty} \frac{2^{2n-1}}{\binom{2n}{n}} \frac{O_n}{n^3} = 7\zeta(3)\ln 2 - \frac{\pi^4}{32} - 8G(1)\,,$$
$$\alpha_3(1) = \sum_{n=1}^{\infty} \frac{2^{2n-1}}{\binom{2n}{n}} \frac{1}{n^4} = \frac{\pi^2}{2}(\ln 2)^2 - \frac{7}{2}\zeta(3)\ln 2 + \frac{\pi^4}{96} + 4G(1)\,,$$

where we use the following notations:

$$O_n := O_n^{(1)},$$
$$G := \beta(2) \ \text{ is the Catalan constant,}$$
$$G(1) := \sum_{n=1}^{\infty} \frac{O_n}{(2n)^3} \ \text{ is the Ramanujan constant (cf. [2] p. 257, [9]).}$$

In particular, the two last identities provide new formulae for Ramanujan's constant $G(1)$ (see section 6).

2 Bell polynomials and "odd" harmonic numbers

Definition 1. The *modified Bell polynomials* are the polynomials

$$P_m \in \mathbb{Q}[x_1, x_2, ..., x_m]$$

defined for all natural numbers m by $P_0 = 1$ and the generating function

$$\exp\left(\sum_{k=1}^{\infty} x_k \frac{z^k}{k}\right) = \sum_{m=0}^{\infty} P_m(x_1, \cdots, x_m)\, z^m \,,$$

The general explicit expression for P_m is

$$P_m(x_1, ..., x_m) = \sum_{k_1+2k_2+\cdots+mk_m=m} \frac{1}{k_1!k_2!\ldots k_m!} \left(\frac{x_1}{1}\right)^{k_1} \left(\frac{x_2}{2}\right)^{k_2} \cdots \left(\frac{x_m}{m}\right)^{k_m} .$$

Example 1. For the first values of m, one has

$$\begin{aligned}
P_0 &= 1\,,\\
P_1 &= x_1\,,\\
P_2 &= \frac{1}{2}x_1^2 + \frac{1}{2}x_2\,,\\
P_3 &= \frac{1}{6}x_1^3 + \frac{1}{2}x_1x_2 + \frac{1}{3}x_3\,,\\
P_4 &= \frac{1}{24}x_1^4 + \frac{1}{4}x_1^2x_2 + \frac{1}{8}x_2^2 + \frac{1}{3}x_1x_3 + \frac{1}{4}x_4\,.
\end{aligned}$$

Notation. For $s \in \mathbb{C}$ with $\Re(s) \geq 1$ and an integer $n \geq 1$, let $O_n^{(s)}$ be the "odd" harmonic sum:

$$O_n^{(s)} = \sum_{k=1}^{n} \frac{1}{(2k-1)^s}\,, \qquad \text{and} \qquad O_n := O_n^{(1)}\,.$$

Proposition 1. For all integers $m \geq 0$ and $n \geq 1$,

$$P_m(O_n, \ldots, O_n^{(m)}) = \frac{n\binom{2n}{n}}{2^{2n-1}} \int_0^{+\infty} e^{-t}(1-e^{-2t})^{n-1} \frac{t^m}{m!} dt\,. \tag{1}$$

Proof. We are going to prove that

$$\sum_{m=0}^{\infty} P_m(O_n, \cdots, O_n^{(m)})\, z^m = \prod_{j=1}^{n} \frac{2j-1}{2j-1-z} = \frac{n\binom{2n}{n}}{2^{2n-1}} \int_0^{+\infty} e^{tz}(1-e^{-2t})^{n-1} e^{-t} dt\,,$$

and then we shall obtain formula (1) by identification of the coefficients of z^m. On one side, one has

$$
\begin{aligned}
\prod_{j=1}^{n} \frac{2j-1}{2j-1-z} &= \prod_{j=1}^{n} (1-\frac{z}{2j-1})^{-1} \\
&= \exp(-\sum_{j=1}^{n} \log(1-\frac{z}{2j-1})) \\
&= \exp(\sum_{j=1}^{n} \sum_{k=1}^{+\infty} \frac{z^k}{k(2j-1)^k}) \\
&= \exp(\sum_{k=1}^{+\infty} \frac{z^k}{k} \sum_{j=1}^{n} \frac{1}{(2j-1)^k}) \,,
\end{aligned}
$$

thus

$$
\prod_{j=1}^{n} \frac{2j-1}{2j-1-z} = \exp(\sum_{k=1}^{\infty} O_n^{(k)} \frac{z^k}{k}) = \sum_{m=0}^{\infty} P_m(O_n, \cdots, O_n^{(m)})\, z^m \,.
$$

On the other side, one has

$$
\begin{aligned}
\prod_{j=1}^{n} \frac{2j-1}{2j-1-z} &= \frac{1}{\sqrt{\pi}} \frac{\Gamma(n+1/2)\Gamma(-z/2+1/2)}{\Gamma(n-z/2+1/2)} \\
&= \frac{1}{\sqrt{\pi}} \frac{\Gamma(n+1/2)}{\Gamma(n)} \frac{\Gamma(n)\Gamma(-z/2+1/2)}{\Gamma(n-z/2+1/2)} \\
&= \frac{n}{2^{2n}} \binom{2n}{n} B(n, -z/2+1/2) \,,
\end{aligned}
$$

where B is the Euler Beta function. Thus, for $0 < |z| < 1$, one has

$$
\prod_{j=1}^{n} \frac{2j-1}{2j-1-z} = \frac{n\binom{2n}{n}}{2^{2n}} \int_0^1 u^{n-1}(1-u)^{-z/2-1/2} du \,,
$$

and making the change of variable $u = 1 - e^{-2t}$, one then obtains:

$$
\prod_{j=1}^{n} \frac{2j-1}{2j-1-z} = \frac{n\binom{2n}{n}}{2^{2n-1}} \int_0^{+\infty} e^{tz}(1-e^{-2t})^{n-1} e^{-t} dt \,,
$$

and finally

$$
\sum_{m=0}^{\infty} P_m(O_n, \cdots, O_n^{(m)})\, z^m = \frac{n\binom{2n}{n}}{2^{2n-1}} \int_0^{+\infty} e^{tz}(1-e^{-2t})^{n-1} e^{-t} dt \,.
$$

□

3 The operators D and S, and the Euler series transformation

Definition 2. Let a be an analytic function in $P = \{x \,|\, \Re(x) \geq 1\}$ defined by

$$a(x) = \int_0^{+\infty} e^{-xt}\widehat{a}(t)dt \quad \text{ for all } x \in P\,,$$

where $\widehat{a} \in \mathcal{C}^1([0,+\infty[)$ is such that there exists $\alpha < 1$, and $C > 0$ with

$$|\widehat{a}(t)| \leq Ce^{\alpha t} \text{ for all } t \in]0,+\infty[\ .$$

For $x \in P$, we define the functions $x \mapsto D(a)(x)$ and $x \mapsto S(a)(x)$ by

$$D(a)(x) = \int_0^{+\infty} \frac{e^{-t}}{1-e^{-t}}(1-e^{-t})^x\, \widehat{a}(t)\, dt\,,$$

$$S(a)(x) = \int_0^{+\infty} \frac{e^{-t}}{1-e^{-t}}(1-e^{-xt})\, \widehat{a}(t)\, dt\,.$$

Proposition 2. For all integers $n \geq 1$, one has

$$S(a)(n) = \sum_{k=1}^{n} a(k)\,,$$

and for all integer $n \geq 0$,

$$D(a)(n+1) = \sum_{k=0}^{n}(-1)^k \binom{n}{k} a(k+1)\,.$$

Proof. The first relation follows from

$$S(a)(n) = \int_0^{+\infty} \frac{e^{-t}-e^{-(n+1)t}}{1-e^{-t}}\widehat{a}(t)dt = \int_0^{+\infty} (\sum_{k=1}^{n} e^{-kt})\widehat{a}(t)dt = \sum_{k=1}^{n} \int_0^{+\infty} e^{-kt}\widehat{a}(t)dt\,.$$

The second relation results from the binomial expansion of $(1-e^{-t})^n$ since

$$\begin{aligned} D(a)(n+1) &= \int_0^{+\infty} e^{-t}(1-e^{-t})^n \widehat{a}(t)dt \\ &= \int_0^{+\infty} e^{-t} \sum_{k=0}^{n}(-1)^k \binom{n}{k} e^{-kt}\widehat{a}(t)dt \\ &= \sum_{k=0}^{n}(-1)^k \binom{n}{k} \int_0^{+\infty} e^{-t}e^{-kt}\widehat{a}(t)dt \\ &= \sum_{k=0}^{n}(-1)^k \binom{n}{k} a(k+1)\,. \end{aligned}$$

□

Example 2. For s with $\Re(s) \geq 1$ and $x \in P$, let

$$a(x) = \frac{1}{(2x-1)^s}\,.$$

One has

$$a(x) = \int_0^{+\infty} e^{-(2x-1)t}\frac{t^{s-1}}{\Gamma(s)}dt = \int_0^{+\infty} e^{-xt}\frac{e^{\frac{t}{2}}(\frac{t}{2})^{s-1}}{2\Gamma(s)}dt\,.$$

Thus, for all integer $n \geq 1$,

$$D(a)(n) = \int_0^{+\infty} e^{-\frac{t}{2}}(1-e^{-t})^{n-1}\frac{(\frac{t}{2})^{s-1}}{2\Gamma(s)}dt = \int_0^{+\infty} e^{-t}(1-e^{-2t})^{n-1}\frac{t^{s-1}}{\Gamma(s)}dt\,. \qquad (2)$$

By (1), one has

$$\int_0^{+\infty} e^{-t}(1-e^{-2t})^{n-1}\frac{t^m}{m!}dt = \frac{2^{2n-1}}{n\binom{2n}{n}}P_m(O_n,\ldots,O_n^{(m)})\,.$$

Thus, if s is an integer, $s = m+1$ with $m \geq 0$, then we get for all integers $n \geq 1$ the following formula

$$D(\frac{1}{(2x-1)^{m+1}})(n) = \frac{2^{2n-1}}{n\binom{2n}{n}}P_m(O_n,\ldots,O_n^{(m)})\,. \qquad (3)$$

Lemma 1. The operators D and S are linked by the following relation:

$$D\left(\frac{1}{x}S(a)\right) = \frac{1}{x}D(a) \text{ for all } x \in P\,.$$

Proof of the lemma. By definition of $S(a)$,

$$\frac{1}{x}S(a)(x) = \int_0^{+\infty} \frac{1-e^{-xt}}{x}\;[\frac{e^{-t}}{1-e^{-t}}\widehat{a}(t)]\,dt\,,$$

integrating by parts, we get

$$\frac{1}{x}S(a)(x) = \int_0^{+\infty} e^{-xt}\left(\int_t^{\infty}\frac{e^{-u}}{1-e^{-u}}\widehat{a}(u)\,du\right)dt\,,$$

this gives

$$\widehat{\frac{1}{x}S(a)}(t) = \int_t^{\infty}\frac{e^{-u}}{1-e^{-u}}\widehat{a}(u)\,du\,.$$

Thus

$$D(\frac{1}{x}S(a))(x) = \int_0^{+\infty} e^{-t}(1-e^{-t})^{x-1}\left(\int_t^{\infty} \frac{e^{-u}}{1-e^{-u}}\widehat{a}(u)\,du\right)\,dt\,,$$

and integrating again by parts, we get

$$D(\frac{1}{x}S(a))(x) = \int_0^{+\infty} \frac{1}{x}(1-e^{-t})^x \frac{e^{-t}}{1-e^{-t}}\widehat{a}(t)\,dt = \frac{1}{x}D(a)(x)\,.$$

□

Proposition 3. For all complex numbers z such that $|z| < \frac{1}{2}$, one has

$$\begin{aligned} \sum_{n=1}^{+\infty} D(a)(n)z^n &= -\sum_{n=1}^{+\infty} a(n)(\frac{z}{z-1})^n\,, \\ \sum_{n=1}^{+\infty} \frac{D(a)(n)}{n}z^n &= -\sum_{n=1}^{+\infty} \frac{1}{n}S(a)(n)(\frac{z}{z-1})^n\,. \end{aligned} \tag{4}$$

Proof. For the first relation we write

$$\begin{aligned} \sum_{n=0}^{+\infty} D(a)(n+1)z^{n+1} &= \int_0^{+\infty} e^{-t}\frac{z}{1-(1-e^{-t})z}\widehat{a}(t)dt \\ &= -\int_0^{+\infty} (\frac{z}{z-1})\frac{e^{-t}}{1-e^{-t}\frac{z}{z-1}}\widehat{a}(t)dt\,. \end{aligned}$$

The expansion

$$(\frac{z}{z-1})\frac{e^{-t}}{1-e^{-t}\frac{z}{z-1}}\widehat{a}(t) = \sum_{n=1}^{+\infty} e^{-nt}(\frac{z}{z-1})^n\widehat{a}(t)$$

gives

$$\begin{aligned} \sum_{n=0}^{+\infty} D(a)(n+1)z^{n+1} &= -\int_0^{+\infty}\sum_{n=1}^{+\infty} e^{-nt}(\frac{z}{z-1})^n\widehat{a}(t)dt \\ &= -\sum_{n=1}^{+\infty}(\frac{z}{z-1})^n\int_0^{+\infty} e^{-nt}\widehat{a}(t)dt\,, \end{aligned}$$

the order of $\int_0^{+\infty}$ and $\sum_{n=1}^{\infty}$ may be interchanged because

$$\int_0^{+\infty}\sum_{n=1}^{+\infty} e^{-nt}(\frac{|z|}{1-|z|})^n|\widehat{a}(t)|dt = (\frac{|z|}{1-|z|})\int_0^{+\infty}\frac{e^{-t}}{1-e^{-t}\frac{|z|}{1-|z|}}|\widehat{a}(t)| < +\infty\,.$$

The second relation (4) is an immediate consequence of the first one by Lemma 1 above. □

Proposition 4. For all integers $p \geq 1$, one has

$$\sum_{n=1}^{\infty} \frac{D(a)(n)}{p^n n^k} = \int_0^{+\infty} \frac{e^{-t}}{1-e^{-t}} \mathrm{Li}_k(\frac{1-e^{-t}}{p})\, \widehat{a}(t)\, dt\,. \tag{5}$$

Proof. Let p be a positive integer, then

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{D(a)(n)}{p^n n^k} &= \sum_{n=1}^{\infty} \frac{1}{p^n n^k} \int_0^{+\infty} \frac{e^{-t}}{1-e^{-t}} (1-e^{-t})^n\, \widehat{a}(t)\, dt \\
&= \int_0^{+\infty} \frac{e^{-t}}{1-e^{-t}} \sum_{n=1}^{\infty} \frac{(1-e^{-t})^n}{p^n n^k}\, \widehat{a}(t)\, dt \\
&= \int_0^{+\infty} \frac{e^{-t}}{1-e^{-t}} \mathrm{Li}_k(\frac{1-e^{-t}}{p})\, \widehat{a}(t)\, dt\,,
\end{aligned}$$

the order of $\int_0^{+\infty}$ and $\sum_{n=1}^{\infty}$ may be interchanged since, by the hypothesis on $\widehat{a}$,

$$|\widehat{a}(t)| \leq C e^{\alpha t} \text{ for all } t \in \,]0, +\infty[\ ,$$

which gives

$$\int_0^{+\infty} \frac{e^{-t}}{1-e^{-t}} \sum_{n=1}^{\infty} \frac{(1-e^{-t})^n}{p^n n^k}\, |\widehat{a}(t)|\, dt \leq C \int_0^{+\infty} \frac{e^{-t}}{1-e^{-t}} \mathrm{Li}_k(\frac{1-e^{-t}}{p}) e^{\alpha t}\, dt < +\infty\,.$$

□

4 The functions α_k and β_k

Definition 3. Let k be a positive integer. The functions α_k and β_k are respectively defined for all $s \in \mathbb{C}$ with $\Re(s) > 0$ by

$$\begin{aligned}
\alpha_k(s) &= \frac{1}{\Gamma(s)} \int_0^{+\infty} \frac{e^{-t}}{1-e^{-2t}} \mathrm{Li}_k\left(1-e^{-2t}\right) t^{s-1}\, dt \quad (\text{for } k \geq 1)\,, \\
\beta_k(s) &= \frac{1}{\Gamma(s)} \int_0^{+\infty} \frac{e^{-t}}{1-e^{-2t}} \mathrm{Li}_k\left(\frac{1-e^{-2t}}{2}\right) t^{s-1}\, dt \quad (\text{for } k \geq 0)\,.
\end{aligned}$$

Example 3.

$$\begin{aligned}
\alpha_1(s) &= \frac{2}{\Gamma(s)} \int_0^{+\infty} \frac{e^{-t}}{1-e^{-2t}} t^s\, dt = 2(1-2^{-s-1}) s \zeta(s+1)\,, \\
\beta_1(s) &= \frac{1}{\Gamma(s)} \int_0^{+\infty} \frac{e^{-t}}{1-e^{-2t}} \left[\ln 2 - \ln(1+e^{-2t})\right] t^{s-1} dt\,.
\end{aligned}$$

Proposition 5. If s is such that $\Re(s) \geq 1$, then

$$\alpha_k(s) = \sum_{n=1}^{\infty} \frac{1}{n^k} D(\frac{1}{(2x-1)^s})(n) \quad (\text{for } k \geq 1)\,,$$
$$\beta_k(s) = \sum_{n=1}^{\infty} \frac{1}{2^n n^k} D(\frac{1}{(2x-1)^s})(n) \quad (\text{for } k \geq 0)\,.$$

Proof. This is an immediate consequence of formula (5) applied to the function $a(x) = \dfrac{1}{(2x-1)^s}$ (for $p = 1, 2$) since $\widehat{a}(t) = e^{\frac{t}{2}} \dfrac{(\frac{t}{2})^{s-1}}{2\Gamma(s)}$ as already seen in Example 2. □

Corollary 1. For all integers $m \geq 0$, then

$$\alpha_k(m+1) = \sum_{n=1}^{\infty} \frac{2^{2n-1}}{\binom{2n}{n} n^{k+1}} P_m(O_n^{(1)}, \ldots, O_n^{(m)}) \quad (\text{for } k \geq 1)\,, \tag{6}$$
$$\beta_k(m+1) = \sum_{n=1}^{\infty} \frac{2^{n-1}}{\binom{2n}{n} n^{k+1}} P_m(O_n^{(1)}, \ldots, O_n^{(m)}) \quad (\text{for } k \geq 0)\,. \tag{7}$$

Proof. This is an immediate consequence of Proposition 5 by formula (3). □

Example 4. Since $\alpha_1(s) = (2 - 2^{-s}) s\zeta(s+1)$, then for all integers $m \geq 1$,

$$(2 - 2^{-m}) m\zeta(m+1) = \sum_{n=1}^{\infty} \frac{2^{2n-1}}{\binom{2n}{n} n^2} P_{m-1}(O_n^{(1)}, \ldots, O_n^{(m-1)})\,. \tag{8}$$

In particular, for $m = 1$, this gives

$$\sum_{n=1}^{\infty} \frac{2^{2n}}{\binom{2n}{n}} \frac{O_n}{n^2} = 7\zeta(3)\,. \tag{9}$$

Example 5. Since $\beta_0(s) = \beta(s)$, one has for all integers $m \geq 1$,

$$\beta(2m) = \sum_{n=1}^{\infty} \frac{2^{n-1}}{\binom{2n}{n} n} P_{2m-1}(O_n^{(1)}, \ldots, O_n^{(2m-1)})\,. \tag{10}$$

In particular, for $m = 1$, a nice formula for Catalan's constant (cf. [2] p. 293, Entry 34) is regained:

$$\sum_{n=1}^{\infty} \frac{2^n}{\binom{2n}{n}} \frac{O_n}{n} = 2G, \tag{11}$$

and for $m = 2$,

$$\sum_{n=1}^{\infty} \frac{2^n}{\binom{2n}{n}} \frac{(O_n)^3}{n} + 3\sum_{n=1}^{\infty} \frac{2^n}{\binom{2n}{n}} \frac{O_n O_n^{(2)}}{n} + 2\sum_{n=1}^{\infty} \frac{2^n}{\binom{2n}{n}} \frac{O_n^{(3)}}{n} = 12\beta(4)\,. \tag{12}$$

Remark 1. In a similar way (cf. [3] § 5.5), one can prove for the couple of functions (ξ_k, η_k) the following identities:

$$\xi_k(s) = \sum_{n=1}^{\infty} \frac{1}{n^k} D(\frac{1}{x^s})(n) \quad (\text{for } \Re(s) \geq 1 \text{ and } k \geq 1)\,,$$
$$\eta_k(s) = \sum_{n=1}^{\infty} \frac{1}{2^n n^k} D(\frac{1}{x^s})(n) \quad (\text{for } \Re(s) \geq 1 \text{ and } k \geq 0)\,,$$

and, furthermore, one has (cf. [3] § 3)

$$D(\frac{1}{x^{m+1}})(n) = \frac{P_m(H_n^{(1)}, \dots, H_n^{(m)})}{n} \text{ with } H_n^{(j)} = \sum_{k=1}^{n} \frac{1}{k^j} \quad (j = 1, 2, \cdots, m)\,,$$

which gives for instance the well-known identities

$$\xi_1(2) = \sum_{n=1}^{\infty} \frac{H_n}{n^2} = 2\zeta(3)\,,$$
$$\eta_1(2) = \sum_{n=1}^{\infty} \frac{H_n}{2^n n^2} = \zeta(3) - \frac{\pi^2}{12} \ln 2 \quad ([2] \text{ p. } 258)\ .$$

4.1 The function β_1

The Euler series transformation (Proposition 3 above) provides an alternative expression for β_1.

Proposition 6. For all $s \in \mathbb{C}$ with $\Re(s) \geq 1$, one has

$$\beta_1(s) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{O_n^{(s)}}{n}\,,$$

hence, for each integer $m \geq 1$,

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{O_n^{(m)}}{n} = \sum_{n=1}^{\infty} \frac{2^{n-1}}{\binom{2n}{n} n^2} P_{m-1}(O_n^{(1)}, \dots, O_n^{(m-1)})\,. \tag{13}$$

Proof. By (4), one has for all $|z| < \frac{1}{2}$,

$$\sum_{n=1}^{+\infty} \frac{D(a)(n)}{n} z^n = -\sum_{n=1}^{+\infty} \frac{1}{n} S(a)(n) (\frac{z}{z-1})^n\,.$$

If the series $\sum_{n=1}^{+\infty} \frac{D(a)(n)}{n} \frac{1}{2^n}$ is convergent, then, by the classical Abel lemma, we get

$$\sum_{n=1}^{+\infty} \frac{D(a)(n)}{n} \frac{1}{2^n} = \sum_{n=1}^{+\infty} \frac{1}{n} S(a)(n) (-1)^{n-1}\,.$$

By formula (5), the series $\sum_{n=1}^{\infty} \frac{1}{2^n n} D(\frac{1}{(2x-1)^s})(n)$ is convergent and

$$\beta_1(s) = \sum_{n=1}^{\infty} \frac{1}{2^n n} D(\frac{1}{(2x-1)^s})(n) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n} S(\frac{1}{(2x-1)^s})(n)\,.$$

Then, using formula (1) for $D(\dfrac{1}{(2x-1)^m})(n)$, one obtains (13). □

Example 6.

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{O_n}{n} = \sum_{n=1}^{\infty} \frac{2^{n-1}}{\binom{2n}{n}} \frac{1}{n^2} = \frac{\pi^2}{16}\,,$$

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{O_n^{(2)}}{n} = \sum_{n=1}^{\infty} \frac{2^{n-1}}{\binom{2n}{n}} \frac{O_n}{n^2} = \frac{7}{4}\zeta(3) - \frac{\pi}{2}G \tag{14}$$

([6] (2.36) and (2.37) with $u = 2$ and $\theta = \dfrac{\pi}{2}$),

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{O_n^{(3)}}{n} = \sum_{n=1}^{\infty} \frac{2^n}{\binom{2n}{n}} \frac{(O_n)^2}{(2n)^2} + \sum_{n=1}^{\infty} \frac{2^n}{\binom{2n}{n}} \frac{O_n^{(2)}}{(2n)^2} = \frac{\pi^4}{64} - G^2 \tag{15}$$

([6] (2.38), (2.39), (2.40) and (C.4) with $u = 2$ and $\theta = \dfrac{\pi}{2}$).

Remark 2. By Remark 1 above, one also has the relation

$$\eta_1(m) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{H_n^{(m)}}{n} = \sum_{n=1}^{\infty} \frac{1}{2^n n^2} P_{m-1}(H_n^{(1)}, \ldots, H_n^{(m-1)}) \tag{16}$$

which is similar to (13) and equivalent to that given by Choi and Srivastava ([4] p. 66, formula (4.29)).

Proposition 7. Let

$$\widetilde{H}_n = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k}.$$

If s is such that $\Re(s) > 1$, then

$$\sum_{n=1}^{\infty} \frac{\widetilde{H}_n}{(2n-1)^s} = (1-2^{-s})\zeta(s)\ln 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n(2n-1)^s} - \beta_1(s)\,. \tag{17}$$

Thus, for each integer $m > 1$,

$$\sum_{n=1}^{\infty} \frac{\widetilde{H}_n}{(2n-1)^m} = (1-2^{-m})\zeta(m)\ln 2 + (-1)^m \ln 2 + 2\sum_{k=1}^{m} (-1)^{m-k}\beta(k) - \beta_1(m)\,. \tag{18}$$

Proof. The first relation is a direct consequence of the following elementary result:

If the series $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$, $\sum_{n=1}^{\infty} a_n b_n$ and $\sum_{n=1}^{\infty} b_n \sum_{k=1}^{n} a_k$ are convergent, then the series $\sum_{n\geq 1} a_n \sum_{k=1}^{n} b_k$ is convergent and we have

$$\sum_{n=1}^{\infty} a_n \sum_{k=1}^{n} b_k = \sum_{n=1}^{\infty} a_n \sum_{n=1}^{\infty} b_n + \sum_{n=1}^{\infty} a_n b_n - \sum_{n=1}^{\infty} b_n \sum_{k=1}^{n} a_k \,.$$

Applied to $a_n = \frac{1}{(2n-1)^s}$ and $b_n = \frac{(-1)^{n-1}}{n}$, this relation gives (17). The second relation is a consequence of the first one by the following observation:

If s is an integer, $s = m$, then we have

$$\sum_{k=1}^{m} \frac{(-1)^k}{(2n-1)^k} = \frac{(-1)^m (2n-1)^{-m} - 1}{2n} \,,$$

hence

$$\frac{(-1)^{n-1}}{n(2n-1)^m} = (-1)^m \frac{(-1)^{n-1}}{n} + 2 \sum_{k=1}^{m} (-1)^{m-k} \frac{(-1)^{n-1}}{(2n-1)^k}$$

which gives formula (18). □

Example 7. Formula (18) gives respectively for $m = 2$ and $m = 3$ the following identities:

$$\sum_{n=1}^{\infty} \frac{\widetilde{H}_n}{(2n-1)^2} = \frac{\pi^2}{8} \ln 2 + \ln 2 - \frac{\pi}{2} + 2G - \frac{7}{4}\zeta(3) + \frac{\pi G}{2} \,, \tag{19}$$

$$\sum_{n=1}^{\infty} \frac{\widetilde{H}_n}{(2n-1)^3} = \frac{7}{8}\zeta(3) \ln 2 - \ln 2 + \frac{\pi}{2} - 2G + \frac{\pi^3}{16} + G^2 - \frac{\pi^4}{64} \,. \tag{20}$$

5 The values $\alpha_k(1)$ and $\beta_k(1)$

The special values of α_k and β_k at $s = 1$ are evaluated in terms of (generalized) log-sine functions (cf. [6], [7]).

Proposition 8. For each integer $k \geq 1$ and $\alpha \in \mathbb{R}$ such that $0 \leq \alpha \leq 1$, let $\mathrm{L}_k(\alpha)$ be the log-sine-type integral[1]:

$$\mathrm{L}_k(\alpha) = \int_0^{\alpha\pi} u \ln^{k-1}\left(2 \sin \frac{u}{2}\right) du = \pi^2 \int_0^{\alpha} x \ln^{k-1}\left(2 \sin \frac{\pi x}{2}\right) dx \,.$$

Then, one has

$$2\alpha_k(1) = \sum_{n=1}^{\infty} \frac{2^{2n}}{\binom{2n}{n}} \frac{1}{n^{k+1}} = 2^{k-1} \sum_{i=1}^{k} (-1)^{i-1} \frac{(\ln 2)^{k-i}}{(i-1)!(k-i)!} \mathrm{L}_i(1) \,, \tag{21}$$

[1] With the notations of [7], $\mathrm{L}_k(\alpha)$ is $-\mathrm{Ls}_{k+1}^{(1)}(\alpha\pi)$.

$$2\beta_k(1) = \sum_{n=1}^{\infty} \frac{2^n}{\binom{2n}{n}} \frac{1}{n^{k+1}} = \sum_{i=1}^{k} (-1)^{i-1} \frac{2^{i-1}(\ln 2)^{k-i}}{(i-1)!(k-i)!} \, \mathrm{L}_i(\frac{1}{2}) \, . \tag{22}$$

Proof. The proof is similar to that given in [11] § 4. Let

$$J_k(x) = \frac{1}{2^k} \sum_{n=1}^{\infty} \frac{(2x)^{2n}}{\binom{2n}{n} n^{k+1}} \, .$$

Then, one has for $k \geq 1$,

$$J_k(x) = \int_0^x \frac{J_{k-1}(u)}{u} du \, .$$

By a classical identity due to Euler (cf. [11], [12]), one also has

$$J_1(x) = (\arcsin x)^2 \, ,$$

hence

$$J_0(x) = \frac{2x \arcsin x}{\sqrt{1-x^2}} .$$

It is easily verified that

$$2\alpha_k(1) = 2^k J_k(1) = 2^k \int_0^1 \frac{J_{k-1}(x)}{x} dx \, ,$$

and

$$2\beta_k(1) = 2^k J_k(\frac{\sqrt{2}}{2}) = 2^k \int_0^{\frac{\sqrt{2}}{2}} \frac{J_{k-1}(x)}{x} dx \, .$$

By $(k-1)$ integrations by parts and the change of variable $x = \sin \frac{u}{2}$, we get

$$J_k(1) = \frac{(-1)^{k-1}}{(k-1)!} \int_0^1 \ln^{k-1}(x) \frac{J_0(x)}{x} dx = \frac{(-1)^{k-1}}{2(k-1)!} \int_0^{\pi} u \ln^{k-1}(\sin \frac{u}{2}) du \, ,$$

and

$$J_k(\frac{\sqrt{2}}{2}) = \frac{(-1)^{k-1}}{(k-1)!} \int_0^{\frac{\sqrt{2}}{2}} \ln^{k-1}(\sqrt{2}x) \frac{J_0(x)}{x} dx = \frac{(-1)^{k-1}}{2(k-1)!} \int_0^{\frac{1}{2}\pi} u \ln^{k-1}(\sqrt{2} \sin \frac{u}{2}) du.$$

It remains to use the binomial expansions of

$$\ln^{k-1}(\sin \frac{u}{2}) = \left[\ln(\frac{1}{2}) + \ln(2 \sin \frac{u}{2}) \right]^{k-1} \, ,$$

and

$$\ln^{k-1}(\sqrt{2} \sin \frac{u}{2}) = \left[\ln(\frac{1}{\sqrt{2}}) + \ln(2 \sin \frac{u}{2}) \right]^{k-1}$$

to obtain formulas (21) and (22). □

Proposition 9. For all α such that $0 \le \alpha \le 1$, we have

$$\text{a) } \mathrm{L}_1(\alpha) = \frac{\pi^2}{2}\alpha^2 ,$$

$$\text{b) } \mathrm{L}_2(\alpha) = \zeta(3) - \sum_{n=1}^{\infty} \frac{\cos(\pi n\alpha)}{n^3} - \alpha\pi \sum_{n=1}^{\infty} \frac{\sin(\pi n\alpha)}{n^2} ,$$

$$\text{c) } \mathrm{L}_3(\alpha) = \frac{\pi^4}{16}(\alpha^4 - \frac{8}{3}\alpha^3 + 2\alpha^2) + 2\sum_{n=1}^{\infty} \frac{H_n}{(n+1)^3}\cos(\pi(n+1)\alpha)$$
$$+ 2\pi\alpha \sum_{n=1}^{\infty} \frac{H_n}{(n+1)^2}\sin(\pi(n+1)\alpha) - \frac{1}{2}\zeta(4) . \tag{23}$$

Proof. The assertion a) is trivially verified and b) is a classical identity (cf. [7], formula (7.53)). It remains to prove c). We use the following expansion :

$$\mathrm{Log}^2(1-z) = 2\sum_{n=1}^{\infty} H_n \frac{z^{n+1}}{n+1}$$

to get

$$\mathrm{Log}^2\left(1 - e^{-i\pi x}\right) = 2\sum_{n=1}^{\infty} H_n \frac{e^{-i\pi x(n+1)}}{n+1} .$$

Since

$$\begin{aligned}\mathrm{Log}^2\left(1 - e^{-i\pi x}\right) &= \mathrm{Log}^2\left(e^{-i\pi x/2}(e^{i\pi x/2} - e^{-i\pi x/2})\right)\\ &= (-i\pi x/2 + i\pi/2 + \ln(2\sin\frac{1}{2}x\pi))^2\\ &= -\frac{\pi^2}{4}(x-1)^2 + \ln^2(2\sin\frac{\pi x}{2}x) + i\Im\left(\mathrm{Log}^2\left(1 - e^{-i\pi x}\right)\right) ,\end{aligned}$$

one has

$$\ln^2(2\sin\frac{\pi x}{2}) = \frac{\pi^2}{4}(x-1)^2 + \Re\left(\mathrm{Log}^2\left(1 - e^{-i\pi x}\right)\right) ,$$

hence

$$\ln^2\left(2\sin\frac{\pi x}{2}\right) = \frac{\pi^2}{4}(x-1)^2 + 2\sum_{n=1}^{\infty} H_n \frac{\cos(\pi(n+1)x)}{n+1} .$$

Integrating, this gives

$$\int_0^{\alpha} x\ln^2\left(2\sin\frac{\pi x}{2}\right)\,dx = \frac{\pi^2}{4}\int_0^{\alpha} x(x-1)^2\,dx$$
$$+ 2\sum_{n=1}^{\infty} \frac{H_n}{n+1}\int_0^{\alpha} x\cos(\pi(n+1)x)\,dx . \tag{24}$$

The permutation of $\sum$ and $\int$ in (24) is justified by the following Lemma 2 and the dominated convergence theorem. The integrals in the right-hand side of (24) are easily computed by

$$\int_0^\alpha x(x-1)^2\,dx = \frac{\alpha^4}{4} - \frac{2\alpha^3}{3} + \frac{\alpha^2}{2}\,,$$

and

$$\int_0^\alpha x\cos(\pi(n+1)x)dx = \frac{\cos(\pi(n+1)\alpha)}{\pi^2(n+1)^2} + \frac{\alpha}{\pi(n+1)}\sin(\pi(n+1)\alpha) - \frac{1}{\pi^2(n+1)^2}\,.$$

Thus, we deduce from (24) the following expression for $\mathrm{L}_3(\alpha)$:

$$\mathrm{L}_3(\alpha) = \frac{\pi^4}{4}(\frac{\alpha^4}{4} - \frac{2\alpha^3}{3} + \frac{\alpha^2}{2}) + 2\sum_{n=1}^\infty \frac{H_n}{(n+1)^3}\cos(\pi(n+1)\alpha)$$
$$+ 2\pi\alpha\sum_{n=1}^\infty \frac{H_n}{(n+1)^2}\sin(\pi(n+1)\alpha) - 2\sum_{n=1}^\infty \frac{H_n}{(n+1)^3}\,.$$

Moreover, one has

$$\sum_{n=1}^\infty \frac{H_n}{(n+1)^3} = \sum_{n=1}^\infty \frac{H_n}{n^3} - \zeta(4) = \frac{5}{4}\zeta(4) - \zeta(4) = \frac{1}{4}\zeta(4)\,,$$

and this gives (23). □

Lemma 2. The partial sums

$$\sum_{n=1}^k H_n \frac{x\cos(\pi(n+1)x)}{n+1}$$

are uniformly bounded for $x \in]0,1[$.

Proof of the lemma. Let $S_n(x) = x\sum_{j=1}^n \cos(\pi(j+1)x)$. A sommation by parts gives

$$\sum_{n=1}^k \frac{H_n}{n+1} x\cos(\pi(n+1)x) = \sum_{n=1}^k S_n(x)(\frac{H_n}{n+1} - \frac{H_{n+1}}{n+2}) + \frac{H_k}{k+1}S_k(x)\,,$$

and one has

$$|S_n(x)| = \left| x\sum_{j=1}^n \cos(\pi(j+1)x)\right| \le \frac{2x}{\sin(\pi x/2)}\,.$$

It follows that, for all $x \in]0,1[$,

$$\left|\sum_{n=1}^k xH_n\frac{\cos(\pi(n+1)x)}{n+1}\right| \le \frac{2x}{\sin(\pi x/2)}(\frac{H_1}{2} - \frac{H_{k+1}}{k+2} + \frac{H_k}{k+1}) \le \frac{Cx}{\sin(\pi x/2)} \le C'\,.$$

□

Example 8. Formulae (21) and (22) give for $k=2$ the following identities:

$$\sum_{n=1}^{\infty} \frac{2^{2n}}{\binom{2n}{n}} \frac{1}{n^3} = \pi^2 \ln 2 - \frac{7}{2}\zeta(3)\,,$$
$$\sum_{n=1}^{\infty} \frac{2^{n}}{\binom{2n}{n}} \frac{1}{n^3} = \frac{\pi^2}{8} \ln 2 + \pi G - \frac{35}{16}\zeta(3)\,,$$

which were known of Ramanujan (cf. [2], p. 269).

6 New formulae for Ramanujan's constant $G(1)$

In Chapter 9 of his notebooks (cf. [2] p. 255, Entry 11), Ramanujan introduced two generating functions[2]:

$$F(x) := \sum_{n=1}^{\infty} \frac{O_n x^{2n}}{(2n)^2} \quad \text{and} \quad G(x) := \sum_{n=1}^{\infty} \frac{O_n x^{2n}}{(2n)^3}\,,$$

then, he writes the following functional relation :

$$\begin{aligned} G(x) + G\left(\frac{1-x}{1+x}\right) &= F(x)\log(x) + F\left(\frac{1-x}{1+x}\right)\log\left(\frac{1-x}{1+x}\right) \\ &\quad - \frac{1}{16}\log^2(x)\log^2\left(\frac{1-x}{1+x}\right) + C\,, \end{aligned} \tag{ii}$$

with

$$C = \frac{\pi}{4}\sum_{n=0}^{\infty} \frac{(-1)^n}{(4n+1)^3} - \frac{\pi}{3\sqrt{3}}\sum_{n=0}^{\infty} \frac{1}{(2n+1)^3}\,.$$

Unfortunately, this beautiful formula for C given by Ramanujan turns out to be erroneous since the constant C in (ii) must be equal to

$$G(1) = \sum_{n=1}^{\infty} \frac{O_n}{(2n)^3} \quad \text{(cf. [2] p. 257, or [9] for more details).}$$

However, the calculation of $\alpha_3(1)$ and $\alpha_2(2)$ provides two interesting formulae for the constant $G(1)$.

Proposition 10. Let $G(1) = \sum_{n=1}^{\infty} \frac{O_n}{(2n)^3}$ be the Ramanujan constant. One has

$$G(1) = \frac{7}{8}\zeta(3)\ln 2 - \frac{\pi^4}{384} - \frac{1}{8}\pi^2(\ln 2)^2 + 2\sum_{n=1}^{\infty} \frac{2^{2n}}{\binom{2n}{n}} \frac{1}{(2n)^4}\,. \tag{25}$$

[2]These functions are respectively quoted ϕ and ψ in the original manuscript : cf. [8] p. 108.

Proof. One has

$$\begin{aligned}
G(1) = \sum_{n=1}^{\infty} \frac{O_n}{(2n)^3} &= \sum_{n=1}^{\infty} \frac{H_{2n} - \frac{1}{2}H_n}{(2n)^3} \\
&= \sum_{n=1}^{\infty} \frac{H_{2n}}{(2n)^3} - \sum_{n=1}^{\infty} \frac{1}{2}\frac{H_n}{(2n)^3} \\
&= \sum_{n=1}^{\infty} \frac{1+(-1)^n}{2}\frac{H_n}{n^3} - \sum_{n=1}^{\infty} \frac{1}{2^4}\frac{H_n}{n^3} \\
&= \frac{1}{2}\sum_{n=1}^{\infty} \frac{H_n}{n^3} + \frac{1}{2}\sum_{n=1}^{\infty} \frac{(-1)^n H_n}{n^3} - \frac{1}{2^4}\sum_{n=1}^{\infty} \frac{H_n}{n^3} \\
&= \frac{35}{64}\zeta(4) - \frac{1}{2}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n}{n^3}\,.
\end{aligned}$$

Moreover, by (23) with $\alpha = 1$, one also has

$$\begin{aligned}
\mathrm{L}_3(1) &= \frac{25}{8}\zeta(4) - 2\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n}{n^3} \\
&= \frac{\pi^4}{96} + 4G(1)\,. \qquad (26)
\end{aligned}$$

Then, applying formula (21) with $k = 3$, it results from (26) that

$$2\alpha_3(1) = \sum_{n=1}^{\infty} \frac{2^{2n}}{\binom{2n}{n}}\frac{1}{n^4} = \pi^2(\ln 2)^2 - 7\zeta(3)\ln 2 + \frac{\pi^4}{48} + 8G(1) \qquad (27)$$

and this relation is equivalent to (25). □

Remark 3. We have seen before (cf. Example 4, formula (9)) that

$$\alpha_1(2) = \sum_{n=1}^{\infty} \frac{2^{2n-1}}{\binom{2n}{n}}\frac{O_n}{n^2} = \frac{7}{2}\zeta(3)\,,$$

and one also knows (cf. [2], p. 259) that

$$4F(1) = \sum_{n=1}^{\infty} \frac{O_n}{n^2} = \frac{7}{4}\zeta(3)\,.$$

Thus

$$F(1) = \sum_{n=1}^{\infty} \frac{O_n}{(2n)^2} = \frac{1}{4}\sum_{n=1}^{\infty} \frac{2^{2n}}{\binom{2n}{n}}\frac{O_n}{(2n)^2}\,. \qquad (28)$$

The calculation of $\alpha_2(2)$ provides a nice expression of the Ramanujan constant $G(1)$ similar to (28).

Proposition 11. Let $G(1) = \sum_{n=1}^{\infty} \frac{O_n}{(2n)^3}$ be the Ramanujan constant. One has the following formula:

$$G(1) = \frac{7}{8}\zeta(3)\ln 2 - \frac{\pi^4}{256} - \frac{1}{2}\sum_{n=1}^{\infty} \frac{2^{2n}}{\binom{2n}{n}} \frac{O_n}{(2n)^3}\,. \tag{29}$$

Proof. Applying (2.44) and (2.45) of [6] with $u = 4$ and $\theta = \pi$, one obtains an expression of $\alpha_2(2)$ involving $\mathrm{L}_3(1)$. By means of (26) previously established, one can simplify this expression to obtain

$$\alpha_2(2) = 7\zeta(3)\ln 2 - \frac{\pi^4}{32} - 8G(1) \tag{30}$$

and this relation is equivalent to (29). □

Remark 4. Since $\frac{7}{8}\zeta(3) = \sum_{n\geq 0} \frac{1}{(2n+1)^3}$ and $\frac{\pi^3}{32} = \sum_{n\geq 0} \frac{(-1)^n}{(2n+1)^3}$, formula (30) may be rewritten

$$G(1) = 2\ln(2)\sum_{n\geq 0} \frac{1}{(4n+1)^3} - (\pi/8 + \ln(2))\sum_{n\geq 0} \frac{(-1)^n}{(2n+1)^3} - \frac{1}{2}\sum_{n=1}^{\infty} \frac{2^{2n}}{\binom{2n}{n}} \frac{O_n}{(2n)^3} \tag{31}$$

which gets closer to the erroneous formula given by Ramanujan for C.

7 Conclusion

With the aim of defining a natural framework for the study of the special values of zeta functions of Arakawa-Kaneko-type, we were led to consider polylogarithmic series in the generic form:

$$F_k(a, z) = \sum_{n=1}^{\infty} D(a)(n)\,\frac{z^n}{n^k} \quad \text{with} \quad k \in \mathbb{N}, \quad z \in \mathbb{C}, \quad \text{and}$$
$$D(a)(n+1) = \sum_{j=0}^{n} (-1)^j \binom{n}{j} a(j+1)\,.$$

We emphasized on four remarkable cases summarized in the following table:

	$z = 1$	$z = 1/2$
$a(x) = x^{-s}$	$\xi_k(s)$	$\eta_k(s)$
$a(x) = (2x-1)^{-s}$	$\alpha_k(s)$	$\beta_k(s)$

In these cases, when s is a positive integer, $D(a)$ admits a suitable expression in terms of Bell polynomial P_m $(m = s-1)$. However, the scope of this approach is limited in practice by the exponential increase in the size of these polynomials.

References

[1] T. Arakawa, M. Kaneko, Multiple zeta values, Poly-Bernoulli numbers and related zeta functions, *Nagoya Math. J.*, **153** (1999), 189-209.

[2] B. C. Berndt, *Ramanujan's Notebooks Part I*, Springer Verlag, New York, 1985.

[3] B. Candelpergher, M.-A. Coppo, A new class of identities involving Cauchy numbers, harmonic numbers and zeta values, *Ramanujan J.*, **27** (2012), 305-328.

[4] J. Choi, H. M. Srivastava, Explicit evaluation of Euler and related sums, *Ramanujan J.* **10** (2005), 51-70.

[5] M-A. Coppo, B. Candelpergher, The Arakawa-Kaneko Zeta function, *Ramanujan J.*, **22** (2010), 153-162.

[6] A. Davydychev, M. Kalmykov, Massive Feynman diagrams and inverse binomial sums, *Nuclear Physics*, **B 699** (2004), 3-64.

[7] L. Lewin, *Polylogarithms and associated functions*, North-Holland, New York, 1981.

[8] S. Ramanujan, *Notebooks*, Vol. 2, Tata Institute of Fundamental Research, Bombay, 1957.

[9] R. Sitaramachandrarao, A formula of S. Ramanujan, *J. Number Theory*, **25** (1987), 1-19.

[10] P. T. Young, Symmetries of Bernoulli polynomial series and Arakawa-Kaneko zeta functions, article in press, *J. Number Theory*, (2014), http://dx.doi.org/10.1016/j.jnt.2014.02.025.

[11] Zhang N-Y., K. S. Williams, Values of the riemann zeta function and integrals involving $\log(2\sin\frac{\theta}{2})$, *Pacific J. Math.*, **168** (1995), 271-289.

[12] I. J. Zucker, On the series $\sum \binom{2k}{k}^{-1} k^{-n}$ and related sums, *J. Number Theory*, **20** (1985), 92-102.