Inverse binomial series and values of Arakawa-Kaneko zeta functions

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Abstract

In this article, we consider a variety of evaluations of series of polygarithmic nature. In particular, we express the special values at positive integers of certain zeta functions of Arakawa-Kaneko type in terms of series involving inverse binomial sums studied by Kalmykov and Davydychev in relation with the Feynman diagrams.

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1 Introduction

The function β defined for $\Re(s) > 0$ by the Dirichlet series

$$\beta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^s}$$

has the integral representation

$$\beta(s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} \frac{e^{-t}}{1+e^{-2t}} t^{s-1}\, dt = \frac{1}{\Gamma(s)} \int_0^{+\infty} \frac{e^{-t}}{1-e^{-2t}} \operatorname{Li}_0 \left(\frac{1-e^{-2t}}{2} \right) t^{s-1}\, dt$$

where Li_k denotes the classical polylogarithm $\operatorname{Li}_k(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^k}$. One may also observe that

$$\frac{1}{\Gamma(s)} \int_0^{+\infty} \frac{e^{-t}}{1-e^{-2t}} \operatorname{Li}_1(1-e^{-2t})\, t^{s-1}\, dt = (2-2^{-s}) s\, \zeta(s+1)\,.$$

We define two families of functions α_k and β_k by the Mellin transforms

$$\alpha_k(s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} \frac{e^{-t}}{1-e^{-2t}} \operatorname{Li}_k \left(1-e^{-2t}\right) t^{s-1}\, dt \quad \text{for } \Re(s) > 0 \text{ and } k \geq 1\,,$$

$$\beta_k(s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} \frac{e^{-t}}{1-e^{-2t}} \operatorname{Li}_k \left(\frac{1-e^{-2t}}{2} \right) t^{s-1}\, dt \quad \text{for } \Re(s) > 0 \text{ and } k \geq 0\,,$$

so that

$$\alpha_1(s) = (2-2^{-s}) s \zeta(s+1), \quad \text{and} \quad \beta_0(s) = \beta(s)\,.$$

We notice a complete analogy between the couple of functions (α_k, β_k) and the couple (ξ_k, η_k) defined by

$$\xi_k(s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} \frac{e^{-t}}{1-e^{-t}} \operatorname{Li}_k(1-e^{-t})\, t^{s-1}\, dt \quad \text{for } \Re(s) > 0, \quad k \geq 1\,,$$
$$\eta_k(s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} \frac{e^{-t}}{1-e^{-t}} \operatorname{Li}_k(\frac{1-e^{-t}}{2})\, t^{s-1}\, dt \quad \text{for } \Re(s) > 0, \quad k \geq 0\,,$$

where the function ξ_k was introduced by Arakawa and Kaneko (cf. [1], [4]). One can easily verify that

$$\xi_1(s) = s\, \zeta(s+1) = (2-2^{-s})^{-1} \alpha_1(s), \text{ and } \eta_0(s) = (1-2^{1-s}) \zeta(s)\,.$$

The values $\alpha_k(n)$ et $\beta_k(n)$ where n is a positive integer can be expressed by means of the inverse binomial series studied by Kalmykov and Davydychev (cf. [5]) in relation with the Feynman diagrams. More precisely, we obtain the following identities:

$$\alpha_1(1) = \sum_{n=1}^{\infty} \frac{2^{2n-1}}{\binom{2n}{n}} \frac{1}{n^2} = \frac{\pi^2}{4}\,,$$

$$\beta_1(1) = \sum_{n=1}^{\infty} \frac{2^{n-1}}{\binom{2n}{n}} \frac{1}{n^2} = \frac{\pi^2}{16}\,,$$

$$\alpha_2(1) = \sum_{n=1}^{\infty} \frac{2^{2n-1}}{\binom{2n}{n}} \frac{1}{n^3} = \frac{\pi^2}{2}\ln 2 - \frac{7}{4}\zeta(3)\,,$$

$$\beta_2(1) = \sum_{n=1}^{\infty} \frac{2^{n-1}}{\binom{2n}{n}} \frac{1}{n^3} = \frac{\pi^2}{16}\ln 2 + \frac{\pi}{2}G - \frac{35}{32}\zeta(3)\,,$$

$$\alpha_3(1) = \sum_{n=1}^{\infty} \frac{2^{2n-1}}{\binom{2n}{n}} \frac{1}{n^4} = \frac{\pi^2}{2}(\ln 2)^2 - \frac{7}{2}\zeta(3)\ln 2 + \frac{\pi^4}{96} + 4G(1)\,,$$

$$\alpha_1(2) = \sum_{n=1}^{\infty} \frac{2^{2n-1}}{\binom{2n}{n}} \frac{O_n}{n^2} = \frac{7}{2}\zeta(3)\,,$$

$$\beta_1(2) = \sum_{n=1}^{\infty} \frac{2^{n-1}}{\binom{2n}{n}} \frac{O_n}{n^2} = \frac{7}{4}\zeta(3) - \frac{\pi}{2}G\,,$$

$$\alpha_2(2) = \sum_{n=1}^{\infty} \frac{2^{2n-1}}{\binom{2n}{n}} \frac{O_n}{n^3} = 7\zeta(3)\ln 2 - \frac{\pi^4}{32} - 8G(1)\,,$$

where we use the following notations:

$$O_n := \sum_{k=1}^{n} \frac{1}{2k-1} \text{ is the "odd" harmonic sum,}$$

$$G := \beta(2) \text{ is the Catalan constant,}$$

$$G(1) := \sum_{n=1}^{\infty} \frac{O_n}{(2n)^3} \text{ is the Ramanujan constant (cf. [2] p. 257, [9]).}$$

The first two identities were known of Euler (cf. [8] p. 526), both following ones were known of Ramanujan (cf. [2] p. 269), and the four last ones seem to be new. The computation of $\beta_1(n)$ is of particular interest, for it leads to another kind of new relations deduced by means of a series transformation (cf. section 4.1).

2 Bell polynomials and "odd" harmonic numbers

Definition 1. The *modified Bell polynomials* are the polynomials

$$P_m \in \mathbb{Q}[x_1, x_2, ..., x_m]$$

defined for all natural numbers m by $P_0 = 1$ and the generating function

$$\exp\left(\sum_{k=1}^{\infty} x_k \frac{z^k}{k}\right) = \sum_{m=0}^{\infty} P_m(x_1, \cdots, x_m)\, z^m \,,$$

The general explicit expression for P_m is

$$P_m(x_1, ..., x_m) = \sum_{k_1+2k_2+\cdots+mk_m=m} \frac{1}{k_1!k_2!\ldots k_m!} \left(\frac{x_1}{1}\right)^{k_1} \left(\frac{x_2}{2}\right)^{k_2} \cdots \left(\frac{x_m}{m}\right)^{k_m} .$$

Example 1. For the first values of m, one has

$$\begin{aligned}
P_0 &= 1\,,\\
P_1 &= x_1\,,\\
P_2 &= \frac{1}{2}x_1^2 + \frac{1}{2}x_2\,,\\
P_3 &= \frac{1}{6}x_1^3 + \frac{1}{2}x_1x_2 + \frac{1}{3}x_3\,,\\
P_4 &= \frac{1}{24}x_1^4 + \frac{1}{4}x_1^2x_2 + \frac{1}{8}x_2^2 + \frac{1}{3}x_1x_3 + \frac{1}{4}x_4\,.
\end{aligned}$$

Notation. For $s \in \mathbb{C}$ with $\Re(s) \geq 1$ and an integer $n \geq 1$, let $O_n^{(s)}$ be the "odd" harmonic sum:

$$O_n^{(s)} = \sum_{k=1}^{n} \frac{1}{(2k-1)^s}\,, \qquad \text{and} \qquad O_n := O_n^{(1)}\,.$$

Proposition 1. For all integers $m \geq 0$ and $n \geq 1$,

$$P_m(O_n, \ldots, O_n^{(m)}) = \frac{n\binom{2n}{n}}{2^{2n-1}} \int_0^{+\infty} e^{-t}(1-e^{-2t})^{n-1} \frac{t^m}{m!} dt\,. \tag{1}$$

Proof. We are going to prove that

$$\sum_{m=0}^{\infty} P_m(O_n, \cdots, O_n^{(m)})\, z^m = \prod_{j=1}^{n} \frac{2j-1}{2j-1-z} = \frac{n\binom{2n}{n}}{2^{2n-1}} \int_0^{+\infty} e^{tz}(1-e^{-2t})^{n-1} e^{-t} dt\,,$$

and then we shall obtain formula (1) by identification of the coefficients of z^m. On one side, one has

$$\begin{aligned}\prod_{j=1}^{n}\frac{2j-1}{2j-1-z} &= \prod_{j=1}^{n}(1-\frac{z}{2j-1})^{-1}\\ &= \exp(-\sum_{j=1}^{n}\log(1-\frac{z}{2j-1}))\\ &= \exp(\sum_{j=1}^{n}\sum_{k=1}^{+\infty}\frac{z^k}{k(2j-1)^k})\\ &= \exp(\sum_{k=1}^{+\infty}\frac{z^k}{k}\sum_{j=1}^{n}\frac{1}{(2j-1)^k})\,,\end{aligned}$$

thus

$$\prod_{j=1}^{n}\frac{2j-1}{2j-1-z} = \exp(\sum_{k=1}^{\infty}O_n^{(k)}\frac{z^k}{k}) = \sum_{m=0}^{\infty}P_m(O_n,\cdots,O_n^{(m)})\,z^m\,.$$

On the other side, one has

$$\begin{aligned}\prod_{j=1}^{n}\frac{2j-1}{2j-1-z} &= \frac{1}{\sqrt{\pi}}\frac{\Gamma(n+1/2)\Gamma(-z/2+1/2)}{\Gamma(n-z/2+1/2)}\\ &= \frac{1}{\sqrt{\pi}}\frac{\Gamma(n+1/2)}{\Gamma(n)}\frac{\Gamma(n)\Gamma(-z/2+1/2)}{\Gamma(n-z/2+1/2)}\\ &= \frac{n}{2^{2n}}\binom{2n}{n}B(n,-z/2+1/2)\,,\end{aligned}$$

where B is the Euler Beta function. Thus, for $0<|z|<1$, one has

$$\prod_{j=1}^{n}\frac{2j-1}{2j-1-z} = \frac{n\binom{2n}{n}}{2^{2n}}\int_0^1 u^{n-1}(1-u)^{-z/2-1/2}du\,,$$

and making the change of variable $u=1-e^{-2t}$, one then obtains:

$$\prod_{j=1}^{n}\frac{2j-1}{2j-1-z} = \frac{n\binom{2n}{n}}{2^{2n-1}}\int_0^{+\infty}e^{tz}(1-e^{-2t})^{n-1}e^{-t}dt\,,$$

and finally

$$\sum_{m=0}^{\infty}P_m(O_n,\cdots,O_n^{(m)})\,z^m = \frac{n\binom{2n}{n}}{2^{2n-1}}\int_0^{+\infty}e^{tz}(1-e^{-2t})^{n-1}e^{-t}dt\,.$$

□

3 The operators D and S, and the Euler series transformation

Definition 2. Let a be an analytic function in $P = \{x \,|\, \Re(x) \geq 1\}$ defined by

$$a(x) = \int_0^{+\infty} e^{-xt}\widehat{a}(t)dt \quad \text{ for all } x \in P\,,$$

where $\widehat{a} \in \mathcal{C}^1([0, +\infty[)$ is such that there exists $\alpha < 1$, and $C > 0$ with

$$|\widehat{a}(t)| \leq Ce^{\alpha t} \text{ for all } t \in]0, +\infty[\ .$$

For $x \in P$, we define the functions $x \mapsto D(a)(x)$ and $x \mapsto S(a)(x)$ by

$$D(a)(x) = \int_0^{+\infty} \frac{e^{-t}}{1-e^{-t}}(1-e^{-t})^x\,\widehat{a}(t)\,dt\,,$$

$$S(a)(x) = \int_0^{+\infty} \frac{e^{-t}}{1-e^{-t}}(1-e^{-xt})\,\widehat{a}(t)\,dt\,.$$

Proposition 2. For all integers $n \geq 1$, one has

$$S(a)(n) = \sum_{k=1}^{n} a(k)\,,$$

and for all integer $n \geq 0$,

$$D(a)(n+1) = \sum_{k=0}^{n}(-1)^k\binom{n}{k}a(k+1)\,.$$

Proof. The first relation follows from

$$S(a)(n) = \int_0^{+\infty} \frac{e^{-t}-e^{-(n+1)t}}{1-e^{-t}}\widehat{a}(t)dt = \int_0^{+\infty} (\sum_{k=1}^{n} e^{-kt})\widehat{a}(t)dt = \sum_{k=1}^{n}\int_0^{+\infty} e^{-kt}\widehat{a}(t)dt\,.$$

The second relation results from the binomial expansion of $(1-e^{-t})^n$ since

$$\begin{aligned}
D(a)(n+1) &= \int_0^{+\infty} e^{-t}(1-e^{-t})^n\widehat{a}(t)dt \\
&= \int_0^{+\infty} e^{-t}\sum_{k=0}^{n}(-1)^k\binom{n}{k}e^{-kt}\widehat{a}(t)dt \\
&= \sum_{k=0}^{n}(-1)^k\binom{n}{k}\int_0^{+\infty} e^{-t}e^{-kt}\widehat{a}(t)dt \\
&= \sum_{k=0}^{n}(-1)^k\binom{n}{k}a(k+1)\,.
\end{aligned}$$

□

Example 2. For s with $\Re(s) \geq 1$ and $x \in P$, let

$$a(x) = \frac{1}{(2x-1)^s}\,.$$

One has

$$a(x) = \int_0^{+\infty} e^{-(2x-1)t} \frac{t^{s-1}}{\Gamma(s)} dt = \int_0^{+\infty} e^{-xt} \frac{e^{\frac{t}{2}}(\frac{t}{2})^{s-1}}{2\Gamma(s)} dt\,.$$

Thus for all integer $n \geq 1$,

$$D(a)(n) = \int_0^{+\infty} e^{-\frac{t}{2}}(1-e^{-t})^{n-1} \frac{(\frac{t}{2})^{s-1}}{2\Gamma(s)} dt = \int_0^{+\infty} e^{-t}(1-e^{-2t})^{n-1} \frac{t^{s-1}}{\Gamma(s)} dt\,. \quad (2)$$

By Proposition 1 above, one also has

$$\int_0^{+\infty} e^{-t}(1-e^{-2t})^{n-1} \frac{t^m}{m!} dt = \frac{2^{2n-1}}{n\binom{2n}{n}} P_m(O_n, \ldots, O_n^{(m)})\,.$$

Thus, if s is an integer, $s = m+1$ with $m \geq 0$, then we get for all integers $n \geq 1$ the following formula

$$D(\frac{1}{(2x-1)^{m+1}})(n) = \frac{2^{2n-1}}{n\binom{2n}{n}} P_m(O_n, \ldots, O_n^{(m)})\,. \quad (3)$$

Lemma 1. The operators D and S are linked by the following relation:

$$D\left(\frac{1}{x}S(a)\right) = \frac{1}{x}D(a) \text{ for all } x \in P\,.$$

Proof of the lemma. By definition of $S(a)$,

$$\frac{1}{x}S(a)(x) = \int_0^{+\infty} \frac{1-e^{-xt}}{x} \; [\frac{e^{-t}}{1-e^{-t}}\widehat{a}(t)]\, dt\,,$$

integrating by parts, we get

$$\frac{1}{x}S(a)(x) = \int_0^{+\infty} e^{-xt} \left(\int_t^{\infty} \frac{e^{-u}}{1-e^{-u}}\widehat{a}(u)\, du\right) dt\,,$$

this gives

$$\widehat{\frac{1}{x}S(a)}(t) = \int_t^{\infty} \frac{e^{-u}}{1-e^{-u}}\widehat{a}(u)\, du\,.$$

Thus

$$D(\frac{1}{x}S(a))(x) = \int_0^{+\infty} e^{-t}(1-e^{-t})^{x-1}\left(\int_t^{\infty} \frac{e^{-u}}{1-e^{-u}}\widehat{a}(u)\,du\right)\,dt\,,$$

and integrating again by parts, we get

$$D(\frac{1}{x}S(a))(x) = \int_0^{+\infty} \frac{1}{x}(1-e^{-t})^x \frac{e^{-t}}{1-e^{-t}}\widehat{a}(t)\,dt = \frac{1}{x}D(a)(x)\,.$$

□

Proposition 3. For all complex numbers z such that $|z| < \frac{1}{2}$, one has

$$\sum_{n=1}^{+\infty} D(a)(n)z^n = -\sum_{n=1}^{+\infty} a(n)(\frac{z}{z-1})^n\,, \tag{4}$$

$$\sum_{n=1}^{+\infty} \frac{D(a)(n)}{n}z^n = -\sum_{n=1}^{+\infty} \frac{1}{n}S(a)(n)(\frac{z}{z-1})^n\,. \tag{5}$$

Proof. For the first relation we write

$$\begin{aligned}\sum_{n=0}^{+\infty} D(a)(n+1)z^{n+1} &= \int_0^{+\infty} e^{-t}\frac{z}{1-(1-e^{-t})z}\widehat{a}(t)dt \\ &= -\int_0^{+\infty} (\frac{z}{z-1})\frac{e^{-t}}{1-e^{-t}\frac{z}{z-1}}\widehat{a}(t)dt\,.\end{aligned}$$

The expansion

$$(\frac{z}{z-1})\frac{e^{-t}}{1-e^{-t}\frac{z}{z-1}}\widehat{a}(t) = \sum_{n=1}^{+\infty} e^{-nt}(\frac{z}{z-1})^n\widehat{a}(t)$$

gives

$$\begin{aligned}\sum_{n=0}^{+\infty} D(a)(n+1)z^{n+1} &= -\int_0^{+\infty}\sum_{n=1}^{+\infty} e^{-nt}(\frac{z}{z-1})^n\widehat{a}(t)dt \\ &= -\sum_{n=1}^{+\infty}(\frac{z}{z-1})^n\int_0^{+\infty} e^{-nt}\widehat{a}(t)dt\,,\end{aligned}$$

the order of $\int_0^{+\infty}$ and $\sum_{n=1}^{\infty}$ may be interchanged because

$$\int_0^{+\infty}\sum_{n=1}^{+\infty} e^{-nt}(\frac{|z|}{1-|z|})^n|\widehat{a}(t)|dt = (\frac{|z|}{1-|z|})\int_0^{+\infty}\frac{e^{-t}}{1-e^{-t}\frac{|z|}{1-|z|}}|\widehat{a}(t)| < +\infty\,.$$

The second relation is an immediate consequence of the first one by Lemma 1 above. □

Proposition 4. For all integers $p \geq 1$, one has

$$\sum_{n=1}^{\infty} \frac{D(a)(n)}{p^n n^k} = \int_0^{+\infty} \frac{e^{-t}}{1-e^{-t}} \mathrm{Li}_k(\frac{1-e^{-t}}{p})\, \widehat{a}(t)\, dt\,.$$

Proof. Let p be a positive integer, then

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{D(a)(n)}{p^n n^k} &= \sum_{n=1}^{\infty} \frac{1}{p^n n^k} \int_0^{+\infty} \frac{e^{-t}}{1-e^{-t}} (1-e^{-t})^n\, \widehat{a}(t)\, dt \\ &= \int_0^{+\infty} \frac{e^{-t}}{1-e^{-t}} \sum_{n=1}^{\infty} \frac{(1-e^{-t})^n}{p^n n^k}\, \widehat{a}(t)\, dt \\ &= \int_0^{+\infty} \frac{e^{-t}}{1-e^{-t}} \mathrm{Li}_k(\frac{1-e^{-t}}{p})\, \widehat{a}(t)\, dt\,,\end{aligned}$$

the order of $\int_0^{+\infty}$ and $\sum_{n=1}^{\infty}$ may be interchanged since, by the hypothesis on $\widehat{a}$,

$$|\widehat{a}(t)| \leq C e^{\alpha t} \text{ for all } t \in\,]0, +\infty[\ ,$$

which gives

$$\int_0^{+\infty} \frac{e^{-t}}{1-e^{-t}} \sum_{n=1}^{\infty} \frac{(1-e^{-t})^n}{p^n n^k}\, |\widehat{a}(t)|\, dt \leq C \int_0^{+\infty} \frac{e^{-t}}{1-e^{-t}} \mathrm{Li}_k(\frac{1-e^{-t}}{p}) e^{\alpha t}\, dt < +\infty\,.$$

□

4 The functions α_k and β_k

Definition 3. Let k be a positive integer. The functions α_k and β_k are respectively defined for all $s \in \mathbb{C}$ with $\Re(s) > 0$ by

$$\begin{aligned}\alpha_k(s) &= \frac{1}{\Gamma(s)} \int_0^{+\infty} \frac{e^{-t}}{1-e^{-2t}}\, \mathrm{Li}_k\left(1-e^{-2t}\right) t^{s-1}\, dt \quad (\text{for } k \geq 1)\,, \\ \beta_k(s) &= \frac{1}{\Gamma(s)} \int_0^{+\infty} \frac{e^{-t}}{1-e^{-2t}}\, \mathrm{Li}_k\left(\frac{1-e^{-2t}}{2}\right) t^{s-1}\, dt \quad (\text{for } k \geq 0)\,.\end{aligned}$$

Example 3.

$$\begin{aligned}\alpha_1(s) &= \frac{2}{\Gamma(s)} \int_0^{+\infty} \frac{e^{-t}}{1-e^{-2t}} t^s\, dt = 2(1-2^{-s-1}) s\zeta(s+1)\,, \\ \beta_1(s) &= \frac{1}{\Gamma(s)} \int_0^{+\infty} \frac{e^{-t}}{1-e^{-2t}} \left[\ln 2 - \ln(1+e^{-2t})\right] t^{s-1} dt\,.\end{aligned}$$

Proposition 5. If s is such that $\Re(s) \geq 1$, then

$$\alpha_k(s) = \sum_{n=1}^{\infty} \frac{1}{n^k} D(\frac{1}{(2x-1)^s})(n) \quad (\text{for } k \geq 1)\,,$$
$$\beta_k(s) = \sum_{n=1}^{\infty} \frac{1}{2^n n^k} D(\frac{1}{(2x-1)^s})(n) \quad (\text{for } k \geq 0)\,.$$

Proof. This is an immediate consequence of Proposition 4 applied to the function $a(x) = \dfrac{1}{(2x-1)^s}$ (for $p = 1, 2$) since $\widehat{a}(t) = e^{\frac{t}{2}} \dfrac{(\frac{t}{2})^{s-1}}{2\Gamma(s)}$ as already seen in Example 2. □

Corollary 1. For all integers $m \geq 0$, then

$$\alpha_k(m+1) = \sum_{n=1}^{\infty} \frac{2^{2n-1}}{\binom{2n}{n} n^{k+1}} P_m(O_n^{(1)}, O_n^{(2)}, \ldots, O_n^{(m)}) \quad (\text{for } k \geq 1)\,, \tag{6}$$
$$\beta_k(m+1) = \sum_{n=1}^{\infty} \frac{2^{n-1}}{\binom{2n}{n} n^{k+1}} P_m(O_n^{(1)}, O_n^{(2)}, \ldots, O_n^{(m)}) \quad (\text{for } k \geq 0)\,. \tag{7}$$

Proof. This is an immediate consequence of Proposition 5 and formula (3). □

Example 4. Since $\alpha_1(s) = (2 - 2^{-s})s\zeta(s+1)$, then for all integers $m \geq 1$,

$$(2 - 2^{-m})m\zeta(m+1) = \sum_{n=1}^{\infty} \frac{2^{2n-1}}{\binom{2n}{n} n^2} P_{m-1}(O_n^{(1)}, O_n^{(2)}, \ldots, O_n^{(m-1)})\,. \tag{8}$$

In particular, for $m = 1$, this gives

$$\zeta(3) = \frac{1}{7} \sum_{n=1}^{\infty} \frac{2^{2n}}{\binom{2n}{n}} \frac{O_n}{n^2}\,. \tag{9}$$

Example 5. Since $\beta_0(s) = \beta(s)$, one has

$$\sum_{n=1}^{\infty} \frac{2^n}{\binom{2n}{n}} \frac{O_n}{n} = 2G\,, \tag{10}$$

and for all integers $m \geq 1$,

$$\sum_{n=1}^{\infty} \frac{2^n}{\binom{2n}{n} n} P_{2m}(O_n^{(1)}, O_n^{(2)}, \ldots, O_n^{(2m)}) = \frac{E_{2m}}{(2m)!} \left(\frac{\pi}{2}\right)^{2m+1}\,, \tag{11}$$

where E_{2m} $(m = 1, 2, \cdots)$ are the Euler numbers (cf. [8] 11 (c) p. 544) defined by the generating function:

$$\frac{1}{\cos z} = 1 + \sum_{m=1}^{\infty} E_{2m} \frac{z^{2m}}{(2m)!} = 1 + \frac{z^2}{2!} + \frac{5z^4}{4!} + \frac{61z^6}{6!} + \cdots .$$

In particular, for $m = 1$, this gives

$$\sum_{n=1}^{\infty} \frac{2^n}{\binom{2n}{n}} \frac{(O_n)^2}{n} + \sum_{n=1}^{\infty} \frac{2^n}{\binom{2n}{n}} \frac{O_n^{(2)}}{n} = \frac{\pi^3}{8} = 4\beta(3) .$$

Remark 1. In a similar way (cf. [3] § 5.5), one can prove for the couple of functions (ξ_k, η_k) the following identities:

$$\xi_k(s) = \sum_{n=1}^{\infty} \frac{1}{n^k} D(\frac{1}{x^s})(n) \quad (\text{for } \Re(s) \geq 1 \text{ and } k \geq 1) ,$$
$$\eta_k(s) = \sum_{n=1}^{\infty} \frac{1}{2^n n^k} D(\frac{1}{x^s})(n) \quad (\text{for } \Re(s) \geq 1 \text{ and } k \geq 0) ,$$

and, furthermore, one also has (cf. [3] § 3)

$$D(\frac{1}{x^{m+1}})(n) = \frac{P_m(H_n^{(1)}, \ldots, H_n^{(m)})}{n} \text{ with } H_n^{(j)} = \sum_{k=1}^{n} \frac{1}{k^j} \quad (j = 1, 2, \cdots, m) .$$

This gives for example

$$\xi_1(2) = \sum_{n=1}^{\infty} \frac{H_n}{n^2} = 2\zeta(3) ,$$
$$\eta_1(2) = \sum_{n=1}^{\infty} \frac{H_n}{2^n n^2} = \zeta(3) - \frac{\pi^2}{12} \ln 2 \quad ([2] \text{ p. } 258) .$$

4.1 The function β_1

Proposition 6. For all $s \in \mathbb{C}$ with $\Re(s) \geq 1$, one has

$$\beta_1(s) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{O_n^{(s)}}{n} ,$$

hence, for each integer $m \geq 1$,

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{O_n^{(m)}}{n} = \sum_{n=1}^{\infty} \frac{2^{n-1}}{\binom{2n}{n} n^2} P_{m-1}(O_n^{(1)}, O_n^{(2)}, \ldots, O_n^{(m-1)}) . \tag{12}$$

Proof. By Proposition 3 above, we have for all $|z| < \frac{1}{2}$,

$$\sum_{n=1}^{+\infty} \frac{D(a)(n)}{n} z^n = -\sum_{n=1}^{+\infty} \frac{1}{n} S(a)(n) (\frac{z}{z-1})^n .$$

If the series $\sum_{n=1}^{+\infty} \frac{D(a)(n)}{n} \frac{1}{2^n}$ is convergent, then, by the classical Abel lemma, we get

$$\sum_{n=1}^{+\infty} \frac{D(a)(n)}{n} \frac{1}{2^n} = \sum_{n=1}^{+\infty} \frac{1}{n} S(a)(n) (-1)^{n-1} .$$

By Proposition 4 above, the series $\sum_{n=1}^{\infty} \frac{1}{2^n n} D(\frac{1}{(2x-1)^s})(n)$ is convergent and

$$\beta_1(s) = \sum_{n=1}^{\infty} \frac{1}{2^n n} D(\frac{1}{(2x-1)^s})(n) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n} S(\frac{1}{(2x-1)^s})(n) .$$

Then, using formula (3) for $D(\dfrac{1}{(2x-1)^m})(n)$, one obtains (12). □

Example 6.

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{O_n}{n} = \sum_{n=1}^{\infty} \frac{2^{n-1}}{\binom{2n}{n}} \frac{1}{n^2} = \frac{\pi^2}{16} ,$$

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{O_n^{(2)}}{n} = \sum_{n=1}^{\infty} \frac{2^{n-1}}{\binom{2n}{n}} \frac{O_n}{n^2} = \frac{7}{4}\zeta(3) - \frac{\pi}{2} G \qquad (13)$$

([5] (2.36) and (2.37) with $u = 2$ and $\theta = \dfrac{\pi}{2}$) ,

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{O_n^{(3)}}{n} = \sum_{n=1}^{\infty} \frac{2^n}{\binom{2n}{n}} \frac{(O_n)^2}{(2n)^2} + \sum_{n=1}^{\infty} \frac{2^n}{\binom{2n}{n}} \frac{O_n^{(2)}}{(2n)^2} = \frac{\pi^4}{64} - G^2 \qquad (14)$$

([5] (2.38), (2.39), (2.40) and (C.4) with $u = 2$ and $\theta = \dfrac{\pi}{2}$) .

Remark 2. By Remark 1 above, we get also the relation

$$\eta_1(m) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{H_n^{(m)}}{n} = \sum_{n=1}^{\infty} \frac{1}{2^n n^2} P_{m-1}(H_n^{(1)}, H_n^{(2)}, \ldots, H_n^{(m-1)}) \qquad (15)$$

which is similar to (12) . In particular, one has the following relation:

$$\eta_1(2) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{H_n^{(2)}}{n} = \sum_{n=1}^{\infty} \frac{H_n}{2^n n^2} = \zeta(3) - \frac{\pi^2}{12} \ln 2 .$$

Proposition 7. Let

$$\widetilde{H}_n = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k}.$$

If s is such that $\Re(s) > 1$, then

$$\sum_{n=1}^{\infty} \frac{\widetilde{H}_n}{(2n-1)^s} = (1-2^{-s})\zeta(s)\ln 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n(2n-1)^s} - \beta_1(s)\,. \tag{16}$$

Thus, for each integer $m > 1$,

$$\sum_{n=1}^{\infty} \frac{\widetilde{H}_n}{(2n-1)^m} = (1-2^{-m})\zeta(m)\ln 2 + (-1)^m \ln 2 + 2\sum_{k=1}^{m} (-1)^{m-k}\beta(k) - \beta_1(m)\,. \tag{17}$$

Proof. The first relation is a direct consequence of the following elementary result:

If the series $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$, $\sum_{n=1}^{\infty} a_n b_n$ and $\sum_{n=1}^{\infty} b_n \sum_{k=1}^{n} a_k$ are convergent, then the series $\sum_{n\geq 1} a_n \sum_{k=1}^{n} b_k$ is convergent and we have

$$\sum_{n=1}^{\infty} a_n \sum_{k=1}^{n} b_k = \sum_{n=1}^{\infty} a_n \sum_{n=1}^{\infty} b_n + \sum_{n=1}^{\infty} a_n b_n - \sum_{n=1}^{\infty} b_n \sum_{k=1}^{n} a_k\,.$$

Applied to $a_n = \frac{1}{(2n-1)^s}$ and $b_n = \frac{(-1)^{n-1}}{n}$, this relation gives (16). The second relation is a consequence of the first one by the following observation:

If s is an integer, $s = m$, then we have

$$\sum_{k=1}^{m} \frac{(-1)^k}{(2n-1)^k} = \frac{(-1)^m (2n-1)^{-m} - 1}{2n}\,,$$

hence

$$\frac{(-1)^{n-1}}{n(2n-1)^m} = (-1)^m \frac{(-1)^{n-1}}{n} + 2\sum_{k=1}^{m} (-1)^{m-k} \frac{(-1)^{n-1}}{(2n-1)^k}$$

which gives formula (17). □

Example 7. Formula (17) gives respectively for $m = 2$ and $m = 3$ the following identities:

$$\sum_{n=1}^{\infty} \frac{\widetilde{H}_n}{(2n-1)^2} = \frac{\pi^2}{8}\ln 2 + \ln 2 - \frac{\pi}{2} + 2G - \frac{7}{4}\zeta(3) + \frac{\pi G}{2}\,, \tag{18}$$

$$\sum_{n=1}^{\infty} \frac{\widetilde{H}_n}{(2n-1)^3} = \frac{7}{8}\zeta(3)\ln 2 - \ln 2 + \frac{\pi}{2} - 2G + \frac{\pi^3}{16} + G^2 - \frac{\pi^4}{64}\,. \tag{19}$$

5 The values $\alpha_k(1)$ and $\beta_k(1)$

Proposition 8. For each integer $k \geq 1$ and $\alpha \in \mathbb{R}$ such that $0 \leq \alpha \leq 1$, let $\mathrm{L}_k(\alpha)$ be the (generalized) log-sine integral

$$\mathrm{L}_k(\alpha) = \int_0^{\alpha\pi} u \ln^{k-1}\left(2\sin\frac{u}{2}\right) du = \pi^2 \int_0^{\alpha} x \ln^{k-1}\left(2\sin\frac{\pi x}{2}\right) dx\,.$$

Then, one has

$$2\alpha_k(1) = \sum_{n=1}^{\infty} \frac{2^{2n}}{\binom{2n}{n}} \frac{1}{n^{k+1}} = 2^{k-1} \sum_{i=1}^{k} (-1)^{i-1} \frac{(\ln 2)^{k-i}}{(i-1)!(k-i)!} \mathrm{L}_i(1)\,, \tag{20}$$

$$2\beta_k(1) = \sum_{n=1}^{\infty} \frac{2^{n}}{\binom{2n}{n}} \frac{1}{n^{k+1}} = \sum_{i=1}^{k} (-1)^{i-1} \frac{2^{i-1}(\ln 2)^{k-i}}{(i-1)!(k-i)!} \mathrm{L}_i(\frac{1}{2})\,. \tag{21}$$

Proof. The proof is similar to that given in [10] § 4. Let

$$J_k(x) = \frac{1}{2^k} \sum_{n=1}^{\infty} \frac{(2x)^{2n}}{\binom{2n}{n} n^{k+1}}\,.$$

Then, one has for $k \geq 1$,

$$J_k(x) = \int_0^x \frac{J_{k-1}(u)}{u} du\,.$$

By a classical identity due to Euler (cf. [8] p. 526, [10]), one also has

$$J_1(x) = (\arcsin x)^2\,,$$

hence

$$J_0(x) = \frac{2x \arcsin x}{\sqrt{1-x^2}}.$$

It is easily verified that

$$2\alpha_k(1) = 2^k J_k(1) = 2^k \int_0^1 \frac{J_{k-1}(x)}{x} dx\,,$$

and

$$2\beta_k(1) = 2^k J_k(\frac{\sqrt{2}}{2}) = 2^k \int_0^{\frac{\sqrt{2}}{2}} \frac{J_{k-1}(x)}{x} dx\,.$$

By $(k-1)$ integrations by parts and the change of variable $x = \sin\frac{u}{2}$, we get

$$J_k(1) = \frac{(-1)^{k-1}}{(k-1)!} \int_0^1 \ln^{k-1}(x) \frac{J_0(x)}{x} dx = \frac{(-1)^{k-1}}{2(k-1)!} \int_0^{\pi} u \ln^{k-1}(\sin\frac{u}{2}) du\,,$$

and

$$J_k(\frac{\sqrt{2}}{2}) = \frac{(-1)^{k-1}}{(k-1)!}\int_0^{\frac{\sqrt{2}}{2}} \ln^{k-1}(\sqrt{2}x)\frac{J_0(x)}{x}dx = \frac{(-1)^{k-1}}{2(k-1)!}\int_0^{\frac{1}{2}\pi} u\ln^{k-1}(\sqrt{2}\sin\frac{u}{2})du.$$

It remains to use the binomial expansions of

$$\ln^{k-1}(\sin\frac{u}{2}) = \left[\ln(\frac{1}{2}) + \ln(2\sin\frac{u}{2})\right]^{k-1},$$

and

$$\ln^{k-1}(\sqrt{2}\sin\frac{u}{2}) = \left[\ln(\frac{1}{\sqrt{2}}) + \ln(2\sin\frac{u}{2})\right]^{k-1}$$

to obtain formulas (20) and (21). □

Proposition 9. For all α such that $0 \le \alpha \le 1$, we have

$$\text{a) } \mathrm{L}_1(\alpha) = \frac{\pi^2}{2}\alpha^2,$$

$$\text{b) } \mathrm{L}_2(\alpha) = \zeta(3) - \sum_{n=1}^{\infty}\frac{\cos(\pi n\alpha)}{n^3} - \alpha\pi\sum_{n=1}^{\infty}\frac{\sin(\pi n\alpha)}{n^2},$$

$$\text{c) } \mathrm{L}_3(\alpha) = \frac{\pi^4}{16}(\alpha^4 - \frac{8}{3}\alpha^3 + 2\alpha^2) + 2\sum_{n=1}^{\infty}\frac{H_n}{(n+1)^3}\cos(\pi(n+1)\alpha)$$
$$+ 2\pi\alpha\sum_{n=1}^{\infty}\frac{H_n}{(n+1)^2}\sin(\pi(n+1)\alpha) - \frac{1}{2}\zeta(4). \quad (22)$$

Proof. The assertion a) is trivially verified and b) is a classical identity (cf. [6], formula (7.53)). It remains to prove c). We use the following expansion :

$$\mathrm{Log}^2(1-z) = 2\sum_{n=1}^{\infty} H_n\frac{z^{n+1}}{n+1}$$

to get

$$\mathrm{Log}^2\left(1 - e^{-i\pi x}\right) = 2\sum_{n=1}^{\infty} H_n\frac{e^{-i\pi x(n+1)}}{n+1}.$$

Since

$$\begin{aligned}\mathrm{Log}^2\left(1 - e^{-i\pi x}\right) &= \mathrm{Log}^2\left(e^{-i\pi x/2}(e^{i\pi x/2} - e^{-i\pi x/2})\right)\\ &= (-i\pi x/2 + i\pi/2 + \ln(2\sin\frac{1}{2}x\pi))^2\\ &= -\frac{\pi^2}{4}(x-1)^2 + \ln^2(2\sin\frac{\pi x}{2}x) + i\Im\left(\mathrm{Log}^2\left(1 - e^{-i\pi x}\right)\right),\end{aligned}$$

one has

$$\ln^2(2\sin\frac{\pi x}{2}) = \frac{\pi^2}{4}(x-1)^2 + \Re\left(\mathrm{Log}^2\left(1-e^{-i\pi x}\right)\right)\,,$$

hence

$$\ln^2\left(2\sin\frac{\pi x}{2}\right) = \frac{\pi^2}{4}(x-1)^2 + 2\sum_{n=1}^{\infty} H_n \frac{\cos(\pi(n+1)x)}{n+1}\,.$$

Integrating, this gives

$$\int_0^\alpha x\ln^2\left(2\sin\frac{\pi x}{2}\right)\,dx = \frac{\pi^2}{4}\int_0^\alpha x(x-1)^2\,dx + 2\sum_{n=1}^{\infty}\frac{H_n}{n+1}\int_0^\alpha x\cos(\pi(n+1)x)\,dx\,. \tag{23}$$

The permutation of $\sum$ and $\int$ in (23) is justified by the following Lemma 2 and the dominated convergence theorem. The integrals in the right-hand side of (23) are easily computed by

$$\int_0^\alpha x(x-1)^2\,dx = \frac{\alpha^4}{4} - \frac{2\alpha^3}{3} + \frac{\alpha^2}{2}\,,$$

and

$$\int_0^\alpha x\cos(\pi(n+1)x)dx = \frac{\cos(\pi(n+1)\alpha)}{\pi^2(n+1)^2} + \frac{\alpha}{\pi(n+1)}\sin(\pi(n+1)\alpha) - \frac{1}{\pi^2(n+1)^2}\,.$$

Thus, we deduce from (23) the following expression for $\mathrm{L}_3(\alpha)$:

$$\mathrm{L}_3(\alpha) = \frac{\pi^4}{4}(\frac{\alpha^4}{4} - \frac{2\alpha^3}{3} + \frac{\alpha^2}{2}) + 2\sum_{n=1}^{\infty}\frac{H_n}{(n+1)^3}\cos(\pi(n+1)\alpha) + 2\pi\alpha\sum_{n=1}^{\infty}\frac{H_n}{(n+1)^2}\sin(\pi(n+1)\alpha) - 2\sum_{n=1}^{\infty}\frac{H_n}{(n+1)^3}\,.$$

Moreover, one has

$$\sum_{n=1}^{\infty}\frac{H_n}{(n+1)^3} = \sum_{n=1}^{\infty}\frac{H_n}{n^3} - \zeta(4) = \frac{5}{4}\zeta(4) - \zeta(4) = \frac{1}{4}\zeta(4)\,,$$

and this gives (22). □

Lemma 2. The partial sums

$$\sum_{n=1}^{k} H_n \frac{x\cos(\pi(n+1)x)}{n+1}$$

are uniformly bounded for $x \in]0,1[$.

Proof of the lemma. Let $S_n(x) = x\sum_{j=1}^{n}\cos(\pi(j+1)x)$. A sommation by parts gives

$$\sum_{n=1}^{k}\frac{H_n}{n+1}x\cos(\pi(n+1)x) = \sum_{n=1}^{k}S_n(x)(\frac{H_n}{n+1} - \frac{H_{n+1}}{n+2}) + \frac{H_k}{k+1}S_k(x)\,,$$

and one has

$$|S_n(x)| = \left|x\sum_{j=1}^{n}\cos(\pi(j+1)x)\right| \leq \frac{2x}{\sin(\pi x/2)}\,.$$

It follows that, for all $x \in]0,1[$,

$$\left|\sum_{n=1}^{k}xH_n\frac{\cos(\pi(n+1)x)}{n+1}\right| \leq \frac{2x}{\sin(\pi x/2)}(\frac{H_1}{2} - \frac{H_{k+1}}{k+2} + \frac{H_k}{k+1}) \leq \frac{Cx}{\sin(\pi x/2)} \leq C'\,.$$

□

Example 8. Formulae (20) and (21) give for $k = 2$ the following relations:

$$\sum_{n=1}^{\infty}\frac{2^{2n}}{\binom{2n}{n}}\frac{1}{n^3} = \pi^2\ln 2 - \frac{7}{2}\zeta(3)\,,$$
$$\sum_{n=1}^{\infty}\frac{2^{n}}{\binom{2n}{n}}\frac{1}{n^3} = \frac{\pi^2}{8}\ln 2 + \pi G - \frac{35}{16}\zeta(3)\,.$$

These two identities were known of Ramanujan (cf. [2], p. 269).

6 New formulae for Ramanujan's constant $G(1)$

In Chapter 9 of his notebooks (cf. [2] p. 255, Entry 11), Ramanujan introduced the two generating functions[1]

$$F(x) := \sum_{n=1}^{\infty}\frac{O_n x^{2n}}{(2n)^2} \quad \text{and} \quad G(x) := \sum_{n=1}^{\infty}\frac{O_n x^{2n}}{(2n)^3}\,,$$

then, he writes the following functional relation :

$$G(x) + G\left(\frac{1-x}{1+x}\right) = F(x)\log(x) + F\left(\frac{1-x}{1+x}\right)\log\left(\frac{1-x}{1+x}\right) - \frac{1}{16}\log^2(x)\log^2\left(\frac{1-x}{1+x}\right) + C\,, \tag{ii}$$

[1] These functions are respectively quoted ϕ and ψ in the original manuscript : cf. [7] p. 108.

with

$$C=\frac{\pi}{4}\sum_{n=0}^{\infty}\frac{(-1)^n}{(4n+1)^3}-\frac{\pi}{3\sqrt{3}}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^3}\,.$$

Unfortunately, this beautiful formula found by Ramanujan for C turns out to be erroneous since the constant C in (ii) must be equal to

$$G(1)=\sum_{n=1}^{\infty}\frac{O_n}{(2n)^3}\qquad \text{(cf. [2] p. 257, or [9] for more details).}$$

However, we now show how the calculation of $\alpha_3(1)$ and $\alpha_2(2)$ provides two interesting formulae for the constant $G(1)$.

Proposition 10. Let $G(1)=\sum_{n=1}^{\infty}\frac{O_n}{(2n)^3}$ be the Ramanujan constant. One has

$$G(1)=\frac{7}{8}\zeta(3)\ln 2-\frac{\pi^4}{384}-\frac{1}{8}\pi^2(\ln 2)^2+2\sum_{n=1}^{\infty}\frac{2^{2n}}{\binom{2n}{n}}\frac{1}{(2n)^4}\,. \tag{24}$$

Proof. First, we prove the two following identities:

$$G(1)=\frac{35}{64}\zeta(4)-\frac{1}{2}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n}{n^3}\,, \tag{25}$$

and

$$\mathrm{L}_3(1)=\frac{\pi^4}{96}+4G(1)\,. \tag{26}$$

One has

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{O_n}{(2n)^3}&=\sum_{n=1}^{\infty}\frac{H_{2n}-\frac{1}{2}H_n}{(2n)^3}\\ &=\sum_{n=1}^{\infty}\frac{H_{2n}}{(2n)^3}-\sum_{n=1}^{\infty}\frac{1}{2}\frac{H_n}{(2n)^3}\\ &=\sum_{n=1}^{\infty}\frac{1+(-1)^n}{2}\frac{H_n}{n^3}-\sum_{n=1}^{\infty}\frac{1}{2^4}\frac{H_n}{n^3}\\ &=\frac{1}{2}\sum_{n=1}^{\infty}\frac{H_n}{n^3}+\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^nH_n}{n^3}-\frac{1}{2^4}\sum_{n=1}^{\infty}\frac{H_n}{n^3}\\ &=\frac{35}{64}\zeta(4)-\frac{1}{2}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n}{n^3}\,.\end{aligned}$$

Moreover, by Proposition 9 c) with $\alpha=1$, one also has

$$\mathrm{L}_3(1)=\frac{25}{8}\zeta(4)-2\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n}{n^3}=\frac{\pi^4}{96}+4G(1)\,.$$

Then, applying Proposition 8, formula (20), it results from (26) that

$$2\alpha_3(1) = \sum_{n=1}^{\infty} \frac{2^{2n}}{\binom{2n}{n}} \frac{1}{n^4} = \pi^2(\ln 2)^2 - 7\zeta(3)\ln 2 + \frac{\pi^4}{48} + 8G(1)\,. \tag{27}$$

□

Remark 3. We have seen before (cf. Example 4, formula (9)) that

$$\alpha_1(2) = \sum_{n=1}^{\infty} \frac{2^{2n-1}}{\binom{2n}{n}} \frac{O_n}{n^2} = \frac{7}{2}\zeta(3)\,,$$

and one also knows (cf. [2], p. 259) that

$$4F(1) = \sum_{n=1}^{\infty} \frac{O_n}{n^2} = \frac{7}{4}\zeta(3)\,.$$

Thus

$$F(1) = \sum_{n=1}^{\infty} \frac{O_n}{(2n)^2} = \frac{1}{4}\sum_{n=1}^{\infty} \frac{2^{2n}}{\binom{2n}{n}} \frac{O_n}{(2n)^2}\,. \tag{28}$$

The calculation of $\alpha_2(2)$ provides a nice expression of the Ramanujan constant $G(1)$ similar to (28).

Proposition 11. Let $G(1) = \sum_{n=1}^{\infty} \frac{O_n}{(2n)^3}$ be the Ramanujan constant. One has the following formula:

$$G(1) = \frac{7}{8}\zeta(3)\ln 2 - \frac{\pi^4}{256} - \frac{1}{2}\sum_{n=1}^{\infty} \frac{2^{2n}}{\binom{2n}{n}} \frac{O_n}{(2n)^3}\,. \tag{29}$$

Proof. Applying (2.44) and (2.45) of [5] with $u = 4$ and $\theta = \pi$, one obtains an expression of $\alpha_2(2)$ involving $\mathrm{L}_3(1)$. By means of (26) previously established, one can simplify this expression to obtain

$$\alpha_2(2) = 7\zeta(3)\ln 2 - \frac{\pi^4}{32} - 8G(1) \tag{30}$$

which is equivalent to (29). □

Remark 4. Since $\frac{7}{8}\zeta(3) = \sum_{n\geq 0} \frac{1}{(2n+1)^3}$ and $\frac{\pi^3}{32} = \sum_{n\geq 0} \frac{(-1)^n}{(2n+1)^3}$, formula (29) may be rewritten

$$G(1) = 2\ln(2)\sum_{n\geq 0} \frac{1}{(4n+1)^3} - (\pi/8 + \ln(2))\sum_{n\geq 0} \frac{(-1)^n}{(2n+1)^3} - \frac{1}{2}\sum_{n=1}^{\infty} \frac{2^{2n}}{\binom{2n}{n}} \frac{O_n}{(2n)^3}$$

which gets closer to the mysterious erroneous formula given by Ramanujan for C.

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