Inverse binomial series and a constant of Ramanujan

Marc-Antoine Coppo et Bernard Candelpergher
Université de Nice Sophia Antipolis
Laboratoire Jean Alexandre Dieudonné
Parc Valrose
F-06108 Nice Cedex 2
FRANCE

`Bernard.CANDELPERGHER@unice.fr`
`Marc-Antoine.COPPO@unice.fr`

Article submitted to the *Ramanujan Journal*
may 2014

Abstract

In this article, we use a binomial transformation to link, through Bell's polynomials, certain "odd" harmonic series with the inverse binomial series studied by Kalmykov and Davydychev in relation with the Feynman diagrams. Surprisingly, this connection allows us to deduce some new and remarkable identities for the constant $C = \sum_{n\geq 1} \frac{1}{(2n)^3}(1 + \frac{1}{3} + \cdots + \frac{1}{2n-1})$ considered by S. Ramanujan in his notebooks.

Mathematical Subject Classification (2010): 11M06, 11M41, 11Y60, 33B30.

Keywords: Bell polynomials, Harmonic numbers, Inverse binomial series, Binomial transformation, Polylogarithms, Log-sine integrals, Mellin transform, Zeta values, Euler sums.

1 Introduction

The function β defined for $\Re(s) > 0$ by the Dirichlet series

$$\beta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^s}$$

has the integral representation

$$\beta(s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} \frac{e^{-t}}{1+e^{-2t}} t^{s-1}\, dt = \frac{1}{\Gamma(s)} \int_0^{+\infty} \frac{e^{-t}}{1-e^{-2t}} \operatorname{Li}_0\left(\frac{1-e^{-2t}}{2}\right) t^{s-1}\, dt$$

where Li_k denotes the classical polylogarithm $\operatorname{Li}_k(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^k}$. One may also observe that

$$\frac{1}{\Gamma(s)} \int_0^{+\infty} \frac{e^{-t}}{1-e^{-2t}} \operatorname{Li}_1(1-e^{-2t})\, t^{s-1}\, dt = (2-2^{-s}) s\, \zeta(s+1)\,.$$

We define two families of functions α_k and β_k by the Mellin transforms

$$\alpha_k(s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} \frac{e^{-t}}{1-e^{-2t}} \operatorname{Li}_k\left(1-e^{-2t}\right) t^{s-1}\, dt \qquad \text{for } \Re(s) > 0 \text{ and } k \geq 1\,,$$

$$\beta_k(s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} \frac{e^{-t}}{1-e^{-2t}} \operatorname{Li}_k\left(\frac{1-e^{-2t}}{2}\right) t^{s-1}\, dt \qquad \text{for } \Re(s) > 0 \text{ and } k \geq 0\,,$$

so that

$$\alpha_1(s) = (2-2^{-s}) s\zeta(s+1), \qquad \text{and} \qquad \beta_0(s) = \beta(s)\,.$$

We notice a complete analogy between the couple of functions (α_k, β_k) and the couple (ξ_k, η_k) defined by

$$\xi_k(s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} \frac{e^{-t}}{1-e^{-t}} \operatorname{Li}_k(1-e^{-t})\, t^{s-1}\, dt \qquad \text{for } \Re(s) > 0, \quad k \geq 1\,,$$
$$\eta_k(s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} \frac{e^{-t}}{1-e^{-t}} \operatorname{Li}_k(\frac{1-e^{-t}}{2})\, t^{s-1}\, dt \qquad \text{for } \Re(s) > 0, \quad k \geq 0\,,$$

where the function ξ_k was introduced by Arakawa and Kaneko (cf. [1], [4]). One can easily verify that

$$\xi_1(s) = s\,\zeta(s+1) = (2-2^{-s})^{-1}\alpha_1(s), \text{ and } \eta_0(s) = (1-2^{1-s})\zeta(s)\,.$$

The values $\alpha_k(n)$ et $\beta_k(n)$ where n is a positive integer can be expressed by means of the inverse binomial series studied by Kalmykov and Davydychev (cf. [5]) in relation with the Feynman diagrams. More precisely, we obtain the following identities:

$$\alpha_1(1) = \sum_{n=1}^{\infty} \frac{2^{2n-1}}{\binom{2n}{n}} \frac{1}{n^2} = \frac{\pi^2}{4},$$
$$\beta_1(1) = \sum_{n=1}^{\infty} \frac{2^{n-1}}{\binom{2n}{n}} \frac{1}{n^2} = \frac{\pi^2}{16},$$
$$\alpha_2(1) = \sum_{n=1}^{\infty} \frac{2^{2n-1}}{\binom{2n}{n}} \frac{1}{n^3} = \frac{\pi^2}{2}\ln 2 - \frac{7}{4}\zeta(3),$$
$$\beta_2(1) = \sum_{n=1}^{\infty} \frac{2^{n-1}}{\binom{2n}{n}} \frac{1}{n^3} = \frac{\pi^2}{16}\ln 2 + \frac{\pi}{2}G - \frac{35}{32}\zeta(3),$$
$$\alpha_3(1) = \sum_{n=1}^{\infty} \frac{2^{2n-1}}{\binom{2n}{n}} \frac{1}{n^4} = \frac{\pi^2}{2}(\ln 2)^2 - \frac{7}{2}\zeta(3)\ln 2 + \frac{\pi^4}{96} + 4G(1),$$
$$\alpha_1(2) = \sum_{n=1}^{\infty} \frac{2^{2n-1}}{\binom{2n}{n}} \frac{O_n}{n^2} = \frac{7}{2}\zeta(3),$$
$$\beta_1(2) = \sum_{n=1}^{\infty} \frac{2^{n-1}}{\binom{2n}{n}} \frac{O_n}{n^2} = \frac{7}{4}\zeta(3) - \frac{\pi}{2}G,$$
$$\alpha_2(2) = \sum_{n=1}^{\infty} \frac{2^{2n-1}}{\binom{2n}{n}} \frac{O_n}{n^3} = 7\zeta(3)\ln 2 - \frac{\pi^4}{32} - 8G(1),$$

where we use the following notations:

$$O_n := \sum_{k=1}^{n} \frac{1}{2k-1} \text{ is the "odd" harmonic sum,}$$
$$G := \beta(2) \text{ is the Catalan constant,}$$
$$G(1) := \sum_{n=1}^{\infty} \frac{O_n}{(2n)^3} \text{ is the Ramanujan constant (cf. [2] p. 257, [9]).}$$

The first two identities were known of Euler (cf. [8] p. 526), both following ones were known of Ramanujan (cf. [2] p. 269), and the four last ones seem to be new. The computation of $\beta_1(n)$ is of particular interest, for it leads to another kind of new relations deduced by means of a series transformation (cf. section 4.1).

2 Bell polynomials and "odd" harmonic numbers

Definition 1. The *modified Bell polynomials* are the polynomials

$$P_m \in \mathbb{Q}[x_1, x_2, ..., x_m]$$

defined for all natural numbers m by $P_0 = 1$ and the generating function

$$\exp\left(\sum_{k=1}^{\infty} x_k \frac{z^k}{k}\right) = \sum_{m=0}^{\infty} P_m(x_1, \cdots, x_m)\, z^m\,,$$

The general explicit expression for P_m is

$$P_m(x_1, ..., x_m) = \sum_{k_1+2k_2+\cdots+mk_m=m} \frac{1}{k_1!k_2!\ldots k_m!}\left(\frac{x_1}{1}\right)^{k_1}\left(\frac{x_2}{2}\right)^{k_2}\cdots\left(\frac{x_m}{m}\right)^{k_m}.$$

Example 1. For the first values of m, one has

$$\begin{aligned}
P_0 &= 1\,,\\
P_1 &= x_1\,,\\
P_2 &= \frac{1}{2}x_1^2 + \frac{1}{2}x_2\,,\\
P_3 &= \frac{1}{6}x_1^3 + \frac{1}{2}x_1x_2 + \frac{1}{3}x_3\,,\\
P_4 &= \frac{1}{24}x_1^4 + \frac{1}{4}x_1^2x_2 + \frac{1}{8}x_2^2 + \frac{1}{3}x_1x_3 + \frac{1}{4}x_4\,.
\end{aligned}$$

Notation. For $s \in \mathbb{C}$ with $\Re(s) \geq 1$ and an integer $n \geq 1$, let $O_n^{(s)}$ be the "odd" harmonic sum:

$$O_n^{(s)} = \sum_{k=1}^{n} \frac{1}{(2k-1)^s}\,, \qquad \text{and} \qquad O_n := O_n^{(1)}\,.$$

Proposition 1. For all integers $m \geq 0$ and $n \geq 1$,

$$P_m(O_n, \ldots, O_n^{(m)}) = \frac{n\binom{2n}{n}}{2^{2n-1}} \int_0^{+\infty} e^{-t}(1-e^{-2t})^{n-1}\frac{t^m}{m!}dt\,. \tag{1}$$

Proof. We are going to prove that

$$\sum_{m=0}^{\infty} P_m(O_n, \cdots, O_n^{(m)})\, z^m = \prod_{j=1}^{n} \frac{2j-1}{2j-1-z} = \frac{n\binom{2n}{n}}{2^{2n-1}} \int_0^{+\infty} e^{tz}(1-e^{-2t})^{n-1}e^{-t}dt\,,$$

and then we shall obtain formula (1) by identification of the coefficients of z^m. On one side, one has

$$\begin{aligned}\prod_{j=1}^{n}\frac{2j-1}{2j-1-z} &= \prod_{j=1}^{n}(1-\frac{z}{2j-1})^{-1}\\ &= \exp(-\sum_{j=1}^{n}\log(1-\frac{z}{2j-1}))\\ &= \exp(\sum_{j=1}^{n}\sum_{k=1}^{+\infty}\frac{z^k}{k(2j-1)^k})\\ &= \exp(\sum_{k=1}^{+\infty}\frac{z^k}{k}\sum_{j=1}^{n}\frac{1}{(2j-1)^k})\,,\end{aligned}$$

thus

$$\prod_{j=1}^{n}\frac{2j-1}{2j-1-z} = \exp(\sum_{k=1}^{\infty}O_n^{(k)}\frac{z^k}{k}) = \sum_{m=0}^{\infty}P_m(O_n,\cdots,O_n^{(m)})\,z^m\,.$$

On the other side, one has

$$\begin{aligned}\prod_{j=1}^{n}\frac{2j-1}{2j-1-z} &= \frac{1}{\sqrt{\pi}}\frac{\Gamma(n+1/2)\Gamma(-z/2+1/2)}{\Gamma(n-z/2+1/2)}\\ &= \frac{1}{\sqrt{\pi}}\frac{\Gamma(n+1/2)}{\Gamma(n)}\frac{\Gamma(n)\Gamma(-z/2+1/2)}{\Gamma(n-z/2+1/2)}\\ &= \frac{n}{2^{2n}}\binom{2n}{n}B(n,-z/2+1/2)\,,\end{aligned}$$

where B is the Euler Beta function. Thus, for $0<|z|<1$, one has

$$\prod_{j=1}^{n}\frac{2j-1}{2j-1-z} = \frac{n\binom{2n}{n}}{2^{2n}}\int_0^1 u^{n-1}(1-u)^{-z/2-1/2}du\,,$$

and making the change of variable $u=1-e^{-2t}$, one then obtains:

$$\prod_{j=1}^{n}\frac{2j-1}{2j-1-z} = \frac{n\binom{2n}{n}}{2^{2n-1}}\int_0^{+\infty} e^{tz}(1-e^{-2t})^{n-1}e^{-t}dt\,,$$

and finally

$$\sum_{m=0}^{\infty}P_m(O_n,\cdots,O_n^{(m)})\,z^m = \frac{n\binom{2n}{n}}{2^{2n-1}}\int_0^{+\infty} e^{tz}(1-e^{-2t})^{n-1}e^{-t}dt\,.$$

$\square$

3 The operators D and S, and the Euler series transformation

Definition 2. Let a be an analytic function in $P = \{x \,|\, \Re(x) \geq 1\}$ defined by

$$a(x) = \int_0^{+\infty} e^{-xt}\widehat{a}(t)dt \quad \text{ for all } x \in P\,,$$

where $\widehat{a} \in \mathcal{C}^1([0, +\infty[)$ is such that there exists $\alpha < 1$, and $C > 0$ with

$$|\widehat{a}(t)| \leq Ce^{\alpha t} \text{ for all } t \in]0, +\infty[\ .$$

For $x \in P$, we define the functions $x \mapsto D(a)(x)$ and $x \mapsto S(a)(x)$ by

$$D(a)(x) = \int_0^{+\infty} \frac{e^{-t}}{1-e^{-t}}(1-e^{-t})^x\, \widehat{a}(t)\, dt\,,$$

$$S(a)(x) = \int_0^{+\infty} \frac{e^{-t}}{1-e^{-t}}(1-e^{-xt})\, \widehat{a}(t)\, dt\,.$$

Proposition 2. For all integers $n \geq 1$, one has

$$S(a)(n) = \sum_{k=1}^{n} a(k)\,,$$

and for all integer $n \geq 0$,

$$D(a)(n+1) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} a(k+1)\,.$$

Proof. The first relation follows from

$$S(a)(n) = \int_0^{+\infty} \frac{e^{-t} - e^{-(n+1)t}}{1-e^{-t}}\widehat{a}(t)dt = \int_0^{+\infty} (\sum_{k=1}^{n} e^{-kt})\widehat{a}(t)dt = \sum_{k=1}^{n} \int_0^{+\infty} e^{-kt}\widehat{a}(t)dt\,.$$

The second relation results from the binomial expansion of $(1-e^{-t})^n$ since

$$\begin{aligned} D(a)(n+1) &= \int_0^{+\infty} e^{-t}(1-e^{-t})^n\widehat{a}(t)dt \\ &= \int_0^{+\infty} e^{-t}\sum_{k=0}^{n}(-1)^k\binom{n}{k}e^{-kt}\widehat{a}(t)dt \\ &= \sum_{k=0}^{n}(-1)^k\binom{n}{k}\int_0^{+\infty} e^{-t}e^{-kt}\widehat{a}(t)dt \\ &= \sum_{k=0}^{n}(-1)^k\binom{n}{k}a(k+1)\,. \end{aligned}$$

□

Example 2. For s with $\Re(s) \geq 1$ and $x \in P$, let

$$a(x) = \frac{1}{(2x-1)^s} .$$

One has

$$a(x) = \int_0^{+\infty} e^{-(2x-1)t} \frac{t^{s-1}}{\Gamma(s)} dt = \int_0^{+\infty} e^{-xt} \frac{e^{\frac{t}{2}} (\frac{t}{2})^{s-1}}{2\Gamma(s)} dt .$$

Thus for all integer $n \geq 1$,

$$D(a)(n) = \int_0^{+\infty} e^{-\frac{t}{2}} (1-e^{-t})^{n-1} \frac{(\frac{t}{2})^{s-1}}{2\Gamma(s)} dt = \int_0^{+\infty} e^{-t} (1-e^{-2t})^{n-1} \frac{t^{s-1}}{\Gamma(s)} dt . \quad (2)$$

By Proposition 1 above, one also has

$$\int_0^{+\infty} e^{-t} (1-e^{-2t})^{n-1} \frac{t^m}{m!} dt = \frac{2^{2n-1}}{n\binom{2n}{n}} P_m(O_n, \dots, O_n^{(m)}) .$$

Thus, if s is an integer, $s = m+1$ with $m \geq 0$, then we get for all integers $n \geq 1$ the following formula

$$D(\frac{1}{(2x-1)^{m+1}})(n) = \frac{2^{2n-1}}{n\binom{2n}{n}} P_m(O_n, \dots, O_n^{(m)}) . \quad (3)$$

Lemma 1. The operators D and S are linked by the following relation:

$$D\left(\frac{1}{x} S(a)\right) = \frac{1}{x} D(a) \text{ for all } x \in P .$$

Proof of the lemma. By definition of $S(a)$,

$$\frac{1}{x} S(a)(x) = \int_0^{+\infty} \frac{1-e^{-xt}}{x} \, [\frac{e^{-t}}{1-e^{-t}} \widehat{a}(t)] \, dt ,$$

integrating by parts, we get

$$\frac{1}{x} S(a)(x) = \int_0^{+\infty} e^{-xt} \left(\int_t^{\infty} \frac{e^{-u}}{1-e^{-u}} \widehat{a}(u) \, du \right) dt ,$$

this gives

$$\widehat{\frac{1}{x} S(a)}(t) = \int_t^{\infty} \frac{e^{-u}}{1-e^{-u}} \widehat{a}(u) \, du .$$

Thus

$$D(\frac{1}{x}S(a))(x) = \int_0^{+\infty} e^{-t}(1-e^{-t})^{x-1}\left(\int_t^{\infty} \frac{e^{-u}}{1-e^{-u}}\widehat{a}(u)\, du\right) dt\,,$$

and integrating again by parts, we get

$$D(\frac{1}{x}S(a))(x) = \int_0^{+\infty} \frac{1}{x}(1-e^{-t})^x \frac{e^{-t}}{1-e^{-t}}\widehat{a}(t)\, dt = \frac{1}{x}D(a)(x)\,.$$

□

Proposition 3. For all complex numbers z such that $|z| < \frac{1}{2}$, one has

$$\sum_{n=1}^{+\infty} D(a)(n) z^n = -\sum_{n=1}^{+\infty} a(n)(\frac{z}{z-1})^n\,, \tag{4}$$

$$\sum_{n=1}^{+\infty} \frac{D(a)(n)}{n} z^n = -\sum_{n=1}^{+\infty} \frac{1}{n}S(a)(n)(\frac{z}{z-1})^n\,. \tag{5}$$

Proof. For the first relation we write

$$\begin{aligned}\sum_{n=0}^{+\infty} D(a)(n+1)z^{n+1} &= \int_0^{+\infty} e^{-t}\frac{z}{1-(1-e^{-t})z}\widehat{a}(t)dt \\ &= -\int_0^{+\infty} (\frac{z}{z-1})\frac{e^{-t}}{1-e^{-t}\frac{z}{z-1}}\widehat{a}(t)dt\,.\end{aligned}$$

The expansion

$$(\frac{z}{z-1})\frac{e^{-t}}{1-e^{-t}\frac{z}{z-1}}\widehat{a}(t) = \sum_{n=1}^{+\infty} e^{-nt}(\frac{z}{z-1})^n\widehat{a}(t)$$

gives

$$\begin{aligned}\sum_{n=0}^{+\infty} D(a)(n+1)z^{n+1} &= -\int_0^{+\infty}\sum_{n=1}^{+\infty} e^{-nt}(\frac{z}{z-1})^n\widehat{a}(t)dt \\ &= -\sum_{n=1}^{+\infty}(\frac{z}{z-1})^n \int_0^{+\infty} e^{-nt}\widehat{a}(t)dt\,,\end{aligned}$$

the order of $\int_0^{+\infty}$ and $\sum_{n=1}^{\infty}$ may be interchanged because

$$\int_0^{+\infty}\sum_{n=1}^{+\infty} e^{-nt}(\frac{|z|}{1-|z|})^n|\widehat{a}(t)|dt = (\frac{|z|}{1-|z|})\int_0^{+\infty} \frac{e^{-t}}{1-e^{-t}\frac{|z|}{1-|z|}}|\widehat{a}(t)| < +\infty\,.$$

The second relation is an immediate consequence of the first one by Lemma 1 above. □

Proposition 4. For all integers $p \geq 1$, one has

$$\sum_{n=1}^{\infty} \frac{D(a)(n)}{p^n n^k} = \int_0^{+\infty} \frac{e^{-t}}{1-e^{-t}} \mathrm{Li}_k(\frac{1-e^{-t}}{p})\, \widehat{a}(t)\, dt\,.$$

Proof. Let p be a positive integer, then

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{D(a)(n)}{p^n n^k} &= \sum_{n=1}^{\infty} \frac{1}{p^n n^k} \int_0^{+\infty} \frac{e^{-t}}{1-e^{-t}} (1-e^{-t})^n\, \widehat{a}(t)\, dt \\
&= \int_0^{+\infty} \frac{e^{-t}}{1-e^{-t}} \sum_{n=1}^{\infty} \frac{(1-e^{-t})^n}{p^n n^k}\, \widehat{a}(t)\, dt \\
&= \int_0^{+\infty} \frac{e^{-t}}{1-e^{-t}} \mathrm{Li}_k(\frac{1-e^{-t}}{p})\, \widehat{a}(t)\, dt\,,
\end{aligned}$$

the order of $\int_0^{+\infty}$ and $\sum_{n=1}^{\infty}$ may be interchanged since, by the hypothesis on $\widehat{a}$,

$$|\widehat{a}(t)| \leq C e^{\alpha t} \text{ for all } t \in \,]0, +\infty[\ ,$$

which gives

$$\int_0^{+\infty} \frac{e^{-t}}{1-e^{-t}} \sum_{n=1}^{\infty} \frac{(1-e^{-t})^n}{p^n n^k}\, |\widehat{a}(t)|\, dt \leq C \int_0^{+\infty} \frac{e^{-t}}{1-e^{-t}} \mathrm{Li}_k(\frac{1-e^{-t}}{p}) e^{\alpha t}\, dt < +\infty\,.$$

□

4 The functions α_k and β_k

Definition 3. Let k be a positive integer. The functions α_k and β_k are respectively defined for all $s \in \mathbb{C}$ with $\Re(s) > 0$ by

$$\begin{aligned}
\alpha_k(s) &= \frac{1}{\Gamma(s)} \int_0^{+\infty} \frac{e^{-t}}{1-e^{-2t}}\, \mathrm{Li}_k\left(1-e^{-2t}\right) t^{s-1}\, dt \quad (\text{for } k \geq 1)\,, \\
\beta_k(s) &= \frac{1}{\Gamma(s)} \int_0^{+\infty} \frac{e^{-t}}{1-e^{-2t}}\, \mathrm{Li}_k\left(\frac{1-e^{-2t}}{2}\right) t^{s-1}\, dt \quad (\text{for } k \geq 0)\,.
\end{aligned}$$

Example 3.

$$\begin{aligned}
\alpha_1(s) &= \frac{2}{\Gamma(s)} \int_0^{+\infty} \frac{e^{-t}}{1-e^{-2t}} t^s\, dt = 2(1-2^{-s-1}) s\zeta(s+1)\,, \\
\beta_1(s) &= \frac{1}{\Gamma(s)} \int_0^{+\infty} \frac{e^{-t}}{1-e^{-2t}} \left[\ln 2 - \ln(1+e^{-2t})\right] t^{s-1} dt\,.
\end{aligned}$$

Proposition 5. If s is such that $\Re(s) \geq 1$, then

$$\alpha_k(s) = \sum_{n=1}^{\infty} \frac{1}{n^k} D(\frac{1}{(2x-1)^s})(n) \quad (\text{for } k \geq 1)\,,$$
$$\beta_k(s) = \sum_{n=1}^{\infty} \frac{1}{2^n n^k} D(\frac{1}{(2x-1)^s})(n) \quad (\text{for } k \geq 0)\,.$$

Proof. This is an immediate consequence of Proposition 4 applied to the function $a(x) = \dfrac{1}{(2x-1)^s}$ (for $p = 1, 2$) since $\widehat{a}(t) = e^{\frac{t}{2}} \dfrac{(\frac{t}{2})^{s-1}}{2\Gamma(s)}$ as already seen in Example 2. □

Corollary 1. For all integers $m \geq 0$, then

$$\alpha_k(m+1) = \sum_{n=1}^{\infty} \frac{2^{2n-1}}{\binom{2n}{n} n^{k+1}} P_m(O_n^{(1)}, O_n^{(2)}, \ldots, O_n^{(m)}) \quad (\text{for } k \geq 1)\,, \tag{6}$$
$$\beta_k(m+1) = \sum_{n=1}^{\infty} \frac{2^{n-1}}{\binom{2n}{n} n^{k+1}} P_m(O_n^{(1)}, O_n^{(2)}, \ldots, O_n^{(m)}) \quad (\text{for } k \geq 0)\,. \tag{7}$$

Proof. This is an immediate consequence of Proposition 5 and formula (3). □

Example 4. Since $\alpha_1(s) = (2 - 2^{-s})s\zeta(s+1)$, then for all integers $m \geq 1$,

$$(2 - 2^{-m})m\zeta(m+1) = \sum_{n=1}^{\infty} \frac{2^{2n-1}}{\binom{2n}{n} n^2} P_{m-1}(O_n^{(1)}, O_n^{(2)}, \ldots, O_n^{(m-1)})\,. \tag{8}$$

In particular, for $m = 1$, this gives

$$\zeta(3) = \frac{1}{7} \sum_{n=1}^{\infty} \frac{2^{2n}}{\binom{2n}{n}} \frac{O_n}{n^2}\,. \tag{9}$$

Example 5. Since $\beta_0(s) = \beta(s)$, one has

$$\sum_{n=1}^{\infty} \frac{2^n}{\binom{2n}{n}} \frac{O_n}{n} = 2G\,, \tag{10}$$

and for all integers $m \geq 1$,

$$\sum_{n=1}^{\infty} \frac{2^n}{\binom{2n}{n} n} P_{2m}(O_n^{(1)}, O_n^{(2)}, \ldots, O_n^{(2m)}) = \frac{E_{2m}}{(2m)!} \left(\frac{\pi}{2}\right)^{2m+1}\,, \tag{11}$$

where E_{2m} $(m = 1, 2, \cdots)$ are the Euler numbers (cf. [8] 11 (c) p. 544) defined by the generating function:

$$\frac{1}{\cos z} = 1 + \sum_{m=1}^{\infty} E_{2m} \frac{z^{2m}}{(2m)!} = 1 + \frac{z^2}{2!} + \frac{5z^4}{4!} + \frac{61z^6}{6!} + \cdots .$$

In particular, for $m = 1$, this gives

$$\sum_{n=1}^{\infty} \frac{2^n}{\binom{2n}{n}} \frac{(O_n)^2}{n} + \sum_{n=1}^{\infty} \frac{2^n}{\binom{2n}{n}} \frac{O_n^{(2)}}{n} = \frac{\pi^3}{8} = 4\beta(3) .$$

Remark 1. In a similar way (cf. [3] § 5.5), one can prove for the couple of functions (ξ_k, η_k) the following identities:

$$\xi_k(s) = \sum_{n=1}^{\infty} \frac{1}{n^k} D(\frac{1}{x^s})(n) \quad (\text{for } \Re(s) \geq 1 \text{ and } k \geq 1) ,$$
$$\eta_k(s) = \sum_{n=1}^{\infty} \frac{1}{2^n n^k} D(\frac{1}{x^s})(n) \quad (\text{for } \Re(s) \geq 1 \text{ and } k \geq 0) ,$$

and, furthermore, one also has (cf. [3] § 3)

$$D(\frac{1}{x^{m+1}})(n) = \frac{P_m(H_n^{(1)}, \ldots, H_n^{(m)})}{n} \text{ with } H_n^{(j)} = \sum_{k=1}^{n} \frac{1}{k^j} \quad (j = 1, 2, \cdots, m) .$$

This gives for example

$$\xi_1(2) = \sum_{n=1}^{\infty} \frac{H_n}{n^2} = 2\zeta(3) ,$$
$$\eta_1(2) = \sum_{n=1}^{\infty} \frac{H_n}{2^n n^2} = \zeta(3) - \frac{\pi^2}{12} \ln 2 \quad ([2] \text{ p. } 258) \; .$$

4.1 The function β_1

Proposition 6. For all $s \in \mathbb{C}$ with $\Re(s) \geq 1$, one has

$$\beta_1(s) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{O_n^{(s)}}{n} ,$$

hence, for each integer $m \geq 1$,

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{O_n^{(m)}}{n} = \sum_{n=1}^{\infty} \frac{2^{n-1}}{\binom{2n}{n} n^2} P_{m-1}(O_n^{(1)}, O_n^{(2)}, \ldots, O_n^{(m-1)}) . \qquad (12)$$

Proof. By Proposition 3 above, we have for all $|z| < \frac{1}{2}$,

$$\sum_{n=1}^{+\infty} \frac{D(a)(n)}{n} z^n = -\sum_{n=1}^{+\infty} \frac{1}{n} S(a)(n) (\frac{z}{z-1})^n \,.$$

If the series $\sum_{n=1}^{+\infty} \frac{D(a)(n)}{n} \frac{1}{2^n}$ is convergent, then, by the classical Abel lemma, we get

$$\sum_{n=1}^{+\infty} \frac{D(a)(n)}{n} \frac{1}{2^n} = \sum_{n=1}^{+\infty} \frac{1}{n} S(a)(n)(-1)^{n-1} \,.$$

By Proposition 4 above, the series $\sum_{n=1}^{\infty} \frac{1}{2^n n} D(\frac{1}{(2x-1)^s})(n)$ is convergent and

$$\beta_1(s) = \sum_{n=1}^{\infty} \frac{1}{2^n n} D(\frac{1}{(2x-1)^s})(n) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n} S(\frac{1}{(2x-1)^s})(n) \,.$$

Then, using formula (3) for $D(\dfrac{1}{(2x-1)^m})(n)$, one obtains (12). $\square$

Example 6.

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{O_n}{n} = \sum_{n=1}^{\infty} \frac{2^{n-1}}{\binom{2n}{n}} \frac{1}{n^2} = \frac{\pi^2}{16} \,,$$

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{O_n^{(2)}}{n} = \sum_{n=1}^{\infty} \frac{2^{n-1}}{\binom{2n}{n}} \frac{O_n}{n^2} = \frac{7}{4}\zeta(3) - \frac{\pi}{2} G \tag{13}$$

([5] (2.36) and (2.37) with $u = 2$ and $\theta = \dfrac{\pi}{2}$) ,

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{O_n^{(3)}}{n} = \sum_{n=1}^{\infty} \frac{2^n}{\binom{2n}{n}} \frac{(O_n)^2}{(2n)^2} + \sum_{n=1}^{\infty} \frac{2^n}{\binom{2n}{n}} \frac{O_n^{(2)}}{(2n)^2} = \frac{\pi^4}{64} - G^2 \tag{14}$$

([5] (2.38), (2.39), (2.40) and (C.4) with $u = 2$ and $\theta = \dfrac{\pi}{2}$) .

Remark 2. By Remark 1 above, we get also the relation

$$\eta_1(m) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{H_n^{(m)}}{n} = \sum_{n=1}^{\infty} \frac{1}{2^n n^2} P_{m-1}(H_n^{(1)}, H_n^{(2)}, \ldots, H_n^{(m-1)}) \tag{15}$$

which is similar to (12) . In particular, one has the following relation:

$$\eta_1(2) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{H_n^{(2)}}{n} = \sum_{n=1}^{\infty} \frac{H_n}{2^n n^2} = \zeta(3) - \frac{\pi^2}{12} \ln 2 \,.$$

Proposition 7. Let

$$\widetilde{H}_n = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k}.$$

If s is such that $\Re(s) > 1$, then

$$\sum_{n=1}^{\infty} \frac{\widetilde{H}_n}{(2n-1)^s} = (1-2^{-s})\zeta(s)\ln 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n(2n-1)^s} - \beta_1(s)\,. \tag{16}$$

Thus, for each integer $m > 1$,

$$\sum_{n=1}^{\infty} \frac{\widetilde{H}_n}{(2n-1)^m} = (1-2^{-m})\zeta(m)\ln 2 + (-1)^m \ln 2 + 2\sum_{k=1}^{m}(-1)^{m-k}\beta(k) - \beta_1(m)\,. \tag{17}$$

Proof. The first relation is a direct consequence of the following elementary result:

If the series $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$, $\sum_{n=1}^{\infty} a_n b_n$ and $\sum_{n=1}^{\infty} b_n \sum_{k=1}^{n} a_k$ are convergent, then the series $\sum_{n\geq 1} a_n \sum_{k=1}^{n} b_k$ is convergent and we have

$$\sum_{n=1}^{\infty} a_n \sum_{k=1}^{n} b_k = \sum_{n=1}^{\infty} a_n \sum_{n=1}^{\infty} b_n + \sum_{n=1}^{\infty} a_n b_n - \sum_{n=1}^{\infty} b_n \sum_{k=1}^{n} a_k\,.$$

Applied to $a_n = \frac{1}{(2n-1)^s}$ and $b_n = \frac{(-1)^{n-1}}{n}$, this relation gives (16). The second relation is a consequence of the first one by the following observation:

If s is an integer, $s = m$, then we have

$$\sum_{k=1}^{m} \frac{(-1)^k}{(2n-1)^k} = \frac{(-1)^m(2n-1)^{-m} - 1}{2n}\,,$$

hence

$$\frac{(-1)^{n-1}}{n(2n-1)^m} = (-1)^m \frac{(-1)^{n-1}}{n} + 2\sum_{k=1}^{m}(-1)^{m-k}\frac{(-1)^{n-1}}{(2n-1)^k}$$

which gives formula (17). □

Example 7. Formula (17) gives respectively for $m = 2$ and $m = 3$ the following identities:

$$\sum_{n=1}^{\infty} \frac{\widetilde{H}_n}{(2n-1)^2} = \frac{\pi^2}{8}\ln 2 + \ln 2 - \frac{\pi}{2} + 2G - \frac{7}{4}\zeta(3) + \frac{\pi G}{2}\,, \tag{18}$$

$$\sum_{n=1}^{\infty} \frac{\widetilde{H}_n}{(2n-1)^3} = \frac{7}{8}\zeta(3)\ln 2 - \ln 2 + \frac{\pi}{2} - 2G + \frac{\pi^3}{16} + G^2 - \frac{\pi^4}{64}\,. \tag{19}$$

5 The values $\alpha_k(1)$ and $\beta_k(1)$

Proposition 8. For each integer $k \geq 1$ and $\alpha \in \mathbb{R}$ such that $0 \leq \alpha \leq 1$, let $\mathrm{L}_k(\alpha)$ be the (generalized) log-sine integral

$$\mathrm{L}_k(\alpha) = \int_0^{\alpha\pi} u \ln^{k-1}\left(2\sin\frac{u}{2}\right) du = \pi^2 \int_0^{\alpha} x \ln^{k-1}\left(2\sin\frac{\pi x}{2}\right) dx \,.$$

Then, one has

$$2\alpha_k(1) = \sum_{n=1}^{\infty} \frac{2^{2n}}{\binom{2n}{n}} \frac{1}{n^{k+1}} = 2^{k-1} \sum_{i=1}^{k} (-1)^{i-1} \frac{(\ln 2)^{k-i}}{(i-1)!(k-i)!} \mathrm{L}_i(1) \,, \tag{20}$$

$$2\beta_k(1) = \sum_{n=1}^{\infty} \frac{2^{n}}{\binom{2n}{n}} \frac{1}{n^{k+1}} = \sum_{i=1}^{k} (-1)^{i-1} \frac{2^{i-1}(\ln 2)^{k-i}}{(i-1)!(k-i)!} \mathrm{L}_i(\frac{1}{2}) \,. \tag{21}$$

Proof. The proof is similar to that given in [10] § 4. Let

$$J_k(x) = \frac{1}{2^k} \sum_{n=1}^{\infty} \frac{(2x)^{2n}}{\binom{2n}{n} n^{k+1}} \,.$$

Then, one has for $k \geq 1$,

$$J_k(x) = \int_0^x \frac{J_{k-1}(u)}{u} du \,.$$

By a classical identity due to Euler (cf. [8] p. 526, [10]), one also has

$$J_1(x) = (\arcsin x)^2 \,,$$

hence

$$J_0(x) = \frac{2x \arcsin x}{\sqrt{1-x^2}}.$$

It is easily verified that

$$2\alpha_k(1) = 2^k J_k(1) = 2^k \int_0^1 \frac{J_{k-1}(x)}{x} dx \,,$$

and

$$2\beta_k(1) = 2^k J_k(\frac{\sqrt{2}}{2}) = 2^k \int_0^{\frac{\sqrt{2}}{2}} \frac{J_{k-1}(x)}{x} dx \,.$$

By $(k-1)$ integrations by parts and the change of variable $x = \sin\frac{u}{2}$, we get

$$J_k(1) = \frac{(-1)^{k-1}}{(k-1)!} \int_0^1 \ln^{k-1}(x) \frac{J_0(x)}{x} dx = \frac{(-1)^{k-1}}{2(k-1)!} \int_0^{\pi} u \ln^{k-1}(\sin\frac{u}{2}) du \,,$$

and

$$J_k(\frac{\sqrt{2}}{2}) = \frac{(-1)^{k-1}}{(k-1)!}\int_0^{\frac{\sqrt{2}}{2}} \ln^{k-1}(\sqrt{2}x)\frac{J_0(x)}{x}dx = \frac{(-1)^{k-1}}{2(k-1)!}\int_0^{\frac{1}{2}\pi} u\ln^{k-1}(\sqrt{2}\sin\frac{u}{2})du.$$

It remains to use the binomial expansions of

$$\ln^{k-1}(\sin\frac{u}{2}) = \left[\ln(\frac{1}{2}) + \ln(2\sin\frac{u}{2})\right]^{k-1},$$

and

$$\ln^{k-1}(\sqrt{2}\sin\frac{u}{2}) = \left[\ln(\frac{1}{\sqrt{2}}) + \ln(2\sin\frac{u}{2})\right]^{k-1}$$

to obtain formulas (20) and (21). □

Proposition 9. For all α such that $0 \le \alpha \le 1$, we have

$$\begin{aligned}
&\text{a) } \mathrm{L}_1(\alpha) = \frac{\pi^2}{2}\alpha^2\,,\\
&\text{b) } \mathrm{L}_2(\alpha) = \zeta(3) - \sum_{n=1}^{\infty}\frac{\cos(\pi n\alpha)}{n^3} - \alpha\pi\sum_{n=1}^{\infty}\frac{\sin(\pi n\alpha)}{n^2}\,,\\
&\text{c) } \mathrm{L}_3(\alpha) = \frac{\pi^4}{16}(\alpha^4 - \frac{8}{3}\alpha^3 + 2\alpha^2) + 2\sum_{n=1}^{\infty}\frac{H_n}{(n+1)^3}\cos(\pi(n+1)\alpha)\\
&\quad + 2\pi\alpha\sum_{n=1}^{\infty}\frac{H_n}{(n+1)^2}\sin(\pi(n+1)\alpha) - \frac{1}{2}\zeta(4)\,. \qquad (22)
\end{aligned}$$

Proof. The assertion a) is trivially verified and b) is a classical identity (cf. [6], formula (7.53)). It remains to prove c). We use the following expansion :

$$\mathrm{Log}^2(1-z) = 2\sum_{n=1}^{\infty} H_n\frac{z^{n+1}}{n+1}$$

to get

$$\mathrm{Log}^2\left(1 - e^{-i\pi x}\right) = 2\sum_{n=1}^{\infty} H_n\frac{e^{-i\pi x(n+1)}}{n+1}\,.$$

Since

$$\begin{aligned}
\mathrm{Log}^2\left(1 - e^{-i\pi x}\right) &= \mathrm{Log}^2\left(e^{-i\pi x/2}(e^{i\pi x/2} - e^{-i\pi x/2})\right)\\
&= (-i\pi x/2 + i\pi/2 + \ln(2\sin\frac{1}{2}x\pi))^2\\
&= -\frac{\pi^2}{4}(x-1)^2 + \ln^2(2\sin\frac{\pi x}{2}x) + i\Im\left(\mathrm{Log}^2\left(1 - e^{-i\pi x}\right)\right)\,,
\end{aligned}$$

one has

$$\ln^2(2\sin\frac{\pi x}{2}) = \frac{\pi^2}{4}(x-1)^2 + \Re\left(\mathrm{Log}^2\left(1-e^{-i\pi x}\right)\right) ,$$

hence

$$\ln^2\left(2\sin\frac{\pi x}{2}\right) = \frac{\pi^2}{4}(x-1)^2 + 2\sum_{n=1}^{\infty} H_n \frac{\cos(\pi(n+1)x)}{n+1} .$$

Integrating, this gives

$$\int_0^\alpha x\ln^2\left(2\sin\frac{\pi x}{2}\right)\,dx = \frac{\pi^2}{4}\int_0^\alpha x(x-1)^2\,dx + 2\sum_{n=1}^{\infty}\frac{H_n}{n+1}\int_0^\alpha x\cos(\pi(n+1)x)\,dx . \tag{23}$$

The permutation of $\sum$ and $\int$ in (23) is justified by the following Lemma 2 and the dominated convergence theorem. The integrals in the right-hand side of (23) are easily computed by

$$\int_0^\alpha x(x-1)^2\,dx = \frac{\alpha^4}{4} - \frac{2\alpha^3}{3} + \frac{\alpha^2}{2} ,$$

and

$$\int_0^\alpha x\cos(\pi(n+1)x)dx = \frac{\cos(\pi(n+1)\alpha)}{\pi^2(n+1)^2} + \frac{\alpha}{\pi(n+1)}\sin(\pi(n+1)\alpha) - \frac{1}{\pi^2(n+1)^2} .$$

Thus, we deduce from (23) the following expression for $\mathrm{L}_3(\alpha)$:

$$\mathrm{L}_3(\alpha) = \frac{\pi^4}{4}(\frac{\alpha^4}{4} - \frac{2\alpha^3}{3} + \frac{\alpha^2}{2}) + 2\sum_{n=1}^{\infty}\frac{H_n}{(n+1)^3}\cos(\pi(n+1)\alpha) + 2\pi\alpha\sum_{n=1}^{\infty}\frac{H_n}{(n+1)^2}\sin(\pi(n+1)\alpha) - 2\sum_{n=1}^{\infty}\frac{H_n}{(n+1)^3} .$$

Moreover, one has

$$\sum_{n=1}^{\infty}\frac{H_n}{(n+1)^3} = \sum_{n=1}^{\infty}\frac{H_n}{n^3} - \zeta(4) = \frac{5}{4}\zeta(4) - \zeta(4) = \frac{1}{4}\zeta(4) ,$$

and this gives (22). □

Lemma 2. The partial sums

$$\sum_{n=1}^{k} H_n \frac{x\cos(\pi(n+1)x)}{n+1}$$

are uniformly bounded for $x \in]0,1[$.

Proof of the lemma. Let $S_n(x) = x\sum_{j=1}^{n}\cos(\pi(j+1)x)$. A sommation by parts gives

$$\sum_{n=1}^{k}\frac{H_n}{n+1}x\cos(\pi(n+1)x) = \sum_{n=1}^{k}S_n(x)(\frac{H_n}{n+1}-\frac{H_{n+1}}{n+2}) + \frac{H_k}{k+1}S_k(x)\,,$$

and one has

$$|S_n(x)| = \left|x\sum_{j=1}^{n}\cos(\pi(j+1)x)\right| \le \frac{2x}{\sin(\pi x/2)}\,.$$

It follows that, for all $x \in]0,1[$,

$$\left|\sum_{n=1}^{k}xH_n\frac{\cos(\pi(n+1)x)}{n+1}\right| \le \frac{2x}{\sin(\pi x/2)}(\frac{H_1}{2}-\frac{H_{k+1}}{k+2}+\frac{H_k}{k+1}) \le \frac{Cx}{\sin(\pi x/2)} \le C'\,.$$

□

Example 8. Formulae (20) and (21) give for $k=2$ the following relations:

$$\sum_{n=1}^{\infty}\frac{2^{2n}}{\binom{2n}{n}}\frac{1}{n^3} = \pi^2\ln 2 - \frac{7}{2}\zeta(3)\,,$$
$$\sum_{n=1}^{\infty}\frac{2^{n}}{\binom{2n}{n}}\frac{1}{n^3} = \frac{\pi^2}{8}\ln 2 + \pi G - \frac{35}{16}\zeta(3)\,.$$

These two identities were known of Ramanujan (cf. [2], p. 269).

6 New formulae for Ramanujan's constant $G(1)$

In Chapter 9 of his notebooks (cf. [2] p. 255, Entry 11), Ramanujan introduced the two generating functions[1]

$$F(x) := \sum_{n=1}^{\infty}\frac{O_n x^{2n}}{(2n)^2} \quad \text{and} \quad G(x) := \sum_{n=1}^{\infty}\frac{O_n x^{2n}}{(2n)^3}\,,$$

then, he writes the following functional relation :

$$\begin{aligned} G(x) + G\left(\frac{1-x}{1+x}\right) &= F(x)\log(x) + F\left(\frac{1-x}{1+x}\right)\log\left(\frac{1-x}{1+x}\right) \\ &\quad - \frac{1}{16}\log^2(x)\log^2\left(\frac{1-x}{1+x}\right) + C\,, \end{aligned} \tag{ii}$$

[1] These functions are respectively quoted ϕ and ψ in the original manuscript : cf. [7] p. 108.

with

$$C = \frac{\pi}{4}\sum_{n=0}^{\infty}\frac{(-1)^n}{(4n+1)^3} - \frac{\pi}{3\sqrt{3}}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^3}\,.$$

Unfortunately, this beautiful formula found by Ramanujan for C turns out to be erroneous since the constant C in (ii) must be equal to

$$G(1) = \sum_{n=1}^{\infty}\frac{O_n}{(2n)^3} \qquad \text{(cf. [2] p. 257, or [9] for more details).}$$

However, we now show how the calculation of $\alpha_3(1)$ and $\alpha_2(2)$ provides two interesting formulae for the constant $G(1)$.

Proposition 10. Let $G(1) = \sum_{n=1}^{\infty}\frac{O_n}{(2n)^3}$ be the Ramanujan constant. One has

$$G(1) = \frac{7}{8}\zeta(3)\ln 2 - \frac{\pi^4}{384} - \frac{1}{8}\pi^2(\ln 2)^2 + 2\sum_{n=1}^{\infty}\frac{2^{2n}}{\binom{2n}{n}}\frac{1}{(2n)^4}\,. \tag{24}$$

Proof. First, we prove the two following identities:

$$G(1) = \frac{35}{64}\zeta(4) - \frac{1}{2}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n}{n^3}\,, \tag{25}$$

and

$$\mathrm{L}_3(1) = \frac{\pi^4}{96} + 4G(1)\,. \tag{26}$$

One has

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{O_n}{(2n)^3} &= \sum_{n=1}^{\infty}\frac{H_{2n} - \frac{1}{2}H_n}{(2n)^3} \\
&= \sum_{n=1}^{\infty}\frac{H_{2n}}{(2n)^3} - \sum_{n=1}^{\infty}\frac{1}{2}\frac{H_n}{(2n)^3} \\
&= \sum_{n=1}^{\infty}\frac{1+(-1)^n}{2}\frac{H_n}{n^3} - \sum_{n=1}^{\infty}\frac{1}{2^4}\frac{H_n}{n^3} \\
&= \frac{1}{2}\sum_{n=1}^{\infty}\frac{H_n}{n^3} + \frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^n H_n}{n^3} - \frac{1}{2^4}\sum_{n=1}^{\infty}\frac{H_n}{n^3} \\
&= \frac{35}{64}\zeta(4) - \frac{1}{2}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n}{n^3}\,.
\end{aligned}$$

Moreover, by Proposition 9 c) with $\alpha = 1$, one also has

$$\mathrm{L}_3(1) = \frac{25}{8}\zeta(4) - 2\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n}{n^3} = \frac{\pi^4}{96} + 4G(1)\,.$$

Then, applying Proposition 8, formula (20), it results from (26) that

$$2\alpha_3(1) = \sum_{n=1}^{\infty} \frac{2^{2n}}{\binom{2n}{n}} \frac{1}{n^4} = \pi^2(\ln 2)^2 - 7\zeta(3)\ln 2 + \frac{\pi^4}{48} + 8G(1)\,. \tag{27}$$

□

Remark 3. We have seen before (cf. Example 4, formula (9)) that

$$\alpha_1(2) = \sum_{n=1}^{\infty} \frac{2^{2n-1}}{\binom{2n}{n}} \frac{O_n}{n^2} = \frac{7}{2}\zeta(3)\,,$$

and one also knows (cf. [2], p. 259) that

$$4F(1) = \sum_{n=1}^{\infty} \frac{O_n}{n^2} = \frac{7}{4}\zeta(3)\,.$$

Thus

$$F(1) = \sum_{n=1}^{\infty} \frac{O_n}{(2n)^2} = \frac{1}{4}\sum_{n=1}^{\infty} \frac{2^{2n}}{\binom{2n}{n}} \frac{O_n}{(2n)^2}\,. \tag{28}$$

The calculation of $\alpha_2(2)$ provides a nice expression of the Ramanujan constant $G(1)$ similar to (28).

Proposition 11. Let $G(1) = \sum_{n=1}^{\infty} \frac{O_n}{(2n)^3}$ be the Ramanujan constant. One has the following formula:

$$G(1) = \frac{7}{8}\zeta(3)\ln 2 - \frac{\pi^4}{256} - \frac{1}{2}\sum_{n=1}^{\infty} \frac{2^{2n}}{\binom{2n}{n}} \frac{O_n}{(2n)^3}\,. \tag{29}$$

Proof. Applying (2.44) and (2.45) of [5] with $u = 4$ and $\theta = \pi$, one obtains an expression of $\alpha_2(2)$ involving $\mathrm{L}_3(1)$. By means of (26) previously established, one can simplify this expression to obtain

$$\alpha_2(2) = 7\zeta(3)\ln 2 - \frac{\pi^4}{32} - 8G(1) \tag{30}$$

which is equivalent to (29). □

Remark 4. Since $\frac{7}{8}\zeta(3) = \sum_{n\geq 0} \frac{1}{(2n+1)^3}$ and $\frac{\pi^3}{32} = \sum_{n\geq 0} \frac{(-1)^n}{(2n+1)^3}$, formula (29) may be rewritten

$$G(1) = 2\ln(2)\sum_{n\geq 0} \frac{1}{(4n+1)^3} - (\pi/8 + \ln(2))\sum_{n\geq 0} \frac{(-1)^n}{(2n+1)^3} - \frac{1}{2}\sum_{n=1}^{\infty} \frac{2^{2n}}{\binom{2n}{n}} \frac{O_n}{(2n)^3}$$

which gets closer to the mysterious erroneous formula given by Ramanujan for C.

References

[1] T. Arakawa, M. Kaneko, Multiple zeta values, Poly-Bernoulli numbers and related zeta functions, *Nagoya Math. J.*, **153** (1999), 189-209.

[2] B. C. Berndt, *Ramanujan's Notebooks Part I*, Springer Verlag, New York, 1985.

[3] B. Candelpergher, M.-A. Coppo, A new class of identities involving Cauchy numbers, harmonic numbers and zeta values, *Ramanujan J.*, **27** (2012), 305-328.

[4] M-A. Coppo, B. Candelpergher, The Arakawa-Kaneko Zeta function, *Ramanujan J.*, **22** (2010), 153-162.

[5] A. Davydychev, M. Kalmykov, Massive Feynman diagrams and inverse binomial sums, *Nuclear Physics*, **B 699** (2004), 3-64.

[6] L. Lewin, *Polylogarithms and associated functions*, North-Holland, New York, 1981.

[7] S. Ramanujan, *Notebooks*, Vol. 2, Tata Institute of Fundamental Research, Bombay, 1957.

[8] R. Roy, *Sources in the Development of Mathematics. Infinite Series and Products from the Fifteenth to the Twenty-first Century*, Cambridge University Press, New York, 2011.

[9] R. Sitaramachandrarao, A formula of S. Ramanujan, *J. of Number Theory*, **25** (1987), 1-19.

[10] Zhang N-Y., K. S. Williams, Values of the riemann zeta function and integrals involving $\log(2\sin\frac{\theta}{2})$, *Pacific J. Math.*, **168** (1995), 271-289.