Triple scale analysis of periodic solutions and resonance of some asymmetric non linear vibrating systems

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Abstract

We consider *small solutions* of a vibrating mechanical system with smooth non-linearities for which we provide an approximate solution by using a triple scale analysis; a rigorous proof of convergence of the triple scale method is included; for the forced response, a stability result is needed in order to prove convergence in a neighbourhood of a primary resonance. The amplitude of the response with respect to the frequency forcing is described and it is related to the frequency of a free periodic vibration.

Keywords: triple scale expansion; periodic solutions; nonlinear vibrations; normal modes

1 Introduction

In this article, we perform a triple scale analysis of small periodic solutions of free vibrations of a discrete structure without damping and with a local smooth non-linearity; then we consider a similar system with damping and a periodic forcing in a resonance situation.

Several experimental studies show that it is possible to detect defects in a structure by considering its vibro-acoustic response to an external actuation; there is a vast literature in applied physics. We recall some papers related to the use of the frequency response for non destructive testing; in particular generation of higher harmonics, cross-modulation of a high frequency by a low frequency (often called intermodulations in telecommunication): [EDK99], [MCG02]; in [DGLV03], "a vibro-acoustic method, based on frequency modulation, is developed in order to detect defects on aluminium and concrete beams"; experiments have been performed on a real bridge by G. Vanderborck with four prestressed cables: two undamaged cables, a damaged one and a safe one but damaged at the anchor. With routine experimental checking of the lowest natural frequency, the presence of the damaged cable had only been found by comparison with data collected 15 years ago; the one damaged at the anchor was not found; see details in [LVdb04], [RV05].

However the analysis *per se* of non linear vibration is also an important topic from the academic and industrial viewpoint. In this work, we are interested in the behaviour due to a local non linear stress-strain law; first, we consider free vibration and then forced response of a damped system with excitation frequency close to a frequency of the free system ; so, this local stress-strain law is assumed to be: $N = k\tilde{u} + c\tilde{u}^2 + d\tilde{u}^3$, where N is the normal force and $\tilde{u}$ is the elongation. The elastodynamic problem of continuum mechanics leads after discretization by finite elements to a system of non linear differential equations of second order, thus, this paper deals with such systems with several degrees of freedom. We determine an asymptotic expansion of *small periodic solutions* of a discrete structure; we use the method of triple scale [Nay81] and compare these results with a numerical integration program; also, we perform a numerical Fourier transform to determine the frequencies and compare with that of the linear system.

Our approach is only valid in the low frequency range and we have bypassed the propagation of acoustic waves in the structure; this point has been studied in [JL09],[JL12]. The case of rigid contact which is also important from the point of view of theory and applications has been addressed in several papers, for example [JL01], and a synthesis in [BBL13] ; a numerical method to compute periodic solutions is proposed in [LL11] .Asymptotic expansions have been used for a long time; such methods are introduced in the famous memoir of Poincaré [Poi99]; a classic general book on asymptotic methods is [BM55] with french and English translations [BM62, BM61]; introductory material is in [Nay81], [Mil06]; a detailed account of the averaging method with precise proofs of convergence may be found in [SV85]; an analysis of several methods including multiple scale expansion may be found in [Mur91]; the case of vibrations with unilateral springs have been presented in [JR09, JR10, VLP08, HR09a, HR09b, HFR09]; this topic has been presented by H. Hazim at "Congrès Smai" in 2009; more details are to be found in his thesis defended at University of Nice Sophia-Antipolis in 2010. In a forthcoming paper, such a non-smooth case will be considered as well as a numerical algorithm based on the fixed point method used in [Rou11]. The case of vibrations with weak grazing unilateral contact has been presented by S. Junca and Ly Tong at 4th Canadian Conference on Nonlinear Solid Mechanics 2013; in [JPS04] a numerical approach for large solutions of piecewise linear systems is proposed. A review paper for so called "non linear normal modes" may be found in [KPGV09]; it includes numerous papers published by the mechanical engineering community; several application fields have been addressed by this community; for example in [Mik10] "nonlinear vibro-absorption problem, the cylindrical shell nonlinear dynamics and the vehicle suspension nonlinear dynamics are analysed". Preliminary versions of these results may be found in [BR09] and have been presented in conferences [Bra10, Bra]; a proof of convergence of double scale expansion is to be found in the preliminary work [BR13].

In the present text and in the conclusion, we compare the use of double or triple scale expansion. We emphasize that the use of three time scales, instead of two times scales presented in the preliminary work [BR13], provides a much improved insight in the behavior of the forced response close to resonance. *In this paper*, as an introduction, in a first step, we consider *small solutions* of a system with one degree of freedom; we compare free vibration frequency and the frequency of the periodic forcing for which the amplitude is maximal. Then we address a system with several degrees of freedom, we look for periodic free vibrations (so called non linear normal modes in the mechanical engineering community); we compare this frequency with the response to a periodic forcing close to resonance.

2 One degree of freedom, quadratic and cubic non linearity

We consider a stress-strain law with a strong cubic non linearity:

$$N = k\tilde{u} + \Phi(\tilde{u}, \epsilon) \text{ with } \Phi(\tilde{u}, \epsilon) = mc\tilde{u}^2 + \frac{md}{\epsilon}\tilde{u}^3$$

where ϵ is a small parameter which is also involved in the size of the solution; m is the mass, k the linear rigidity of the spring and $\tilde{u}$ the change of length of the spring; the choice of this scaling provides frequencies which are amplitude dependent at first order.

2.1 Free vibration, triple scale expansion up to second order

Using second Newton law, free vibrations of a mass attached to such a spring are governed by:

$$\ddot{\tilde{u}} + \omega^2\tilde{u} + c\tilde{u}^2 + \frac{d}{\epsilon}\tilde{u}^3 = 0. \tag{1}$$

Remark 2.1.

- *We intend to look for a small solution therefore, we consider a change of function* $\boxed{\tilde{u} = \epsilon u}$ *and obtain the transformed equation:*

  $$\ddot{u} + \omega^2 u + \epsilon c u^2 + \epsilon d u^3 = 0.$$

  In this form, this is a Duffing equation for which exists a vast literature, for example see the expository book [KB2011].

- *For the scaling we have chosen, when we use double scale analysis, we remarked in [BR09] that the approximation that we obtain does not involve explicitly the coefficient c of the quadratic term; this coefficient is only involved in the proof of the validity of the expansion. In particular the frequency shift only involves the coefficient d of the cubic term.*

- *However when we use three time scales, the coefficient of the quadratic term is involved in the frequency shift.*

- *On the other hand, if we would let $\epsilon \to +\infty$ in* (1) *, we would get a singular perturbation problem; this is not considered here.*

As we look for a *small* solution with a triple scale analysis for time; we set

$$T_0 = \omega t, \quad T_1 = \epsilon t, \quad T_2 = \epsilon^2 t, \text{ hence } D_0 u = \frac{\partial u}{\partial T_0}, \quad D_1 u = \frac{\partial u}{\partial T_1} \text{ and } D_2 u = \frac{\partial u}{\partial T_2} \tag{2}$$

and we obtain

$$\frac{du}{dt} = \omega D_0 u + \epsilon D_1 u + \epsilon^2 D_2 u$$
$$\frac{d^2u}{dt^2} = \omega^2 D_0^2 u + 2\epsilon\omega D_0 D_1 u + 2\epsilon^2\omega D_0 D_2 u + \epsilon^2 D_1^2 u + 2\epsilon^3 D_1 D_2 u + \epsilon^4 D_2^2 u.$$

As we look for a small solution we consider initial data $\tilde{u}(0) = \epsilon a + \epsilon^2 v_1 + \mathcal{O}(\epsilon^3)$ and $\dot{\tilde{u}}(0) = \mathcal{O}(\epsilon^3)$; or $u(0) = a + \epsilon v_1 + \mathcal{O}(\epsilon^2)$ and $\dot{u}(0) = \mathcal{O}(\epsilon^2)$; we expand the solution with the *ansatz*

$$u(t) = u(T_0, T_1, T_2) = u^{(1)}(T_0, T_1, T_2) + \epsilon u^{(2)}(T_0, T_1, T_2) + \epsilon^2 r(T_0, T_1, T_2); \tag{3}$$

so we obtain:

$$\begin{aligned}\frac{du}{dt} = \frac{du^{(1)}}{dt} + \epsilon\frac{du^{(2)}}{dt} + \epsilon^2\frac{dr}{dt} = \frac{du^{(1)}}{dt} + \epsilon\frac{du^{(2)}}{dt} + \epsilon^2 D_0 r + \epsilon^2(\frac{dr}{dt} - \omega D_0 r)\\ = [\omega D_0 u^{(1)} + \epsilon D_1 u^{(1)} + \epsilon D_2 u^{(1)}] + \epsilon[\omega D_0 u^{(2)} + \epsilon D_1 u^{(2)} + \epsilon D_2 u^{(2)}]\\ + \epsilon^2[\omega D_0 r + \epsilon D_1 r + \epsilon^2 D_2 r]\end{aligned}$$

and with the formula

$$\mathcal{D}_3 r = \frac{1}{\epsilon}\left(\frac{d^2 r}{dt^2} - \omega^2 D_0^2 r\right) = 2\omega D_0 D_1 r + \epsilon\left[2\omega D_0 D_2 r + D_1^2 r + 2\epsilon D_1 D_2 r,\right] + \epsilon^3 D_2^2 r,$$

we get

$$\begin{aligned}\frac{d^2u}{dt^2} = \frac{d^2u^{(1)}}{dt^2} + \epsilon\frac{d^2u^{(2)}}{dt^2} + \epsilon^2\frac{d^2r}{dt^2} = \frac{d^2u^{(1)}}{dt^2} + \epsilon\frac{d^2u^{(2)}}{dt^2} + \epsilon^2 D_0^2 r + \epsilon^3\mathcal{D}_3 r\\ = \omega^2 D_0^2 u^{(1)} + \epsilon\left[2\omega D_0 D_1 u^{(1)} + \omega^2 D_0^2 u^{(2)}\right]\\ + \epsilon^2\left[2\omega D_0 D_2 u^{(1)} + D_1^2 u^{(1)} + 2\omega D_0 D_1 u^{(2)} + D_0^2 r\right]\\ + \epsilon^3\left[2D_1 D_2 u^{(1)} + 2\omega D_0 D_2 u^{(2)} + D_1^2 u^{(2)} + \mathcal{D}_3 r\right]\\ + \epsilon^4\left[D_2^2 u^{(1)} + 2D_1 D_2 u^{(2)} + \epsilon D_2^2 u^{(2)}\right].\end{aligned} \tag{4}$$

We plug expansions (3),(4) into (1); by identifying the powers of ϵ in the expansion of equation (1), we obtain:

$$\left\{\begin{array}{l} D_0^2 u^{(1)} + u^{(1)} = 0\\ \omega^2\left[D_0^2 u^{(2)} + u^{(2)}\right] = S_2\\ \omega^2\left[D_0^2 r + r\right] = S_3\end{array}\right. \tag{5}$$

with

$$\begin{aligned}&S_2 = -cu^{(1)^2} - du^{(1)3} - 2\omega D_0 D_1 u^{(1)} \quad \text{and}\\ &S_3 = -2cu^{(1)}u^{(2)} - 3du^{(1)2}u^{(2)} - 2\omega D_0 D_2 u^{(1)} - D_1^2 u^{(1)} - 2\omega D_0 D_1 u^{(2)} - \epsilon R(\epsilon, r, u^{(1)}, u^{(2)}),\end{aligned}$$

with

$$\begin{aligned}R(\epsilon, r, u^{(1)}, u^{(2)}) = 2D_1 D_2 u^{(1)} + 2\omega D_0 D_2 u^{(2)} + D_1^2 u^{(2)}\\ + cu^{(2)2} + 2cru^{(1)} + 3du^{(1)}u^{(2)2} + 3du^{(1)2}r + \mathcal{D}_3 r\\ + \epsilon\left(D_2^2 u^{(1)} + 2D_1 D_2 u^{(2)} + \epsilon D_2^2 u^{(2)}\right) + \epsilon\rho(u^{(1)}, u^{(2)}, r, \epsilon)\end{aligned}$$

and with ρ, a polynomial in r:

$$\begin{aligned}\rho(u^{(1)}, u^{(2)}, r, \epsilon) = 2cru^{(2)} + du^{(2)3} + +6du^{(1)}u^{(2)}r\\ + \epsilon(cr^2 + 3du^{(2)2}r + 3du^{(1)}r^2) + \epsilon^2[3du^{(2)}r^2 + \epsilon dr^3].\end{aligned}$$

For convenience, we perform the change of variable $\theta(T_0, T_1, T_2) = T_0 + \beta(T_1, T_2)$; we notice that $D_0\theta = 1; D_1\theta = D_1\beta$ and $D_2\theta = D_2\beta$; we solve the first equation of (5) with $D_0 u^{(1)}(0) = 0$, we get:

$$u^{(1)} = a(T_1, T_2)\ \cos(\theta). \tag{6}$$

Remark 2.2. *We notice that a and β are not constants but functions of time scales T_1 and T_2 because u depends on these times scales. The dependence of these functions with respect to T_1 and T_2 will be determined by solving the equations of the following orders and eliminating the so-called secular terms.*

First, we determine the dependence on T_1; with simple manipulation of the second equation of (5), we obtain

$$S_2 = -\frac{ca^2}{2}(\cos(2\theta) + 1) - \frac{da^3}{4}\cos(3\theta) + \cos(\theta)\left(\frac{-3da^3}{4} + 2\omega a D_1\beta\right) + 2\omega D_1 a \sin(\theta)$$

we gather terms at angular frequency ω:

$$S_2 = -\frac{3da^3}{4}cos(\theta) + 2\omega\left[D_1 a \sin(\theta) + a D_1\beta cos(\theta)\right] + S_2^\sharp \quad \text{where}$$

$$S_2^\sharp = \frac{-ca^2}{2}(1 + \cos(2\theta)) - \frac{da^3}{4}\cos(3\theta)$$

It appears some terms at the frequency of the system, these terms provide a solution $u^{(2)}$ of the equation (73) which is non periodic and non bounded over long time intervals. We will eliminate these so-called secular terms by imposing:

$$D_1 a = 0 \text{ and } \quad D_1\beta = \frac{3da^2}{8\omega} \tag{7}$$

the solution of the second equation of (5), is:

$$u^{(2)} = \frac{-ca^2}{2\omega^2} \ + \ \frac{ca^2}{6\omega^2}\cos(2\theta) + \frac{da^3}{32\omega^2}\cos(3\theta). \tag{8}$$

Remark 2.3. *We have omitted the term at frequency ω which is redundant with $u^{(1)}$; however this choice is connected to the value of the initial condition; see Remark 2.5.*

For the third equation of (5), the unknown is r; this equation includes non linearities; we do not solve it but we show that the solution is bounded on an interval dependent on ϵ. We use the values of $u^{(1)}, u^{(2)}$ in S_3. Intermediate computations:

$$u^{(1)}u^{(2)} = \frac{-5ca^3}{12\omega^2}\cos(\theta) + \frac{ca^3}{12\omega^2}\cos(3\theta) \ + \frac{da^4}{64\omega^2}(\cos(2\theta) + \cos(4\theta)).$$

$$(u^{(1)})^2u^{(2)} = \frac{-5ca^4}{24\omega^2} + \frac{da^5}{128\omega^2}\cos(\theta) - \frac{ca^4}{6\omega^2}\cos(2\theta) + \frac{da^5}{64\omega^2}\cos(3\theta) + \frac{ca^4}{24\omega^2}\cos(4\theta) + \frac{da^5}{128\omega^2}\cos(5\theta)$$

The right hand side, after some manipulations is:

$$S_3 = \sin(\theta)\left(2\omega D_2 a + 2D_1 a D_1\beta + a D_1^2\beta\right) + \cos(\theta)\left(2\omega a D_2\beta - D_1^2 a + a(D_1\beta)^2 + \frac{5c^2a^3}{6\omega^2} - \frac{3d^2a^5}{128\omega^2}\right) + S_3^\sharp - \epsilon R(r,\epsilon,u^{(1)},u^{(2)})$$

with

$$S_3^\sharp = \frac{5dca^4}{8\omega^2} + \sin(2\theta)\left(\frac{4ca}{3\omega}D_1 a\right) + \cos(2\theta)\left(\frac{4ca^2}{3\omega}D_1\beta + \frac{15cda^4}{32\omega^2}\right) + \sin(3\theta)\left(\frac{9da^2}{16\omega}D_1 a\right) + \cos(3\theta)\left(\frac{-c^2a^3}{6\omega^2} - \frac{3d^2a^5}{64\omega^2} + \frac{9da^3}{16\omega}D_1\beta\right) + \cos(4\theta)(\frac{-5cda^4}{32\omega^2}) + \cos(5\theta)(\frac{-3d^2a^5}{128\omega^2}).$$

By imposing

$$2\omega D_2 a + 2D_1 a D_1\beta + a D_1^2\beta = 0$$
$$2\omega a D_2\beta - D_1^2 a + a(D_1\beta)^2 + \frac{5c^2a^3}{6\omega^2} - \frac{3d^2a^5}{128\omega^2} = 0$$

we get that $S_3 = S_3^\sharp - \epsilon R(\epsilon, u^{(1)}, u^{(2)}, r)$ no longer contains any term at frequency ω.

As $D_1 a = 0$ and $D_1\beta = \frac{3da^2}{8\omega}$, we obtain

$$2\omega a D_2\beta + a\left(\frac{9d^2a^4}{64\omega^2}\right) + \frac{5c^2a^3}{6\omega^2} - \frac{3d^2a^5}{128\omega^2} = 0.$$

So,

$$D_2 a(T_2) = 0 \quad and \quad D_2\beta(T_2) = (-\frac{5c^2a^2}{12\omega^3} - \frac{15d^2a^4}{256\omega^3}). \tag{9}$$

As a and β do not depend on T_0, we note that:

$$\begin{cases} \dfrac{da}{dt} = \epsilon D_1 a + \epsilon^2 D_2 a + \mathcal{O}(\epsilon^3) \\ \dfrac{d\beta}{dt} = \epsilon D_1\beta + \epsilon^2 D_2\beta + \mathcal{O}(\epsilon^3), \end{cases} \tag{10}$$

thus taking into account (7) and to (9), we obtain:

$$\frac{da}{dt} = 0 \quad \text{and} \quad \frac{d\beta}{dt} = \epsilon\frac{3da^2}{8\omega} + \epsilon^2(\frac{-5c^2a^2}{12\omega^3} - \frac{15d^2a^4}{256\omega^3}) \tag{11}$$

therefore, the solution of these equations is:

$$a = cte \quad \text{and} \quad \beta(t) = \left[\epsilon\frac{3da^2}{8\omega} + \epsilon^2(-\frac{5c^2a^2}{12\omega^3} - \frac{15d^2a^4}{256\omega^3})\right] t. \tag{12}$$

The constant of integration is chosen to be zero as the initial velocity satisfies $\dot{u}(0) = \mathcal{O}(\epsilon^3)$.

In order to show that, r is bounded, after eliminating terms at angular frequency ω, we go back to the t variable in the third equations of (5).

$$\frac{d^2 r}{dt^2} + \omega^2 r = \tilde{S}_3 \tag{13}$$

with $\tilde{S}_3 = S_3^\sharp(t,\epsilon) - \epsilon\tilde{R}(r,\epsilon,u^{(1)},u^{(2)})$ where

$$\begin{aligned} S_3^\sharp(t,\epsilon) = \frac{5dca^4}{8\omega^2} &+ \cos(2(\omega t+\beta(t)))\left(\frac{15cda^4}{32\omega^2} + \frac{cda^4}{2\omega^2}\right) + \sin(2(\omega t+\beta(t)))(\frac{cda^4}{2\omega^2}) \\ &+ \cos(3(\omega t+\beta(t)))\left(\frac{-c^2a^3}{6\omega^2} - \frac{3d^2a^5}{64\omega^2} + \frac{27d^2a^5}{128\omega^2}\right) + \sin(3(\omega t+\beta(t)))(\frac{9d^2a^5}{128\omega^2}) \\ &+ \cos(4(\omega t+\beta(t)))(\frac{-3cda^4}{32\omega^2}) + \cos(5(\omega t+\beta(t)))(\frac{-3d^2a^5}{128\omega^2}) \end{aligned}$$

$$\text{and} \quad \tilde{R} = R(\epsilon,r,u^{(1)},u^{(2)}) - \mathcal{D}_3 r.$$

in which the remainder $\tilde{R}$, the functions $u^{(1)}, u^{(2)}$ and their partial derivatives with respect to T_1, T_2 are expressed with the variable t.

Proposition 2.1. *There exists $\gamma > 0$ such that for all $t \leq t_\epsilon = \frac{\gamma}{\epsilon^2}$, the solution $\tilde{u} = \epsilon u$ of (1) has the following expansion,*

$$\begin{cases} \tilde{u}(t) = \epsilon a\ \cos(\nu_\epsilon t) + \epsilon^2\left(\frac{-ca^2}{2\omega^2} + \frac{ca^2}{6\omega^2}\cos(2\nu_\epsilon t) + \frac{da^3}{32\omega^2}\cos(3\nu_\epsilon t)\right) + \epsilon^3 r(\epsilon,t) \\ \tilde{u}(0) = \epsilon a + \epsilon^2(\frac{-ca^2}{3\omega^2} + \frac{da^3}{32\omega^2}) + O(\epsilon^3), \dot{u}(0) = O(\epsilon^2) \end{cases} \tag{14}$$

with

$$\nu_\epsilon = \omega + \epsilon\frac{3da^2}{8\omega} + \epsilon^2\left(-\frac{5c^2a^2}{12\omega^3} - \frac{15d^2a^4}{256\omega^3}\right) + \mathcal{O}(\epsilon^3) \tag{15}$$

and r is uniformly bounded in $C^2(0,t_\epsilon)$.

Proof. Let us use lemma 5.1 with equation (13); set $S = S_3^\sharp$; as we have enforced (11), it is a periodic bounded function orthogonal to $e^{\pm it}$, it satisfies lemma hypothesis; similarly set $g = \tilde{R}$; it is a polynomial in variable r with coefficients which are bounded functions, so it is a lipschitzian function on bounded subsets and satisfies lemma hypothesis. □

Remark 2.4. *We notice that if we increase c, there is a change of convexity of the mapping $a \mapsto \nu_\epsilon$; this is an effect which cannot be noticed by just obtaining a first order approximation of the frequency with a double scale approximation of the solution as in [BR13]. See numerical results at the end of subsection 2.3.*

Remark 2.5. *We can notice that we can also derive the solution which satisfies $u(0) = \epsilon a$ by adding to the solution $-\epsilon^2(\frac{-ca^2}{3\omega^2} + \frac{da^3}{32\omega^2})\cos(\nu_\epsilon t)$*

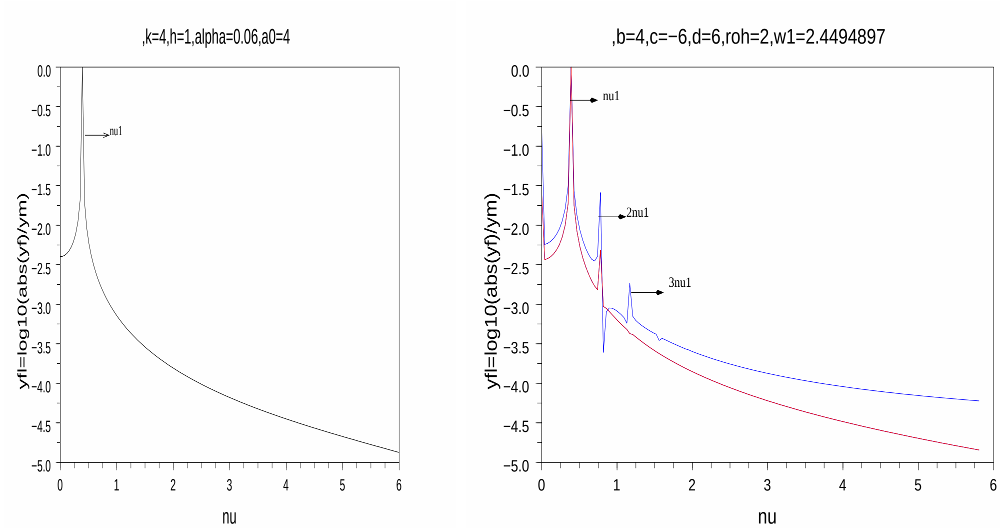

Figure 1: Dynamic frequency shift(fft) linear(left) and a non linear element with two methods(numerical(blue), asymptotic expansion(red))

2.2 Numerical Results

In the figure 1, we find plots of the Fourier transform of solutions; on the left, the linear case, we notice one frequency and on the right, three frequencies in the non linear case. the Fourier transform displays the frequencies, $\nu_1 = 0.164$; $2\nu_1 = 0.329$; $3\nu_1 = 0.493$

We notice good correlation between analytical results of asymptotic expansion and an integration step by step (with Scilab program ODE and numerical fast Fourier transform).

2.3 Forced vibration, triple scale expansion up to second order

2.3.1 Derivation of the expansion

Here we consider a similar system with a sinusoidal forcing at a frequency close to the free frequency; in the linear case without damping, it is well known that the solution is no longer bounded when the forcing frequency goes to the free frequency. Here, we consider the mechanical system of previous section but with periodic forcing and we include some damping term; the scaling of the forcing term is chosen so that the expansion works properly; this is a known point, for example see [Nay86].

$$\ddot{\tilde{u}} + \omega^2 \tilde{u} + \epsilon \lambda \dot{\tilde{u}} + c\tilde{u}^2 + \frac{d}{\epsilon}\tilde{u}^3 = \epsilon^2 F_m \cos(\tilde{\omega_\epsilon} t), \tag{16}$$

where $F_m = \frac{F}{m}$ with the mass m; we assume positive damping, $\lambda > 0$ and excitation frequency ω is *close* to an eigenfrequency of the linear system in the following way:

$$\tilde{\omega_\epsilon} = \omega + \epsilon\sigma. \tag{17}$$

Remark 2.6. • *We look for a small solution with a triple scale expansion; as for the free vibrations, we consider a change of function* $\boxed{\tilde{u} = \epsilon u}$ *and obtain the transformed equation*

$$\ddot{u} + \omega^2 u + \epsilon\lambda\dot{u} + \epsilon c u^2 + \epsilon d u^3 = \epsilon F_m \cos(\tilde{\omega_\epsilon} t).$$

- *To simplify the computations, the fast scale T_0 is chosen to be ϵ dependent.*

We set:

$$T_0 = \tilde{\omega}_\epsilon t, \quad T_1 = \epsilon t \text{ and } T_2 = \epsilon^2 t, \text{ therefore } D_0 u = \frac{\partial u}{\partial T_0}, \quad D_1 u = \frac{\partial u}{\partial T_1} \text{ and } D_2 u = \frac{\partial u}{\partial T_2},$$

so

$$\begin{aligned} &\frac{du}{dt} = \tilde{\omega}_\epsilon D_0 u + \epsilon D_1 u + \epsilon^2 D_2 u \quad \text{and} \\ &\frac{d^2u}{dt^2} = \tilde{\omega}_\epsilon^2 D_0^2 u + 2\epsilon\tilde{\omega}_\epsilon D_0 D_1 u + 2\epsilon^2 D_0 D_2 u + \epsilon^2 D_1^2 u + 2\epsilon^3 D_1 D_2 u + \epsilon^4 D_2^2 u. \end{aligned} \tag{18}$$

With (17), (18) and the following *ansatz*, we look for a small solution:

$$u(t) = u(T_0, T_1, T_2) = u^{(1)}(T_0, T_1, T_2) + \epsilon u^{(2)}(T_0, T_1, T_2) + \epsilon^2 r(T_0, T_1, T_2) \tag{19}$$

we obtain:

$$\begin{aligned} \frac{du}{dt} &= \frac{du^{(1)}}{dt} + \epsilon\frac{du^{(2)}}{dt} + \epsilon^2\frac{dr}{dt} = \frac{du^{(1)}}{dt} + \epsilon\frac{du^{(2)}}{dt} + \epsilon^2 D_0 r + \epsilon^2(\frac{dr}{dt} - D_0 r) \\ &= [(\omega + \epsilon\sigma)D_0 u^{(1)} + \epsilon D_1 u^{(1)} + \epsilon D_2 u^{(1)}] + \epsilon[(\omega + \epsilon\sigma)D_0 u^{(2)} + \epsilon D_1 u^{(2)} + \epsilon^2 D_2 u^{(2)}] \\ &\qquad + \epsilon^2 \omega D_0 r + \epsilon^2(\frac{dr}{dt} - \omega D_0 r) \end{aligned}$$

where we remark that $\frac{dr}{dt} - \omega D_0 r = \epsilon D_1 r + \epsilon^2 D_2 r$ is of degree 1 in ϵ. For the second derivative, as for the case without forcing, we introduce

$$\begin{aligned} \mathcal{D}_3 r &= \frac{1}{\epsilon}(\frac{d^2r}{dt^2} - \omega^2 D_0^2 r) \\ &= 2\tilde{\omega} D_0 D_1 r + \epsilon\left[2\tilde{\omega} D_0 D_2 r + D_1^2 r + 2\epsilon D_2 D_1 r\right] + \epsilon^3 D_2^2 r \end{aligned}$$

and we get

$$\begin{aligned} \frac{d^2u}{dt^2} &= \frac{d^2u^{(1)}}{dt^2} + \epsilon\frac{d^2u^{(2)}}{dt^2} + \epsilon^2\frac{d^2r}{dt^2} = \frac{d^2u^{(1)}}{dt^2} + \epsilon\frac{d^2u^{(2)}}{dt^2} + \epsilon^2\tilde{\omega}^2 D_0^2 r + \epsilon^3 \mathcal{D}_3 r \\ &= \tilde{\omega}^2 D_0^2 u^{(1)} + \epsilon\left[2\tilde{\omega} D_0 D_1 u^{(1)} + \tilde{\omega}^2 D_0^2 u^{(2)}\right] \\ &\quad + \epsilon^2\left[2\tilde{\omega} D_0 D_2 u^{(1)} + D_1^2 u^{(1)} + 2\tilde{\omega} D_0 D_1 u^{(2)} + \tilde{\omega}^2 D_0^2 r\right] \\ &\quad + \epsilon^3\left[2D_1 D_2 u^{(1)} + 2\tilde{\omega} D_0 D_2 u^{(2)} + D_1^2 u^{(2)} + \mathcal{D}_3 r\right] \\ &\quad + \epsilon^4\left[D_2^2 u^{(1)} + 2D_1 D_2 u^{(2)} + \epsilon D_2^2 u^{(2)}\right]. \end{aligned}$$

We plug previous expansions into (16); we obtain:

$$\left\{ \begin{array}{l} D_0^2 u^{(1)} + u^{(1)} = 0 \\ \omega^2 \left(D_0^2 u^{(2)} + u^{(2)} \right) = S_2 \\ \omega^2 \left(D_0^2 r + r \right) = S_3 \end{array} \right. \tag{20}$$

with

$$S_2 = -c{u^{(1)}}^2 - du^{(1)3} - 2\omega D_0 D_1 u^{(1)} - \lambda\omega D_0 u^{(1)} - 2\omega\sigma D_0^2 u^{(1)} + F_m \cos(T_0) \quad \text{and} \tag{21}$$

$$S_3 = -2cu^{(1)}u^{(2)} - 3du^{(1)2}u^{(2)} - 2\omega D_0 D_2 u^{(1)} - D_1^2 u^{(1)} - 2\omega D_0 D_1 u^{(2)} - \sigma^2 D_0^2 u^{(1)} - 2\sigma D_0 D_1 u^{(1)} \tag{22}$$

$$- 2\omega\sigma D_0^2 u^{(2)} - \lambda\omega D_0 u^{(2)} - \lambda D_1 u^{(1)} - \lambda\sigma D_0 u^{(1)} - \epsilon R(\epsilon, r, u^{(1)}, u^{(2)}) \tag{23}$$

with

revoir

$$\begin{aligned} R(\epsilon, r, u^{(1)}, u^{(2)}) &= 2D_1 D_2 u^{(1)} + 2\omega D_0 D_2 u^{(2)} + D_1^2 u^{(2)} \\ &+ cu^{(2)2} + 2cu^{(1)}r + 3du^{(1)}u^{(2)2} + 3du^{(1)2}r + \lambda(\omega D_0 r + D_2 u^{(1)} + D_1 u^{(2)} + \epsilon D_2 u^{(2)}) \\ &+ \epsilon \left(D_2^2 u^{(1)} + 2D_1 D_2 u^{(2)} + \epsilon D_2^2 u^{(2)} \right) + \mathcal{D}_3 r + \lambda(\frac{dr}{dt} - \omega D_0 r) + \epsilon\rho(u^{(1)}, u^{(2)}, r, \epsilon) \end{aligned}$$

and

$$\begin{aligned} \rho(u^{(1)}, u^{(2)}, r, \epsilon) &= 2cru^{(2)} + du^{(2)3} + +6du^{(1)}u^{(2)}r \\ &+ \epsilon(cr^2 + 3du^{(2)2}r + 3du^{(1)}r^2) + \epsilon^2[3du^{(2)}r^2 + \epsilon dr^3]. \end{aligned}$$

We solve the first equation of (20):

$$u^{(1)} = a(T_1, T_2) \ \cos\theta \tag{24}$$

where we have set $\theta(T_0, T_1, T_2) = T_0 + \beta(T_1, T_2)$; we use $\cos(T_0) = \cos(\theta)\cos(\beta) + \sin(\theta)\sin(\beta)$ and we obtain

$$\begin{aligned} S_2 = -\frac{ca^2}{2}(\cos(2\theta) + 1) - \frac{da^3}{4}\cos(3\theta) + \sin(\theta)\left[2\omega D_1 a + \lambda\omega a + F_m \sin(\beta)\right] \\ + \cos(\theta)\left[2\omega a D_1\beta - \frac{3da^3}{4} + 2\omega a\sigma + F_m\cos(\beta)\right] \end{aligned}$$

$$\begin{aligned} \text{or} \quad S_2 = \cos(\theta)\left[\frac{-3da^3}{4} + F_m\cos(\beta)\right] + 2\omega[D_1 a \sin(\theta) + a(D_1\beta + \sigma)\cos(\theta)] \\ + \sin(\theta)\left[\lambda\omega a + F_m\sin(\beta)\right] + S_2^\sharp \end{aligned}$$

$$\text{with} \quad S_2^\sharp = -\frac{ca^2}{2}(\cos(2\theta) + 1) - \frac{da^3}{4}\cos(3\theta).$$

By imposing

$$\left\{\begin{array}{l} 2\omega D_1 a + \lambda\omega a = -F_m \sin(\beta) \\ 2\omega a D_1\beta + 2\omega a\sigma - \frac{3da^3}{4} = -F_m\cos(\beta), \end{array}\right. \tag{25}$$

the solution of the second equation of (20) is:

$$u^{(2)} = \frac{-ca^2}{2\omega^2} \;+\; \frac{ca^2}{6\omega^2}\cos(2\theta) + \frac{da^3}{32\omega^2}\cos(3\theta) \tag{26}$$

where we have omitted the term at the frequency ω is which redundant with $u^{(1)}$.

The third equation of (20) includes non linearities, the unknown is r, we do not solve it, but we show that the solution is bounded on an interval which is ϵ dependent; the right hand side is:

$$\begin{aligned} S_3 = \sin\theta\left[2\omega D_2 a + \lambda a D_1\beta + 2D_1 a D_1\beta + a D_1^2\beta + 2\sigma D_1 a + \lambda a\sigma\right] \\ + \cos\theta\left[2\omega a D_2\beta - \lambda D_1 a - D_1^2 a + a(D_1\beta)^2 + \sigma^2 a + 2\sigma a D_1\beta + \frac{5c^2a^3}{6\omega^2} - \frac{3d^2a^5}{128\omega^2}\right] \\ + S_3^\sharp - \epsilon R(\epsilon, r, u^{(1)}, u^{(2)}) \end{aligned}$$

where revoir

$$\begin{aligned} S_3^\sharp = \frac{5cda^4}{8\omega^2} + \sin 2\theta\left[\frac{4ca}{3\omega}D_1 a + \lambda\frac{ca^2}{3\omega}\right] + \cos 2\theta\left[\frac{4ca^2}{3\omega}D_1\beta + \frac{15cda^4}{32\omega^2}\right] + \\ \sin 3\theta\left[\frac{9da^2}{16\omega}D_1 a + \frac{3\lambda da^3}{16\omega}\right] + \cos 3\theta\left[\frac{9da^3}{16\omega}D_1\beta - \frac{c^2a^3}{6\omega^2} - \frac{3d^2a^5}{64\omega^2}\right] + \\ \cos 4\theta\left[\frac{-3cda^4}{32\omega^2}\right] - \frac{3d^2a^5}{128\omega^2}\cos 5\theta \end{aligned} \tag{27}$$

To eliminate the secular terms, we impose:

$$\left\{\begin{array}{l} 2\omega D_2 a + \lambda a D_1\beta + 2D_1 a D_1\beta + a D_1^2\beta + 2\sigma D_1 a + \lambda a\sigma = 0 \\ 2\omega a D_2\beta - \lambda D_1 a - D_1^2 a + a(D_1\beta)^2 + \sigma^2 a + 2\sigma a D_1\beta + \frac{5c^2a^3}{6\omega^2} - \frac{3d^2a^5}{128\omega^2} = 0. \end{array}\right. \tag{28}$$

In the system (25) the expression of $D_1 a, D_1\beta$ can be extracted:

$$\left\{\begin{array}{l} D_1 a = -\frac{F_m\sin(\beta)}{2\omega} - \frac{\lambda a}{2} \\ D_1\beta = -\sigma - \frac{F_m\cos(\beta)}{2a\omega} + \frac{3da^2}{8\omega} \end{array}\right. \tag{29}$$

As the functions a and β do not depend on T_0, the following relations hold:

$$\frac{da}{dt} = \epsilon D_1 a + \epsilon^2 D_2 a + \prime\epsilon^3) \tag{30}$$

$$\frac{d\beta}{dt} = \epsilon D_1\beta + \epsilon^2 D_2\beta + \prime\epsilon^3). \tag{31}$$

We are going to express $\frac{da}{dt}, \frac{d\beta}{dt}$ as functions of a, β. We manipulate equation (28)

$$\left\{\begin{array}{l} 2\omega D_2 a + (\lambda a + 2D_1 a)(\sigma + D_1\beta) - a D_1^2\beta = 0 \\ 2\omega a D_2\beta - \lambda D_1 a - D_1^2 a + a(\sigma + D_1\beta)^2 + \frac{5c^2a^3}{6\omega^2} - \frac{3d^2a^5}{128\omega^2} = 0 \end{array}\right.$$

then, we replace $D_1 a, D_1\beta$ by their expression in (29), we get

$$\begin{cases} 2\omega D_2 a - \frac{F_m \sin(\beta)}{\omega}(\sigma + D_1\beta) - a D_1^2\beta = 0 \\ -2\omega a D_2\beta - \lambda D_1 a - D_1^2 a + a(\sigma + D_1\beta)^2 + \frac{5c^2a^3}{6\omega^2} - \frac{3d^2a^5}{128\omega^2} = 0 \end{cases}$$

and

$$\begin{cases} 2\omega D_2 a - \frac{F_m \sin(\beta)}{\omega}(-\frac{F_m\cos(\beta)}{2a\omega} + \frac{3da^2}{8\omega}) - a D_1^2\beta = 0 \\ -2\omega a D_2\beta - \lambda(-\frac{F_m\sin(\beta)}{2\omega} - \frac{\lambda a}{2}) - D_1^2 a + a(\frac{F_m\cos(\beta)}{2a\omega} - \frac{3da^2}{8\omega})^2 + \frac{5c^2a^3}{6\omega^2} - \frac{3d^2a^5}{128\omega 2} = 0. \end{cases} \tag{32}$$

On the other hand, we can determine $D_1^2 a$ and $D_1^2\beta$ by differentiating (29);

$$\begin{aligned} D_1^2 a &= -\frac{F_m\cos(\beta)D_1\beta}{2\omega} - \frac{\lambda D_1 a}{2} \\ D_1^2\beta &= \frac{F_m\sin(\beta)D_1\beta}{2a\omega} + \left(\frac{F_m\cos(\beta)}{2a^2\omega} + \frac{3da}{4\omega}\right) D_1 a \end{aligned}$$

or with (29)

$$\begin{aligned} D_1^2 a &= -\frac{F_m\cos(\beta)}{2\omega}\left(-\sigma - \frac{F_m\cos(\beta)}{2a\omega} + \frac{3da^2}{8\omega}\right) - \frac{\lambda}{2}\left(-\frac{F_m\sin(\beta)}{2\omega} - \frac{\lambda a}{2}\right) \\ D_1^2\beta &= \frac{F_m\sin(\beta)}{2a\omega}\left(-\sigma - \frac{F_m\cos(\beta)}{2a\omega} + \frac{3da^2}{8\omega}\right) + \left(\frac{F_m\cos(\beta)}{2a^2\omega} + \frac{3da}{4\omega}\right)\left(-\frac{F_m\sin(\beta)}{2\omega} - \frac{\lambda a}{2}\right) \end{aligned}$$

or

$$\begin{aligned} D_1^2 a &= \frac{\sigma F_m\cos(\beta)}{2\omega} + \frac{F_m^2\cos^2(\beta)}{4a\omega^2} - \frac{3da^2F_m\cos(\beta)}{16\omega^2} + \frac{\lambda F_m\sin(\beta)}{4\omega} + \frac{\lambda^2 a}{4} \\ D_1^2\beta &= -\frac{\sigma F_m\sin(\beta)}{2a\omega} - \frac{F_m^2\sin(\beta)\cos(\beta)}{2a^2\omega^2} - \frac{3daF_m\sin(\beta)}{16\omega^2} - \frac{\lambda F_m\cos(\beta)}{4a\omega} - \frac{3d\lambda a^2}{8\omega}. \end{aligned}$$

Then, in (32) we use previous formula

$$\begin{cases} 2\omega D_2 a - \frac{F_m\sin(\beta)}{\omega}(-\frac{F_m\cos(\beta)}{2a\omega} + \frac{3da^2}{8\omega}) \\ \qquad +a\left(-\frac{\sigma F_m\sin(\beta)}{2a\omega} - \frac{F_m^2\sin(\beta)\cos(\beta)}{2a^2\omega^2} - \frac{3daF_m\sin(\beta)}{16\omega^2} - \frac{\lambda F_m\cos(\beta)}{4a\omega} - \frac{3d\lambda a^2}{8\omega}\right) = 0 \\ 2\omega a D_2\beta - \lambda(-\frac{F_m\sin(\beta)}{2\omega} - \frac{\lambda a}{2}) - \left(\frac{\sigma F_m\cos(\beta)}{2\omega} + \frac{F_m^2\cos^2(\beta)}{4a\omega^2} - \frac{3da^2F_m\cos(\beta)}{16\omega^2} + \frac{\lambda F_m\sin(\beta)}{4\omega} + \frac{\lambda^2 a}{4}\right) \\ \qquad +a(\frac{-F_m\cos(\beta)}{2a\omega} + \frac{3da^2}{8\omega})^2 + \frac{5c^2a^3}{6\omega^2} - \frac{3d^2a^5}{128\omega^2} = 0 \end{cases}$$

we manipulate

$$\begin{cases} 2\omega D_2 a - \frac{9da^2F_m\sin(\beta)}{16\omega^2} - \frac{\sigma F_m\sin(\beta)}{2\omega} - \frac{\lambda F_m\cos(\beta)}{4\omega} - \frac{3d\lambda a^3}{8\omega} = 0 \\ 2\omega a D_2\beta + \frac{\lambda F_m\sin(\beta)}{4\omega} + \frac{\lambda^2 a}{4} - \frac{\sigma F_m\cos(\beta)}{2\omega} - \frac{3da^2F_m\cos(\beta)}{16\omega^2} - \frac{15d^2a^5}{128\omega^2} + \frac{5c^2a^3}{6\omega^2} = 0 \end{cases}$$

and we obtain:

$$\begin{cases} D_2 a = \frac{3d\lambda a^3}{16\omega^2} + \frac{\sigma F_m\sin\beta}{4\omega^2} + \frac{\lambda F_m\cos\beta}{8\omega^2} + \frac{9da^2F_m\sin\beta}{32\omega^3} \\ D_2\beta = -\frac{\lambda^2}{8\omega} - \frac{15d^2a^4}{256\omega^3} - \frac{5c^2a^2}{12\omega^3} + \frac{\sigma F_m\cos\beta}{4\omega^2 a} + \frac{3daF_m\cos\beta}{32\omega^3} - \frac{\lambda F_m\sin\beta}{8\omega^2 a}. \end{cases} \tag{33}$$

Now we return to (30) introducing (29) and (33), we obtain:

$$\left\{\begin{array}{l}\frac{da}{dt}=\epsilon\left(-\frac{F_m\sin(\beta)}{2\omega}-\frac{\lambda a}{2}\right)\\ \qquad\qquad+\epsilon^2\left(\frac{3d\lambda a^3}{16\omega^2}+\frac{\sigma F_m\sin\beta}{4\omega^2}+\frac{\lambda F_m\cos\beta}{8\omega^2}+\frac{9da^2F_m\sin\beta}{32\omega^3}\right)+O(\epsilon^3)\\ \frac{d\beta}{dt}=\epsilon\left(-\sigma+\frac{3da^2}{8\omega}-\frac{F_m\cos(\beta)}{2a\omega}\right)+\epsilon^2\Bigg(-\frac{\lambda^2}{8\omega}-\frac{15d^2a^4}{256\omega^3}-\frac{5c^2a^2}{12\omega^3}\\ \qquad\qquad\qquad+\frac{\sigma F_m\cos\beta}{4\omega^2a}+\frac{3daF_m\cos\beta}{32\omega^3}-\frac{\lambda F_m\sin\beta}{8\omega^2a}\Bigg)+O(\epsilon^3)\end{array}\right. \tag{34}$$

Orientation: amplitude and phase equation. Equations (34) ensure that $S_3^\sharp$ has no term at frequency of ω_1 or which goes to ω_1 .
This will allow us to justify this expansion in certain conditions; before we need to consider the stationnary solution of the system (34) and the stability of the solution close to the stationary solution. This equation (34) is an extension for triple scale analysis of a similar equation introduced in a preliminary work with double scale analysis in [BR13].

Remark 2.7. *In this approach, we are using* the method of reconstitution*; this term has been introduced in 1985 in [Nay86] in order to resolve a discrepancy between higher order approximation solutions obtained by multi scales method on the one hand and generalised averaging method on the other hand; it has been discussed in [VLP08] and from the engineering point of view, the controversy has been resolved in [Nay05]; however the present mathematical proof of convergence seems new.*

Remark 2.8. *The previous equations are of importance to derive the solution of the equation* (1)*; their stationary solution will provide an approximate periodic solution of* (1)*.*

2.3.2 Stationnary solution and stability

Let us consider the stationary solution of(34), it satisfies:

$$\left\{\begin{array}{l}g_1(a,\beta,\sigma,\epsilon)=0,\\ g_2(a,\beta,\sigma,\epsilon)=0\end{array}\right. \tag{35}$$

with

$$\left\{\begin{array}{l}g_1=\epsilon(-\frac{F_m\sin(\beta)}{2\omega}-\frac{\lambda a}{2})+\\ \qquad\epsilon^2(\frac{3d\lambda a^3}{16\omega^2}+\frac{\sigma F_m\sin\beta}{4\omega^2}+\frac{\lambda F_m\cos\beta}{8\omega^2}+\frac{9da^2F_m\sin\beta}{32\omega^3})+\mathcal{O}(\epsilon^3)\\ g_2=\epsilon(-\sigma+\frac{3da^2}{8\omega}-\frac{F_m\cos(\beta)}{2a\omega})\\ \qquad+\epsilon^2(-\frac{\lambda^2}{8\omega}-\frac{15d^2a^4}{256\omega^3}-\frac{5c^2a^2}{12\omega^3}+\frac{\sigma F_m\cos\beta}{4\omega^2a}+\frac{3daF_m\cos\beta}{32\omega^3}-\frac{\lambda F_m\sin\beta}{8\omega^2a})+\mathcal{O}(\epsilon^3).\end{array}\right. \tag{36}$$

Now, we study the stability of the solution of (36) in a neighbourhood of this stationary solution noted $(\bar{a},\bar{\beta})$; set $a=\bar{a}+\tilde{a}$ and $\beta=\bar{\beta}+\tilde{\beta}$, the linearised system is written :

$$\begin{pmatrix}\frac{d\tilde{a}}{dt}\\ \frac{d\tilde{\beta}}{dt}\end{pmatrix}=J\begin{pmatrix}\tilde{a}\\ \tilde{\beta}\end{pmatrix}$$

with the jacobian matrix

$$J=\begin{pmatrix}\partial_{\bar{a}}g_1 & \partial_{\beta}g_1\\ \partial_{\bar{a}}g_2 & \partial_{\beta}g_2\end{pmatrix}$$

we compute the partial derivatives:

$$\partial_{\bar{a}} g_1 = \epsilon(-\frac{\lambda}{2}) + \mathcal{O}(\epsilon^2) \qquad \partial_{\bar{a}} g_2 = \epsilon \left(\frac{3d\bar{a}}{4\omega} + \frac{F_m \cos(\beta)}{2a^2\omega} \right) + \mathcal{O}(\epsilon^2)$$

$$\partial_{\beta} g_1 = -\epsilon \frac{F_m \cos(\beta)}{2\omega} + \mathcal{O}(\epsilon^2) \qquad \partial_{\beta} g_{12} = \epsilon \frac{F_m \sin(\beta)}{2a\omega} + \mathcal{O}(\epsilon^2)$$

or:

$$\partial_{\bar{a}} g_1 = \epsilon(-\frac{\lambda}{2}) + \mathcal{O}(\epsilon^2) \qquad \partial_{\bar{a}} g_2 = \epsilon(\frac{\sigma}{\bar{a}} + \frac{9d\bar{a}}{8\omega}) + \mathcal{O}(\epsilon^2)$$

$$\partial_{\gamma} g_1 = \epsilon(\sigma\bar{a} - \frac{3d\bar{a}^3}{8\omega}) + \mathcal{O}(\epsilon^2) \qquad \partial_{\gamma} g_2 = \epsilon(-\frac{\lambda}{2}) + \mathcal{O}(\epsilon^2)$$

The matrix trace is $tr(J) = -\lambda\epsilon$ and the determinant is

$$\det(J) = \epsilon^2 \left[-\frac{\lambda^2}{4} + \sigma^2 - \frac{3d\sigma\bar{a}^2}{2\omega} + \frac{27d^2\bar{a}^4}{64\omega^2} \right] + O(\epsilon^3) \tag{37}$$

the two eigenvalues are negative for ϵ is small enough; when

$$\sigma \leq \frac{3d\bar{a}^2}{4\omega} - \frac{1}{2}\sqrt{\frac{9d^2\bar{a}^4}{16\omega^2} - \lambda^2}$$

then the solution of the linearised system goes to zero; with the theorem of Poincaré-Lyapunov (look in the appendix for the theorem 5.1) when the initial data is close enough to the stationary solution, the solution of the system (34), goes to the stationary solution.

Proposition 2.2. *When*

$$\sigma \leq \frac{3d\bar{a}^2}{4\omega} - \frac{1}{2}\sqrt{\frac{9d^2\bar{a}^4}{16\omega^2} - \lambda^2}$$

and ϵ small enough, the stationary solution $(\bar{a}, \bar{\beta})$ of (34) is stable in the sense of Lyapunov (if the dynamic solution starts close to the stationary solution of (36), it remains close to it and converges to it); to the stationary case corresponds the approximate solution $\tilde{u}_{app} = \epsilon u_{app}$ of (16)

$$\tilde{u}_{app} = \epsilon\bar{a}\cos(\tilde{\omega}_\epsilon t + \bar{\beta}) + \epsilon^2 \left[\frac{-c\bar{a}^2}{2\omega^2} + \frac{c\bar{a}^2}{6\omega^2}\cos(2(\tilde{\omega}_\epsilon t + \bar{\beta})) + \frac{d\bar{a}^3}{32\omega^2}\cos(3(\tilde{\omega}_\epsilon t + \bar{\beta})) \right]$$

with

$$\tilde{\omega}_\epsilon = \omega + \epsilon\sigma$$

It is periodic up to the order two.

Remark 2.9. *The expression of u_{app} uses the remark*

$$u^{(1)} = a\ \cos(T_0 + \beta) = a\ \cos(\tilde{\omega}_\epsilon t + \beta)$$

and similarly for $u^{(2)}$.

With this result of stability, we can state precisely the approximation of the solution of (16)

2.3.3 Convergence of the expansion

Proposition 2.3. *Consider the solution $\tilde{u} = \epsilon u$ of (16) with initial conditions*

$$\tilde{u}(0) = \epsilon a_0 \cos(\beta_0) + \epsilon^2 [\frac{-ca_0^2}{2\omega^2} + \frac{ca_0^2}{6\omega^2}\cos(2\beta_0) + \frac{da^3}{32\omega^2}]\cos(3\beta_0) + \mathcal{O}(\epsilon^3), \tag{38}$$

$$\dot{\tilde{u}}(0) = -\epsilon\omega a_0 \sin(\beta_0) + \epsilon^2 [\frac{-ca_0^2}{2\omega^2}\sin(2\beta_0) - \frac{da_0^3}{32\omega^2}\sin(3\beta_0)] + \mathcal{O}(\epsilon^3) \tag{39}$$

with (a_0, β_0) close of the stationary solution $(\bar{a}, \bar{\beta})$;

$$|a_0 - \bar{a}| \le \epsilon^2 C_1, |\beta - \bar{\beta}| \le \epsilon^2 C_1$$

when $\sigma \le \frac{3d\bar{a}^2}{4\omega} - \frac{1}{2}\sqrt{\frac{9d^2\bar{a}^4}{16\omega^2} - \lambda^2}$ and ϵ small enough, there exists $\varsigma > 0$ such that for all $t < t_\epsilon = \frac{\varsigma}{\epsilon^2}$, the following expansion of $\tilde{u} = \epsilon u$ is satisfied

$$\left\{ \begin{array}{l} \tilde{u}(t) = \epsilon a(t)\cos(\tilde{\omega}_\epsilon t + \beta(t)) + \\ \qquad \epsilon^2 [\frac{-ca^2}{2\omega^2} + \frac{ca^2}{6\omega^2}\cos(2(\tilde{\omega} t + \beta(t))) + \frac{da^3}{32\omega^2}\cos(3(\tilde{\omega} t + \beta(t)))] + \epsilon^3 r(\epsilon, t) \end{array} \right. \tag{40}$$

with $\tilde{\omega}_\epsilon = \omega + \epsilon\sigma$ and r uniformly bounded in $C^2(0, t_\epsilon)$ and with $a,\ \beta$ solution of (34)

Proof. Indeed after eliminating terms at frequency ν_1, we go back to the variable t for the third equation (20).

$$\frac{d^2 r}{dt^2} + \omega^2 r = \tilde{S}_3$$

with
$\tilde{S}_3 = S_3^\sharp(t, \epsilon) - \epsilon\tilde{R}(u^{(1)}, u^{(2)}, r, \epsilon)$ with $\tilde{R} = R - \mathcal{D}_3 r - \lambda(\frac{dr}{dt} - D_0 r)$
with all the terms expressed with the variable t. We express $S_2^\sharp$ in (27) by inserting $D_1 a, D_1\beta$ by their expressions in (25) and using $\theta = \tilde{\omega}_\epsilon t + \beta$; this function is not periodic but is *close* to a periodic function $S_3^\sharp$ by replacing β by $\bar{\beta}$.

As the solution of (34) is stable, for $t \le \frac{\varsigma}{\epsilon^2}$:

$$|\beta(\epsilon t, \epsilon^2 t) - \bar{\beta}| \le \epsilon^2 C_1, \quad |a(\epsilon t, \epsilon^2 t) - \bar{a}| \le \epsilon^2 C_2$$

and

$$|S_3^\sharp - S_3^\natural| \le \epsilon^2 C_3$$

so this difference may be included in the remainder $\tilde{R}$. We use lemma 5.1 of Appendix (already introduced in [BR13]); with $S = S_3^\natural$; it satisfies lemma hypothesis; similarly, we use $R = \tilde{R}$; it satisfies the hypothesis because it is a polynomial in the variables r, u_1, ϵ,with coefficients which are bounded functions, so it is lipschitzian on bounded subsets.

□

Remark 2.10. *The previous proposition states that for well prepared data close to the stationary solution, the triple scales approximation converges in the sense that the difference between the solution and its approximation is equal to $\epsilon^3 r$ where r is a function which remains bounded in $C^2(0, t_\epsilon)$ with $t_\epsilon = \frac{\gamma}{\epsilon}$, for some constant γ, with ϵ going to 0.*

2.3.4 Maximum of the stationary solution, primary resonance

We consider the stationary solution of (34), it satisfies,

$$\left\{\begin{array}{l} g_1(a,\beta,\sigma,\epsilon)=0, \\ g_2(a,\beta,\sigma,\epsilon)=0 \end{array}\right. \tag{41}$$

with formulae (36). We are going to find an expansion of a, β, σ with respect to the small parameter ϵ when $\sigma \mapsto a$ reaches a maximum. The idea is that the functions $(\sigma,\epsilon) \mapsto (a,\beta)$ are defined implicitly by the previous equations; the jacobian matrix is

$$\begin{pmatrix} g_{1a}, & g_{1\beta} & g_{1\sigma} & g_{1\epsilon} \\ g_{2a} & g_{2\beta} & g_{2\sigma} & g_{2\epsilon} \end{pmatrix}$$

and its sub matrix $J_{a\beta}$ is:

$$J(a,\beta)=\begin{pmatrix} g_{1a} & g_{1\beta} \\ g_{2a} & g_{2\beta} \end{pmatrix}$$

in paragraph 2.3.2, we have proved previously that when σ, ϵ are small enough, $J_{a\beta} \neq 0$ and so with the implicit function theorem, in a neighbourhood of the stationary solution, there exists a regular function

$$(\sigma,\epsilon) \longmapsto (a,\beta).$$

We first transform (35) (36) in the following way

$$g_1(a,\beta,\sigma,\epsilon)=(-\frac{F_m\sin(\beta)}{2\omega}-\frac{\lambda a}{2})+ \qquad \epsilon A_1(a,\beta,\sigma)+\mathcal{O}(\epsilon^2)=0 \tag{42}$$

$$g_2(a,\beta,\sigma,\epsilon)=(-\sigma-\frac{3da^2}{8\omega}-\frac{F_m\cos(\beta)}{2a\omega})+ \qquad \epsilon A_2(a,\beta,\sigma)+\mathcal{O}(\epsilon^2)=0 \tag{43}$$

with

$$A_1(a,\beta,\sigma)=\frac{3d\lambda a^3}{16\omega^2}+\frac{\sigma F_m\sin\beta}{4\omega^2}+\frac{\lambda F_m\cos\beta}{8\omega^2}+\frac{9da^2F_m\sin\beta}{32\omega^3}$$

$$A_2(a,\beta,\sigma)=-\frac{\lambda^2}{8\omega}-\frac{15d^2a^4}{256\omega^3}-\frac{5c^2a^2}{12\omega^3} + \frac{\sigma F_m\cos\beta}{4\omega^2 a}+\frac{3daF_m\cos\beta}{32\omega^3}-\frac{\lambda F_m\sin\beta}{8\omega^2 a}$$

We derive a first approximation of $\sin\beta$ and $\cos\beta$ by neglecting terms of order one in ϵ:

$$\left\{\begin{array}{l} \frac{F_m\sin\beta}{2\omega}=-\frac{\lambda a}{2}+\mathcal{O}(\epsilon) \\ \frac{F_m\cos\beta}{2\omega}=\frac{3da^3}{8\omega}-\sigma a+\mathcal{O}(\epsilon) \end{array}\right. \tag{44}$$

Using $\frac{dg_1}{d\sigma}=0$, we get

$$\frac{F_m\cos(\beta)}{2\omega}\frac{\partial\beta}{\partial\sigma}-\frac{\lambda}{2}\frac{\partial a}{\partial\sigma}+\epsilon\frac{dA_1}{d\sigma}+\mathcal{O}(\epsilon)=0 \tag{45}$$

When a is maximum with respect to σ, we get another equation $\frac{\partial a}{\partial \sigma} = 0$; with previous equation, we get a third equation $g_3 = 0$ with

$$g_3(a, \beta, \sigma, \epsilon) = \frac{F_m \cos(\beta)}{2\omega} \frac{\partial \beta}{\partial \sigma} + \epsilon \frac{dA_1}{d\sigma} + \mathcal{O}(\epsilon)$$

We have for $\epsilon = 0$, $\frac{\partial g_3}{\partial a} = 0$, $\frac{\partial g_3}{\partial \sigma} = 0$; we denote $a_0^*, \beta_0^*, \sigma_0^*$ the solution of the 3 equations for $\epsilon = 0$.

We differentiate (44) with respect to σ; when $\frac{\partial a}{\partial \sigma} = 0$, we obtain for the first approximation

$$\left\{ \begin{array}{l} \frac{F_m \cos(\beta_0^*)}{\omega} \frac{\partial \beta_0^*}{\partial \sigma} = 0, \\ -\frac{F_m \sin(\beta_0^*)}{2\omega} \frac{\partial \beta_0^*}{\partial \sigma} + a_0^* = 0 \end{array} \right. \tag{46}$$

and so $\cos(\beta_0^*) = 0,\ \sin(\beta_0^*) = \pm 1$; if we use (42), we notice that a change of sign of $\sin(\beta_0^*)$ changes the sign of a; so we choose $\sin(\beta_0^*) = -1$ and a_0 has the sign of F_m; then with (42), (43), the following equalities hold:

$$a_0^* = \frac{F_m}{\lambda \omega},\ \sigma_0^* = \frac{3da_0^{*2}}{8\omega} = \frac{3dF_m^2}{8\lambda^2\omega^3}; \tag{47}$$

with (46), we get also $\frac{\partial \beta_0^*}{\partial \sigma} = \frac{2\omega a_0^*}{F_m} = \frac{2}{\lambda}$. We remark that c is not involved in these formulas. Then we can compute for $\epsilon = 0$, $\frac{\partial g_3}{\partial a} = 0$; $\frac{\partial g_3}{\partial \beta} = -\frac{F_m \sin(\beta)}{2\omega} \frac{\partial \beta}{\partial \sigma} = -\frac{F_m}{\lambda \omega}$; $\frac{\partial g_3}{\partial \sigma} = 0$. So we obtain that the determinant of the extended matrix

$$J^\clubsuit(a, \beta, \sigma) = \begin{pmatrix} g_{1a} & g_{1\beta} & g_{1,\sigma} \\ g_{2a} & g_{2\beta} & g_{2,\sigma} \\ g_{3a} & g_{3\beta} & g_{3,\sigma} \end{pmatrix}$$

is not zero for $(a_0^*, \beta_0^*, \sigma_0^*)$; so once more, we can use the implicit function theorem to define differentiable functions

$$\epsilon \longmapsto (a^*, \beta^*, \sigma^*)$$

where we denote a^*, β^*, σ^* the solution of the 3 equations.

After this first approximation, we look for an expansion of these functions: $\epsilon \longmapsto (a^*, \beta^*, \sigma^*)$;

$$a^* = a_0^* + \epsilon a_1^* + \mathcal{O}(\epsilon^2),\ \beta^* = \beta_0^* + \epsilon \beta_1^* + \mathcal{O}(\epsilon^2),\ \sigma^* = \sigma_0^* + \epsilon \sigma_1^* + \mathcal{O}(\epsilon^2). \tag{48}$$

We perform some preliminary computations of $A_{1,0}^* = A_1(a_0^*, \beta_0^*, \sigma_0^*),\ A_{2,0}^* = A_2(a_0^*, \beta_0^*, \sigma_0^*)$;

$$A_{1,0}^* = \frac{3d\lambda a_0^{*3}}{16\omega^2} + \frac{\sigma_0^* F_m \sin(\beta_0^*)}{4\omega^2} + \frac{9da_0^{*2} F_m \sin\beta_0^*}{32\omega^3}$$

$$A_{2,0}^* = -\frac{\lambda^2}{8\omega} - \frac{15d^2 a_0^{*4}}{256\omega^3} - \frac{5c^2 a_0^{*2}}{12\omega^3} - \frac{\lambda F_m \sin\beta_0^*}{8\omega^2 a_0}$$

then, we use the values of (47) and we get

$$\begin{array}{ll} A_{1,0}^* = -\frac{F_m \sigma_0^*}{2\omega^2} = -\frac{\lambda a_0^* \sigma_0^*}{2\omega}, & \frac{\partial A_{1,0}^*}{\partial \sigma} = \frac{F_m \sin(\beta_0^*)}{4\omega^2} = -\frac{a_0^* \lambda}{4\omega} \\ A_{2,0}^* = -\frac{15d^2 a_0^{*4}}{256\omega^3} - \frac{5c^2 a_0^{*2}}{12\omega^3} = -\frac{5\sigma_0^{*2}}{12\omega} - \frac{5c^2 a_0^{*2}}{12\omega^3}, & \frac{\partial A_{2,0}^*}{\partial \sigma} = \frac{-F_m \cos(\beta_0^*)}{4\omega^2 a} = 0 \end{array} \tag{49}$$

$$\frac{\partial A^*_{1,0}}{\partial \beta} = \frac{\sigma F_m \cos(\beta^*_0)}{4\omega^2} - \frac{\lambda F_m \sin(\beta^*_0)}{8\omega^2} + \frac{9da^2 F_m \cos(\beta^*_0)}{32\omega^3} = \frac{\lambda F_m}{8\omega^2} = \frac{\lambda^2 a^*_0}{8\omega} \tag{50}$$

$$\frac{\partial A^*_{2,0}}{\partial \beta} = -\frac{\sigma^*_0 F_m \sin(\beta^*_0)}{4\omega^2 a^*_0} - \frac{3da^*_0 F_m \sin(\beta^*_0)}{32\omega^3} - \frac{\lambda F_m \cos(\beta^*_0)}{8\omega^2 a} \tag{51}$$

$$= \frac{\sigma^*_0 F_m}{4\omega^2 a^*_0} + \frac{3da^*_0 F_m}{32\omega^3} = \frac{\sigma^*_0 F_m}{2\omega^2 a^*_0} = \frac{\sigma^*_0 \lambda}{2\omega}; \tag{52}$$

On the other hand, we notice that $\sin(\beta_0 + \epsilon\beta_1 + \mathcal{O}(\epsilon^2)) = -1 + \mathcal{O}(\epsilon^2)$ and with (47), we expand formula (42) to obtain at second order

$$\frac{\lambda a^*_1}{2} = A^*_{1,0} = -\frac{\lambda a^*_0 \sigma^*_0}{2\omega}$$

and therefore

$$a^*_1 = -\frac{a^*_0 \sigma^*_0}{\omega} \tag{53}$$

We compute

$$\frac{\partial g^*_{1,0}}{\partial \sigma} = \epsilon \frac{\partial A^*_{1,0}}{\partial \sigma} + \mathcal{O}(\epsilon^2) = -\epsilon \frac{\lambda a_0}{4\omega} + O(\epsilon^2) \tag{54}$$

$$\frac{\partial g^*_{1,0}}{\partial \beta} = \frac{F_m \cos(\beta)}{2\omega} + \epsilon \frac{\partial A^*_{1,0}}{\partial \beta} + O(\epsilon^2) = -\epsilon \frac{F_m \beta^*_1}{2\omega} - \epsilon \frac{\lambda^2 a^*_0}{8\omega} + \mathcal{O}(\epsilon^2); \tag{55}$$

where we have used $\cos(\beta_0 + \epsilon\beta_1 + \mathcal{O}(\epsilon^2)) = -\epsilon\beta_1 + \mathcal{O}(\epsilon^2)$ and $\frac{\partial a}{\partial \sigma} = 0$

$$\begin{aligned} \frac{dg_1}{d\sigma} &= \frac{\partial g^*_{1,0}}{\partial \sigma} + \frac{\partial g^*_{1,0}}{\partial \beta}\frac{\partial \beta}{\partial \sigma} + \frac{\partial g^*_{1,0}}{\partial \beta}\frac{\partial a}{\partial a} + \mathcal{O}(\epsilon^2) \\ &= \epsilon \left[-\frac{\lambda a^*_0}{4\omega} + \left(-\frac{F_m \beta^*_1}{2\omega} - \frac{\lambda^2 a^*_0}{8\omega} \right) \left(-\frac{2}{\lambda} \right) \right] + \mathcal{O}(\epsilon^2) \\ &= -\epsilon a^*_0 (\beta^*_1 + \frac{\lambda}{2\omega}) + \mathcal{O}(\epsilon^2) \end{aligned} \tag{56}$$

as $\frac{dg_1}{d\sigma} = 0$, we get

$$\beta^*_1 = -\frac{\lambda}{2\omega}. \tag{57}$$

We use these approximations in the second equation (43) to obtain

$$-(\sigma^*_0 + \epsilon\sigma^*_1) + \frac{3da^{*2}_0}{8\omega} + 6\epsilon \frac{da^*_0 a^*_1}{8\omega} + \frac{F_m \beta^*_1}{2a^*_0 \omega} + \epsilon A^*_{2,0} + \mathcal{O}(\epsilon^2) = 0 \tag{58}$$

and hence

$$\sigma_1^* = \frac{3da_0^*a_1^*}{4\omega} + \frac{F_m\beta_1^*}{2a_0^*\omega} + A_{2,0}^* \tag{59}$$
$$= \frac{3da_0^*}{4\omega}\left(\frac{-a_0^*\sigma_0^*}{\omega}\right) + \frac{F_m}{2a_0^*\omega}\left(\frac{-\lambda}{2\omega}\right) + A_{2,0}^*$$
$$= -2\frac{\sigma_0^2}{\omega} - \frac{\lambda^2}{4\omega} + A_{2,0}^*$$
$$= -29\frac{\sigma_0^{*2}}{12\omega} - \frac{5c^2a_0^2}{12\omega^3} - \frac{\lambda^2}{4\omega} \tag{60}$$

We can check the computations by using another way, see Appendix in subsection 5.3 We remark that we get a frequency slightly different of the free vibration frequency associated to the same amplitude.

We have obtained the following important result.

Proposition 2.4. *The stationary solution of* (34) *satisfies*

$$\begin{cases} (\frac{F_m\sin(\beta)}{2\omega} - \frac{\lambda a}{2}) + \quad \epsilon A_1(a,\beta,\sigma) + \mathcal{O}(\epsilon^2) = 0 \\ (\sigma - \frac{3da^2}{8\omega} + \frac{F_m\cos(\beta)}{2a\omega}) + \quad \epsilon A_2(a,\beta,\sigma) + \mathcal{O}(\epsilon^2) = 0 \end{cases} \tag{61}$$

with

$$A_1(a,\beta,\sigma) = \frac{3d\lambda a^3}{16\omega^2} + \frac{\sigma F_m\sin\beta}{4\omega^2} + \frac{\lambda F_m\cos\beta}{8\omega^2} + \frac{9da_1^2F_m\sin\beta}{32\omega^3}$$
$$A_2(a,\beta,\sigma) = -\frac{\lambda^2}{8\omega} - \frac{15d^2a^4}{256\omega^3} - \frac{5c^2a_1^2}{12\omega^3} + \frac{\sigma F_m\cos\beta}{4\omega^2a_1} + c\frac{3daF_m\cos\beta}{32\omega^3} - \frac{\lambda F_m\sin\beta}{8\omega^2a_1}$$

this stationary solution reaches its maximum amplitude for $\sigma = \sigma_0^* + \epsilon\sigma_1^* + \mathcal{O}(\epsilon^2)$, $a^* = a_0^* + \epsilon a_1^* + \mathcal{O}(\epsilon^2)$, $\beta^* = \beta_0^* + \epsilon\beta_1^* + \mathcal{O}(\epsilon^2)$ *with*

$$a_0^* = \frac{F_m}{\lambda\omega},\ \sigma_0^* = \frac{3da_0^{*2}}{8\omega} = \frac{3F_m^2}{8\lambda^2\omega^3}, \quad \beta_0^* = -\frac{\pi}{2} \tag{62}$$

and

$$\sigma_1^* = -\frac{29}{12\omega}\sigma_0^{*2} - \frac{5c^2a_0^{*2}}{12\omega^3} - \frac{\lambda^2}{4\omega} = -\frac{87d^2a_0^4}{256\omega^3} - \frac{5c^2a_0^{*2}}{12\omega^3} - \frac{\lambda^2}{4\omega}, \quad \beta_1^* = \frac{-\lambda}{2\omega}, \quad a_1^* = -\frac{a_0^*\sigma_0^*}{\omega}$$

the periodic forcing is at the angular frequency

$$\tilde{\omega}_\epsilon = \omega + \epsilon\sigma_0^* + \epsilon^2\sigma_1^* + \mathcal{O}(\epsilon^2)$$

it is slightly different of the approximate angular frequency ν_ϵ of the undamped free periodic solution associated to the same amplitude. (15)*; for this frequency, the approximation (of the solution $\tilde{u} = \epsilon u$ of* (16) *up to the order ϵ^2) is periodic:*

$$\begin{cases} \tilde{u}(t) = \epsilon a^*\cos(\tilde{\omega}_\epsilon t + \beta^* t) \\ \qquad +\epsilon^2[\frac{-ca^{*2}}{2\omega^2} + \frac{ca^{*2}}{6\omega^2}\cos(2(\tilde{\omega}_\epsilon t + \beta^*)) + \frac{da^{*3}}{32\omega^2}\cos(3(\tilde{\omega}_\epsilon t + \beta^*))] + \epsilon^3 r(\epsilon,t) \\ \tilde{u}(0) = \epsilon a^* + \epsilon^2[\frac{-ca^{*2}}{3\omega^2} + \frac{da^{*3}}{32\omega^2}] + \mathcal{O}(\epsilon^3), \quad \dot{u}(0) = \mathcal{O}(\epsilon^3) \end{cases} \tag{63}$$

with r bounded in $C^2(0,t_\epsilon)$

Remark 2.11. *We remark that, for ϵ small enough, this value of σ^* is indeed smaller than the maximal value that σ may reach in order that the previous expansion converges as indicated in proposition 2.3.*

Remark 2.12. *We have obtained an expansion of $\tilde{\omega}_\epsilon$ up to order ϵ^2 to be compared with the expansion with a double scale analysis (see in [BR13]); in particular the amplitude dependence on the frequency of the applied force depends on the ratio of c and d; see numerical results below.*

We have justified the basic behaviour of a primary resonance; many other phenomena may appear like subharmonic resonances, see for example [Nay86].

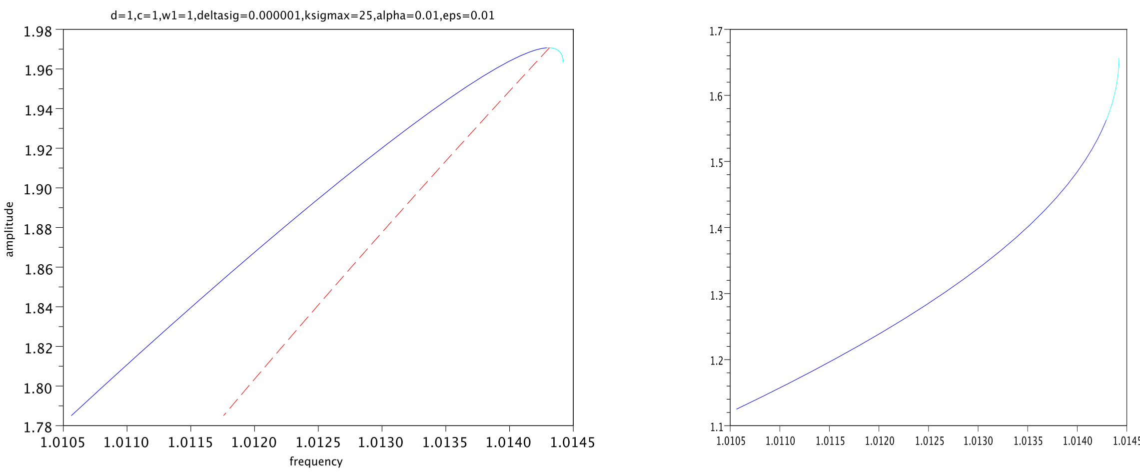

Figure 2: Left: amplitude versus frequency of stationary forced solution in blue and magenta; amplitude of free solution in red. Right: phase versus frequency of stationary forced solution

In figure 2, we use $\epsilon = 0.01, \lambda = 1/2, c = 1, d = 1, \omega = 1, F = 1$. On the left, the solid line displays the amplitude of the solution of this equation with respect to values of the frequency; we have solved (41) with the routine FSOLVE of Scilab; it implements a variant of the hybrid method of Powell. In proposition 2.2, the solution is stable when sigma is small enough; the routine FSOLVE fails to solve the equation when σ is too large; then we have exchanged the use of σ and a. The dotted line plots the amplitude of the free solution with respect to its frequency. On the right, the phase $\gamma = -\beta$ is plotted with respect to the frequency; it is also obtained by solving (41) with the routine FSOLVE.

In figure 3, we use $\epsilon = 0.01, \lambda = 1/2, c = 6, d = 1/4, \omega = 1, F = 1$. On the left the solid line displays the amplitude of the solution with respect to values of the frequency; on the right the phase γ is plotted. We notice that the behaviour is quite different of the previous plots.

Remark 2.13. *We emphasise that the behaviour of the last plots is linked to the ration of c and d; this type of behaviour cannot be obtained with double scale expansion ; see [BR13].*

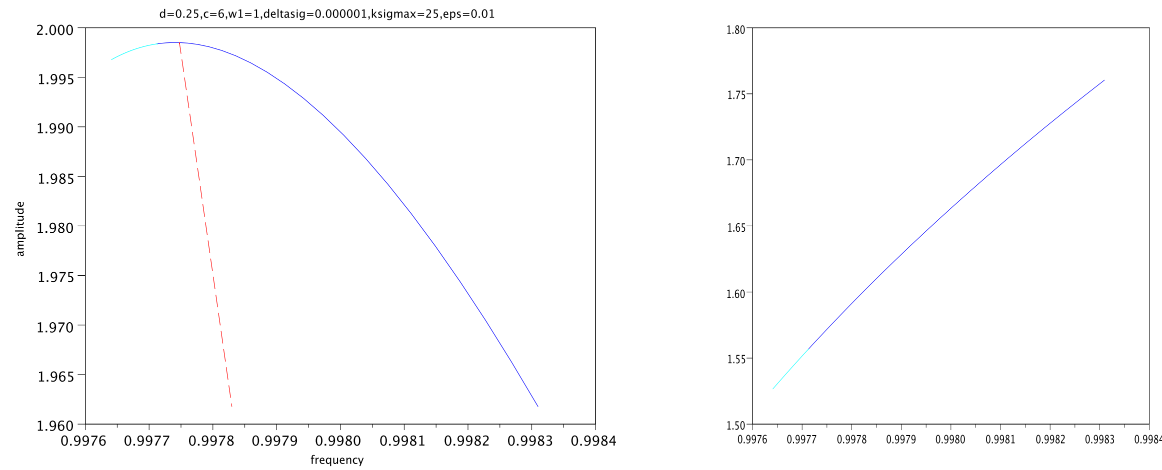

Figure 3: Left: amplitude versus frequency of stationary forced solution in blue and magenta; amplitude of free solution in red. Right: phase versus frequency of stationary forced solution

3 System with local quadratic and cubic non linearity

3.1 Free vibrations, triple scale expansion up to second order

We consider a system of several vibrating masses attached to springs:

$$M\ddot{\tilde{u}} + K\,\tilde{u} + \Phi(\tilde{u}, \epsilon) = 0 \tag{64}$$

The mass matrix M and the rigidity matrix K are assumed to be symmetric and positive definite. We assume that the non linearity is local, all components are zero except for two components $p-1,\ p$ which correspond to the end points of some spring assumed to be non linear:

$$\Phi_{p-1}(\tilde{u}) = c(\tilde{u}_p - \tilde{u}_{p-1})^2 + \frac{d}{\epsilon}(\tilde{u}_p - \tilde{u}_{p-1})^3,\ \Phi_p = -\Phi_{p-1} \tag{65}$$

In order to get an approximate solution, we are going to display the equation in the generalised eigenvector basis:

$$K\phi_k = \omega_k^2 M\phi_k,\ \text{with}\ \phi_k^T\ M\ \phi_l = \delta\phi_{kl}, \quad k, l = 1\ldots, n \tag{66}$$

So we perform the change of functions:

$$\tilde{u} = \sum_{k=1}^{n} \tilde{y}_k\phi_k;\ K\tilde{u} = \sum_{k=1}^{n} \tilde{y}_k K\phi_k = \sum_{k=1}^{n} \tilde{y}_k\omega_k^2 M\phi_k;\ M\ddot{\tilde{u}} = \sum_{k=1}^{n} \ddot{\tilde{y}}_k M\phi_k \tag{67}$$

we obtain

$$\ddot{\tilde{y}}_k + \omega_k^2\tilde{y}_k + \phi_k^T\Phi(\sum_{i=1}^{n}\tilde{y}_i\phi_i, \epsilon) = 0, \quad k = 1\ldots, n$$

As Φ has only 2 components which are not zero, it can be written

$$\ddot{\tilde{y}}_k + \omega_k^2 \tilde{y}_k + (\phi_{k,p-1} - \phi_{k,p})\, \Phi_{p-1}(\sum_{i=1}^{n} \tilde{y}_i \phi_i, \epsilon) = 0, \quad k = 1\ldots,n$$

or more precisely

$$\ddot{\tilde{y}}_k + \omega_k^2 \tilde{y}_k + (\phi_{k,p-1} - \phi_{k,p}) \left[c \left(\sum_{i=1}^{n} \tilde{y}_i (\phi_{i,p} - \phi_{i,p-1}) \right)^2 + \frac{d}{\epsilon} \left(\sum_{i=1}^{n} \tilde{y}_i (\phi_{i,p} - \phi_{i,p-1}) \right)^3 \right] = 0, \quad k = 1\ldots,n \tag{68}$$

Remark 3.1. *As we intend to look for a small solution, we consider a change of function* $\boxed{\tilde{y}_k = \epsilon y_k}$ *and we obtain the transformed equation:*

$$\ddot{y}_k + \omega_k^2 y_k + (\phi_{k,p-1} - \phi_{k,p}) \left[\epsilon c \left(\sum_{i=1}^{n} y_i (\phi_{i,p} - \phi_{i,p-1}) \right)^2 + \epsilon d \left(\sum_{i=1}^{n} y_i (\phi_{i,p} - \phi_{i,p-1}) \right)^3 \right] = 0, \quad k = 1\ldots,n \tag{69}$$

3.1.1 Derivation of an asymptotic expansion

As for the 1 degree of freedom case, we use a triple scale expansion to compute an approximate small solution; more precisely, *we look for a solution close to a normal mode of the associated linear system*; we denote this mode by subscript ω_1; obviously by permuting the coordinates, this subscript could be anyone (different of p, this case would give similar results with slightly different formulae); we set

$$T_0 = \omega_1 t, \quad T_1 = \epsilon t, \quad T_2 = \epsilon^2 t \text{ hence } \quad D_0 y_k = \frac{\partial y_k}{\partial T_0},\ D_1 y_k = \frac{\partial y_k}{\partial T_1} \text{ and } D_2 y_k = \frac{\partial y_k}{\partial T_2} \tag{70}$$

and we use the *ansatz*:

$$y_k(t) = y_k(T_0, T_1, T_2) = y_k^{(1)}(T_0, T_1, T_2) + \epsilon y_k^{(2)}(T_0, T_1, T_2) + \epsilon^2 r_k(T_0, T_1, T_2) \tag{71}$$

So we have:

$$\begin{aligned} \frac{d^2 y_k}{dt^2} &= \omega_1^2 D_0^2 y_k^{(1)} + \epsilon \left[2\omega_1 D_0 D_1 y_k^{(1)} + D_0^2 y_k^{(2)} \right] \\ &+ \epsilon^2 \left[2\omega_1 D_0 D_2 y_k^{(1)} + D_1^2 y_k^{(1)} + 2\omega_1 D_0 D_1 y_k^{(2)} + D_0^2 r \right] \\ &+ \epsilon^3 \left[2 D_1 D_2 y_k^{(1)} + 2\omega_1 D_0 D_2 y_k^{(2)} + D_1^2 y_k^{(2)} + \mathcal{D}_3 r_k \right] \\ &+ \epsilon^4 \left[D_2^2 y_k^{(1)} + 2 D_1 D_2 y_k^{(2)} + \epsilon D_2^2 y_k^{(2)} \right] \end{aligned} \tag{72}$$

with

$$\mathcal{D}_3 r_k = \frac{1}{\epsilon}\left(\frac{d^2 r_k}{dt^2} - \omega_1^2 D_0^2 r_k\right) = 2\omega_1 D_0 D_1 r_k + \epsilon[2\omega_1 D_0 D_2 r_k + D_1^2 r_k] + 2\epsilon^2 D_1 D_2 r_k + \epsilon^3 D_2^2 r_k$$

We plug previous expansions (71) and (72) into (69); by identifying the coefficients of the powers of ϵ in the expansion of (69), we get:

$$\left\{\begin{array}{lll} \omega_1^2 D_0^2 y_k^{(1)} + \omega_k^2 y_k^{(1)} = 0 & , & k = 1\ldots, n \\ \omega_1^2 D_0^2 y_k^{(2)} + \omega_k^2 y_k^{(2)} = S_{2,k} & , & k = 1\ldots, n \\ \omega_1^2 D_0^2 r_k + \omega_k^2 r_k = S_{3,k} & , & k = 1\ldots, n \end{array}\right. \tag{73}$$

where $S_{2,k}, S_{3,k}$ are defined below; to simplify the manipulations, we set $\delta\phi_{kp} = (\phi_{k,p} - \phi_{k,p-1})$;

$$S_{2,k} = -c\delta\phi_{kp}\left(\sum_{l,m} y_l^{(1)}\delta\phi_{lp} y_m^{(1)}\delta\phi_{mp}\right) - d\delta\phi_{kp}\left(\sum_{g,l,o} y_g^{(1)} y_l^{(1)}\delta\phi_{lp} y_o^{(1)}\delta\phi_{gp}\delta\phi_{op}\right) - 2\omega_1 D_0 D_1 y_k^{(1)}$$

$$\begin{aligned} S_{3,k} = -c\delta\phi_{kp}\left(\sum_{l,j} y_l^{(1)} y_j^{(2)}\delta\phi_{lp}\delta\phi_{jp}\right) - d\delta\phi_{kp}\left(\sum_{h,g,l} y_h^{(1)} y_g^{(1)} y_l^{(2)}\delta\phi_{hp}\delta\phi_{gp}\delta\phi_{lp}\right) \\ - 2\omega_1 D_0 D_2 y_k^{(1)} - D_1^2 y_k^{(1)} - 2\omega_1 D_0 D_1 y_k^{(2)} - \epsilon R_k(y_1^{(1)}, y_1^{(2)}, r_k, \epsilon) \end{aligned}$$

with

$$\begin{aligned} R_k(\epsilon, r_k, y_k^{(1)}, y_k^{(2)}) &= 2D_1 D_2 y_k^{(1)} + 2\omega_1 D_0 D_2 y_k^{(2)} + D_1^2 y_k^{(2)} \\ &+ c\delta\phi_{kp}\left(\sum_{l,j} y_j^{(2)} y_j^{(2)}\delta\phi_{lp}\delta\phi_{jp}\right) + c\delta\phi_{kp}\left(\sum_{l,j} y_j^{(1)} r_l\delta\phi_{jp}\delta\phi_{lp}\right) \\ &+ d\delta\phi_{kp}\left(\sum_{h,g,l} y_h^{(1)} y_g^{(2)} y_l^{(2)}\delta\phi_{hp}\delta\phi_{gp}\delta\phi_{lp}\right) + d\delta\phi_{kp}\left(\sum_{h,g,l} y_h^{(1)} y_g^{(1)} r_l\delta\phi_{hp}\delta\phi_{gp}\delta\phi_{lp}\right) \\ &+ \mathcal{D}_3 r_k + \epsilon(D_2^2 y_k^{(1)} + 2D_1 D_2 y_k^{(2)} + \epsilon D_2^2 y_k^{(2)}) + \epsilon\rho(y_k^{(1)}, y_k^{(2)}, r_k, \epsilon) \end{aligned}$$

and with a polynomial in the variables r_n with coefficients $y_l^{(1)}, y_m^{(2)}$,

$$\rho(y_k^{(1)}, y_k^{(2)}, r_k, \epsilon) = c\delta\phi_{kp}\left(\sum_{l,j} y_l^{(2)} r_j \delta\phi_{lp}\delta\phi_{jp}\right) + d\delta\phi_{kp}\left(\sum_{h,g,l} y_h^{(1)} y_g^{(2)} r_l \delta\phi_{hp}\delta\phi_{gp}\delta\phi_{lp}\right)$$
$$+ d\delta\phi_{kp}\left(\sum_{h,g,l} y_h^{(2)} y_g^{(2)} y_l^{(2)} \delta\phi_{hp}\delta\phi_{gp}\delta\phi_{lp}\right)$$
$$+ \epsilon c\Bigg[c\delta\phi_{kp}\left(\sum_{l,j} r_l r_j \delta\phi_{lp}\delta\phi_{jp}\right) + d\delta\phi_{kp}\left(\sum_{h,g,l} y_h^{(2)} y_g^{(2)} r_l \delta\phi_{hp}\delta\phi_{gp}\delta\phi_{lp}\right)$$
$$+ d\delta\phi_{kp}\left(\sum_{h,g,l} y_h^{(1)} r_g r_l \delta\phi_{hp}\delta\phi_{gp}\delta\phi_{lp}\right)\Bigg]$$
$$+ \epsilon^2 d\delta\phi_{kp}\left(\sum_{h,g,l} y_h^{(2)} r_g r_l \delta\phi_{hp}\delta\phi_{gp}\delta\phi_{lp}\right) + \epsilon^3 d\delta\phi_{kp}\left(\sum_{h,g,l} r_h r_g r_l \delta\phi_{hp}\delta\phi_{gp}\delta\phi_{lp}\right) \quad (74)$$

We set $\theta(T_0, T_1, T_2) = T_0 + \beta_1(T_1, T_2)$; we note that $D_0\theta = 1,\ D_1\theta = D_1\beta$ and $D_2\theta = D_2\beta_1$; we solve the first set of equations (73), imposing $O(\epsilon^3)$ initial Cauchy data for $k \neq 1$ and $D_0 y_1^{(1)}(0) = 0$; we get:

$$\begin{cases} y_1^{(1)} = a_1(T_1, T_2)\ \cos(\theta) \\ y_k^{(1)} = 0, \quad k = 2 \dots n \end{cases} \quad (75)$$

Remark 3.2. *We note that a_1 and β_1 are not constants but functions of times T_1 and T_2 because u depends on these times scales. The dependence of these functions with respect to T_1 and T_2 will be determined by solving the equations of the following orders and eliminating secular terms.*

First, we determine the dependence in T_1; we manipulate the right hand sides:

$$S_{2,1} = -\delta\phi_{1p}\Big[\frac{ca_1^2}{2}(1+\cos(2\theta))\delta\phi_{1p}^2 + \frac{da_1^3}{4}(\cos(3\theta) + 3\cos(\theta))\delta\phi_{1p}^3\Big]$$
$$+ 2\omega_1\left[a_1 D_1\beta_1 \cos(\theta) + D_1 a_1 \sin(\theta)\right]$$

$$S_{2,k} = -\delta\phi_{kp}\Big[\frac{ca_1^2}{2}(1+\cos(2\theta))\delta\phi_{1p}^2 + \frac{da_1^3}{4}(\cos(3\theta) + 3\cos(\theta))\delta\phi_{1p}^3\Big],\ \text{for } k \neq 1$$

In $S_{2,1}$, we gather the terms at angular frequency ω_1;

$$S_{2,1} = -3\frac{da_1^3}{4}\cos(\theta))\delta\phi_{1p}^4 + 2\omega_1\left[a_1 D_1\beta_1 \cos(\theta) + D_1 a_1 \sin(\theta)\right] + S_2^\sharp \quad (76)$$

with

$$S^{\sharp}_{2,1} = -\delta\phi_{1p}\Big[\frac{ca_1^2}{2}(1+\cos(2\theta))\delta\phi_{1p}^2 + \frac{da_1^3}{4}\cos(3\theta)\delta\phi_{1p}^3\Big]$$

It appears some terms at the frequency of the system, these terms provide a solution $y_1^{(2)}$ of the equation (73) which is non periodic and non bounded over long time intervals. We will eliminate these terms by imposing:

$$\left\{\begin{array}{l} D_1 a_1 = 0 \\ D_1\beta_1 = \frac{3d\delta\phi_{1p}^4 a_1^2}{8\omega_1} \end{array}\right. \tag{77}$$

and if we assume that ω_1^2 is a simple eigenvalue and $\omega_k^2 \neq 9\omega_1^2$, $\omega_k^2 \neq 4\omega_1^2$ (no internal resonance), the solution of the second equation (73) is:

$$\left\{\begin{array}{l} y_1^{(2)} = \delta\phi_{1p}^3[-\frac{ca_1^2}{2\omega_1^2} + \frac{ca_1^2}{6\omega_1^2}\cos(2\theta)] + \delta\phi_{1p}^4\frac{da_1^3}{32\omega_1^2}\cos(3\theta) \\ y_k^{(2)} = \delta\phi_{kp}\delta\phi_{1p}^2[-\frac{ca_1^2}{2\omega_k^2} + \frac{ca_1^2}{2(4\omega_1^2-\omega_k^2)}\cos(2\theta)] + \delta\phi_{kp}\delta\phi_{1p}^3\frac{da_1^3}{4(9\omega_1^2-\omega_k^2)}\cos(3\theta), \quad k=2,\ldots,n. \end{array}\right. \tag{78}$$

where we have omitted the term at angular frequency ω_1 which is redundant with $y_1^{(1)}$.
For the third set of equations of (73), r is the unknown, this equation contains non-linearities, we do not solve it but we show that the solution is bounded on an interval dependent of ϵ. The right hand side, after some manipulations is:

$$\begin{aligned} S_{3,1} = {} & \sin(\theta)\big(2\omega_1 D_2 a_1 + 2D_1 a_1 D_1\beta_1 + a_1 D_1^2\beta_1\big) \\ & \cos(\theta)\left(2\omega_1 a_1 D_2\beta_1 - D_1^2 a_1 + a_1(D_1\beta_1)^2 + \frac{5c^2\delta\phi_{1p}^6 a_1^3}{6\omega_1^2} - \frac{3d^2\delta\phi_{1p}^8 a_1^5}{128\omega_1^2}\right) \\ & + S^{\sharp}_{3,1} - \epsilon R_1(r_1,\epsilon,y_1^{(1)},y_1^{(2)}) \end{aligned}$$

where

$$\begin{aligned} S^{\sharp}_{3,1} = {} & \frac{5cd\delta\phi_{1p}^7 a_1^4}{8\omega_1^2} + \sin 2\theta\left[\frac{4c\delta\phi_{1p}^3 a_1}{3\omega_1}D_1 a_1\right] + \cos 2\theta\left[\frac{4c\delta\phi_{1p}^3 a_1^2}{3\omega_1}D_1\beta_1 + \frac{15cd\delta\phi_{1p}^7 a_1^4}{32\omega_1^2}\right] \\ & + \sin 3\theta\left[\frac{9d\delta\phi_{1p}^4 a_1^2}{16\omega_1}D_1 a_1\right] + \cos 3\theta\left[-\frac{c^2\delta\phi_{1p}^6 a_1^3}{6\omega_1^2} - \frac{3d^2\delta\phi_{1p}^8 a_1^5}{64\omega_1^2} + \frac{9da_1^3\delta\phi_{1p}^4}{16\omega}D_1\beta_1\right] \\ & + \cos 4\theta\left[-\frac{5cd\delta\phi_{1p}^7 a_1^4}{32\omega_1^2}\right] - \frac{3d^2\delta\phi_{1p}^8 a_1^5}{128\omega_1^2}\cos 5\theta \end{aligned} \tag{79}$$

and

$$S_{3,k} = \cos(\theta)\left(\frac{5c^2\delta\phi_{kp}\delta\phi_{1p}^6 a_1^3}{6\omega_1^2} - \frac{3d^2\delta\phi_{kp}\delta\phi_{1p}^8 a_1^5}{128\omega_1^2}\right) + S^{\sharp}_{3,k} - \epsilon R_k(r_k,\epsilon,y_1^{(1)},y_1^{(2)})$$

where

$$S^{\sharp}_{3,k} = \frac{5cd\delta\phi_{kp}\delta\phi_{1p}^6 a_1^4}{8\omega_1^2} + \sin 2\theta\left[\frac{4c\delta\phi_{kp}\delta\phi_{1p}^2 a_1}{3\omega_1}D_1 a_1\right] + \cos 2\theta\left[\frac{4c\delta\phi_{kp}\delta\phi_{1p}^2 a_1^2}{3\omega_1}D_1\beta_1 + \frac{15cd\delta\phi_{kp}\delta\phi_{1p}^6 a_1^4}{32\omega_1^2}\right]$$
$$+\sin 3\theta\left[\frac{9d\delta\phi_{kp}\delta\phi_{1p}^3 a_1^2}{16\omega_1}D_1 a_1\right] + \cos 3\theta\left[-\frac{c^2\delta\phi_{kp}\delta\phi_{1p}^5 a_1^3}{6\omega_1^2} - \frac{3d^2\delta\phi_{kp}\delta\phi_{1p}^7 a_1^5}{64\omega_1^2} + \frac{9d\delta\phi_{kp}\delta\phi_{11}^3 a_1^3}{16\omega_1}D_1\beta_1\right]$$
$$+\cos 4\theta\left[-\frac{5cd\delta\phi_{kp}\delta\phi_{1p}^6 a_1^4}{32\omega_1^2}\right] - \frac{3d^2\delta\phi_{kp}\delta\phi_{1p}^7 a_1^5}{128\omega_1^2}\cos 5\theta$$

By imposing

$$\begin{cases} 2\omega_1 D_2 a_1 + 2D_1 a_1 D_1\beta_1 + a_1 D_1^2\beta_1 = 0 \\ 2\omega_1 a_1 D_2\beta_1 - D_1^2 a_1 + a_1(D_1\beta_1)^2 + \frac{5c^2\delta\phi_{1p}^6 a_1^3}{6\omega_1^2} - \frac{3d^2\delta\phi_{1p}^8 a_1^5}{128\omega_1^2} = 0 \end{cases}$$

we get that $S_{3,1} = S^{\sharp}_{3,1} - \epsilon R_1(r_1, \epsilon, y_1^{(1)}, y_1^{(2)})$ contains no terms at the frequency of the system.

As $D_1 a_1 = 0$ and $D_1\beta_1 = \frac{-3d\delta\phi_{1p}^4 a_1^2}{8\omega_1}$, we obtain

$$2\omega_1 a_1 D_2\beta_1 + a_1\Big(\frac{3d\delta\phi_{1p}^4 a_1^2}{8\omega_1}\Big)^2 + \frac{5c^2\delta\phi_{1p}^6 a_1^3}{6\omega_1^2} - \frac{3d^2\delta\phi_{1p}^8 a_1^5}{128\omega_1^2} = 0$$

so:

$$D_2 a_1(T_2) = 0 \qquad \text{and} \qquad D_2\beta_1(T_2) = -\frac{5c^2\delta\phi_{1p}^6 a_1^2}{12\omega_1^3} - \frac{15d^2\delta\phi_{1p}^8 a_1^4}{256\omega_1^3} \tag{80}$$

As a, β do not depend on T_0,

$$\begin{cases} \frac{da_1}{dt} = \epsilon D_1 a_1 + \epsilon^2 D_2 a_1 + \mathcal{O}(\epsilon^3) \\ \frac{d\beta}{dt} = \epsilon D_1\beta + \epsilon^2 D_2\beta + \mathcal{O}(\epsilon^3) \end{cases} \tag{81}$$

and taking into account (77) and (80),we obtain:

$$\frac{da_1}{dt} = 0 \quad \text{and} \quad \frac{d\beta_1}{dt} = \epsilon\frac{3d\delta\phi_{1p}^4 a_1^2}{8\omega_1} + \epsilon^2\Big(\frac{-5c^2\delta\phi_{1p}^6 a_1^2}{12\omega_1^3} - \frac{15d^2\delta\phi_{1p}^8 a_1^4}{256\omega_1^3}\Big) \tag{82}$$

As a result, the solution of these equations is:

$$a_1 = cte \qquad \text{and} \qquad \beta_1 = \left[\epsilon\frac{3d\delta\phi_{1p}^4 a_1^2}{8\omega_1} + \epsilon^2\Big(-\frac{5c^2\delta\phi_{1p}^6 a_1^2}{12\omega_1^3} - \frac{15d^2\delta\phi_{1p}^8 a_1^4}{256\omega_1^3}\Big)\right]t \tag{83}$$

In order to show that r_1 is bounded, after eliminating the secular terms, we can go back to the variable t in the equation of r_k, we get:

$$\frac{d^2 r_1}{dt^2} + \omega_1^2 r_1 = \tilde{S}_{3,1} \quad \text{with} \quad \tilde{S}_{3,1} = S^{\sharp}_{3,1}(t,\epsilon) - \epsilon\tilde{R}_1(r_1, \epsilon, y_1^{(1)}, y_1^{(2)})$$
$$\frac{d^2 r_k}{dt^2} + \omega_1^2 r_k = \tilde{S}_{3,k} \quad \text{with} \quad \tilde{S}_{3,k} = S^{\sharp}_{3,k}(t,\epsilon) - \epsilon\tilde{R}_k(r_k, \epsilon, y_k^{(1)}, y_k^{(2)}) \quad k = 2, \ldots, n$$

where $S^{\sharp}_{3,1}$ is in (79) where all time scales T_0, T_1, T_2 are expressed with the time variable t.

$$\tilde{R}_1 = R_1(\epsilon, r_1, y_1^{(1)}, y_1^{(2)}) - \mathcal{D}_3 r_1$$

After these manipulations, we can state a proposition which will be easily proved with technical lemmas of the Appendix.

Proposition 3.1. *We assume that ω_1^2 is a simple eigenvalue and $\omega_k^2 - 9\omega_1^2 \neq 0,\ \omega_k^2 - 4\omega_1^2 \neq 0$ (no internal resonance), then it exists $\varsigma > 0$ such that for all $t \leq t_\epsilon = \frac{\varsigma}{\epsilon^2}$, the solution $\tilde{y}_k = \epsilon y_k$ of (68) with the initial data*

$$\begin{aligned} \tilde{y}_1(0) &= \epsilon a_1 + \epsilon^2\ (-\frac{\check{c}_1 a_1^2}{3\omega_1^2}\ +\ \frac{\check{d}_1 a_1^3}{32\omega_1^2}) + \epsilon^3 r_1(\epsilon, 0), \quad \dot{\tilde{y}}_1(0) = \mathcal{O}(\epsilon) \\ \tilde{y}_k(0) &= \epsilon^2[-\frac{\check{c}_k a_1^2}{2\omega_k^2} + \frac{\check{c}_k a_1^2}{2(4\omega_1^2 - \omega_k^2)} + \frac{\check{d}_k a_1^3}{4(9\omega_1^2 - \omega_k^2)}] + \epsilon^3 r_k(\epsilon, 0), \quad \dot{\tilde{y}}_k(0) = \mathcal{O}(\epsilon) \end{aligned} \tag{84}$$

has the following expansion:

$$\begin{cases} \tilde{y}_1(t) = \epsilon a_1 \cos(\nu_\epsilon t) + \epsilon^2[-\frac{\check{c}_1 a_1^2}{2\omega_1^2} + \frac{\check{c}_1 a_1^2}{6\omega_1^2}\cos(2(\nu_\epsilon t)) + \frac{\check{d}_1 a_1^3}{32\omega_1^2}\cos(3(\nu_\epsilon t))] + \epsilon^3 r_1(\epsilon, t) \\ \tilde{y}_k(t) = \epsilon^2[-\frac{\check{c}_k a_1^2}{2\omega_k^2} + \frac{\check{c}_k a_1^2}{2(4\omega_1^2 - \omega_k^2)}\cos(2(\nu_\epsilon t)) + \frac{\check{d}_k a_1^3}{4(9\omega_1^2 - \omega_k^2)}\cos(3(\nu_\epsilon t))] + \epsilon^3 r_k(\epsilon, t) \end{cases} \tag{85}$$

with r_k uniformly bounded in $C^2(0, t_{\epsilon^2})$ for $k = 1, \dots n$ and the angular frequency

$$\nu_\epsilon = \omega_1 + \epsilon(\frac{3\check{d}_1 a_1^2}{8\omega_1}) + \epsilon^2\left(\frac{-5\check{c}_1^2 a_1^2}{12\omega_1^3} - \frac{15\check{d}_1^2 a_1^4}{256\omega_1^3}\right) + \mathcal{O}(\epsilon^3) \tag{86}$$

with $\delta\phi_{1p} = (\phi_{1,p} - \phi_{1,p-1}), \delta\phi_{kp} = (\phi_{k,p} - \phi_{k,p-1}),\ \check{c}_1 = c(\delta\phi_{1p})^3,\ \check{d}_1 = d(\delta\phi_{1p})^4$ and

$$\check{c}_k = c(\delta\phi_{1p})^2\delta\phi_{kp},\ \check{d}_k = d(\delta\phi_{1p})^3\delta\phi_{kp}$$

Corollary 3.1. *The solution of (64) with initial conditions*

$$\begin{aligned} {}^t\phi_1\tilde{u}(0) &= \epsilon a_1 + \epsilon^2\ (-\frac{\check{c} a_1^2}{3\omega_1^2}\ +\ \frac{\check{d} a_1^3}{32\omega_1^2}) + \epsilon^3 r_1(\epsilon, 0), \quad {}^t\phi_1\dot{u}(0) = \mathcal{O}(\epsilon^2) \\ {}^t\phi_k\tilde{u}(0) &= \epsilon^2[-\frac{\check{c}_k a_1^2}{2\omega_k^2} + \frac{\check{c}_k a_1^2}{2(4\omega_1^2 - \omega_k^2)} + \frac{\check{d}_k a_1^3}{4(9\omega_1^2 - \omega_k^2)}] + \epsilon^3 r_k(\epsilon, 0), \quad {}^t\phi_k\dot{u}(0) = \mathcal{O}(\epsilon^2) \end{aligned} \tag{87}$$

$$\text{is } \tilde{u}(t) = \sum_{k=1}^{n} \tilde{y}_k(t)\phi_k + \epsilon^3 r(t, \epsilon) \tag{88}$$

with the expansion of y_k of previous proposition.

Proof. For the proposition, we use lemma 5.4; set $S_1 = \tilde{S}_{3,1}$, $S_k = S_{3,k}$ for $k = 1, \dots n$; as we have enforced (83), the functions S_k are periodic, bounded, and are orthogonal to $e^{\pm it}$, we have assumed that ω_k and ω_1 are $\mathbb{Z}$ independent for $k \neq 1$; then S satisfies satisfies the lemma hypothesis. Similarly, set $g = \tilde{R}$, its components are polynomials in r with coefficients which are bounded functions, so it is lipschitzian on the bounded subsets it satisfies the hypothesis of the lemma and so the proposition is proved. The corollary is an easy consequence of the proposition and the change of function (67) □

Remark 3.3. *1. We have obtained a periodic asymptotic expansion of a solution of system* (64)*; they are called non linear normal modes in the mechanical community ([KPGV09, JPS04]. If the initial condition is close to an eigenvector Φ_1 up to second order, the component of the solution on this eigenvector has an approximation which has the same form as for the single degree of freedom system; the other components remain small.*

2. *The frequency shift is given by a similar formula with c replaced by $\check{c} = c(\phi_{1,p} - \phi_{1,p-1})^3$, d replaced by $\check{d} = d(\phi_{1,p} - \phi_{1,p-1})^4$; so the frequency shift depends on the position of non-linearity with respect to the components of the associated eigenvector.*

3. *In the spirit of inverse problems, this previous point opens a way to localise the non-linearity.*

4. *We do not study the periodicity of the solution itself but as the system is Hamiltonian, it could be obtained from general results, for example see [MH92].*

5. *In the next section, under the assumption of no internal resonance, we shall derive that the frequencies of the normal mode are close to resonant frequencies for an associated forced system, the so called primary resonance; with some changes, secondary resonance could be derived along similar lines.*

3.1.2 Numerical results

We consider numerical solution of (64) with (65); we have chosen $M = I$; $u = 0$ at both ends, so K is the classical matrix

$$k \begin{pmatrix} 2 & -1 & \ldots\ldots\ldots\ldots & & \\ -1 & 2 & -1 & \ldots\ldots\ldots & \\ 0 & -1 & 2 & -1 & \ldots \\ \ldots\ldots\ldots\ldots & & & & \\ \ldots\ldots\ldots\ldots & & & -1 & 2 \end{pmatrix};$$

$C = \lambda I$ with $\lambda = 1/2$; for numerical balance, we have computed $\frac{u}{\epsilon}$; with the choice $p = 1$ we have $\Phi_1 = \epsilon[cu_1^2 + du_1^3]$ with $c = 1, d = 1$. In figure 3.1.2, for 29 degrees of freedom, we find the Fourier transform of the components; some components have the same transform; the graphs are slightly non symmetric; we find also several curves in phase space for some components of the system.

We remark that up to numerical integration errors, all frequencies are equal and the components are periodic. All these characteristics are coherent with the results obtained by asymptotic expansions: an approximation of a non linear normal mode which is a continuation with respect to ϵ of a linear normal mode.

3.2 Forced, damped vibrations, triple scale expansion

3.2.1 Derivation of an asymptotic expansion

We consider a similar system of forced vibrating masses attached to springs with some damping and submitted to a periodic forcing:

$$M\ddot{\tilde{u}} + \epsilon C\dot{\tilde{u}} + Ku + \Phi(\tilde{u}, \epsilon) = \epsilon^2 F \cos \tilde{\omega}_\epsilon t \qquad (89)$$

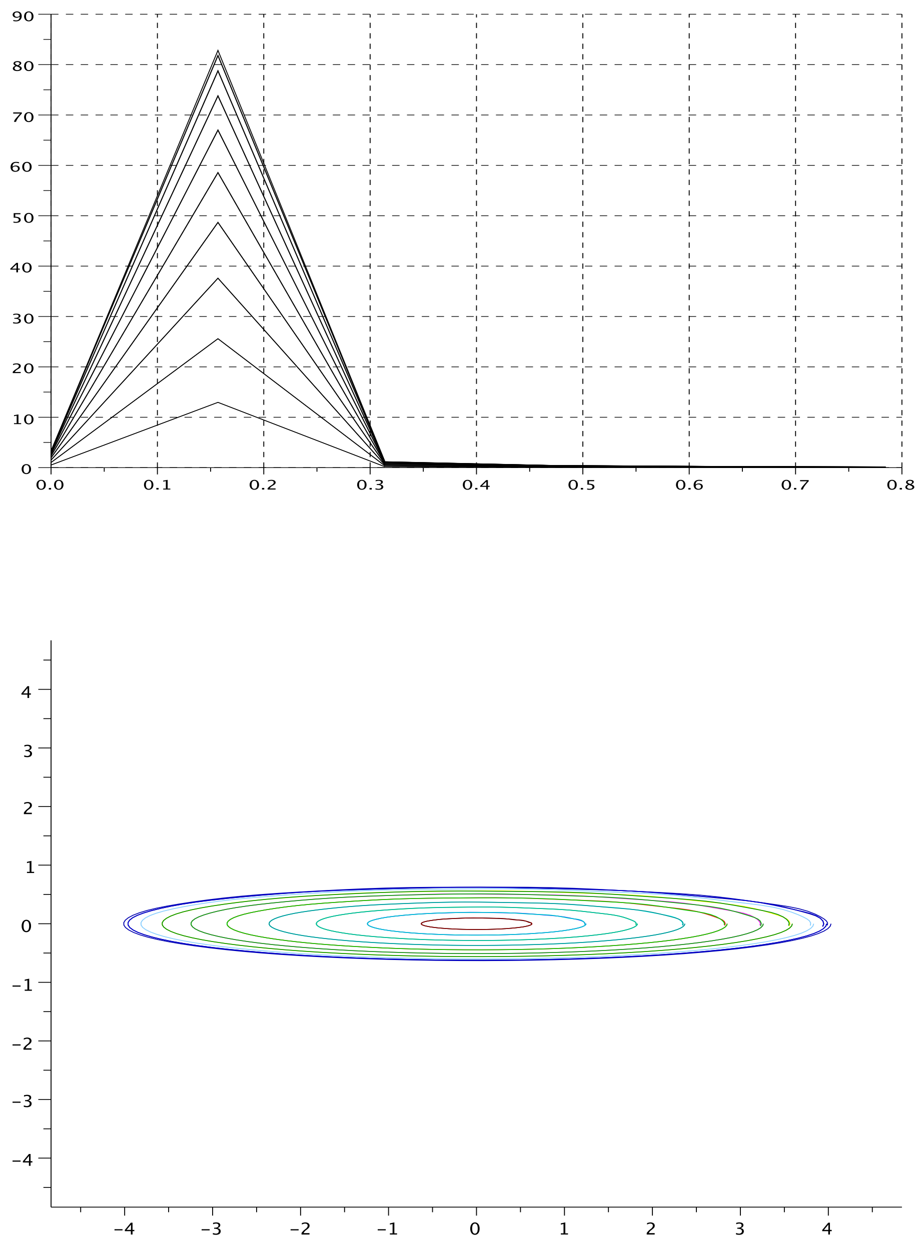

Figure 4: Absolute value of the Fourier transform for (fft) (left); phase portrait(right)

with the same assumptions as in subsection 3.1. We assume that the non linearity is local, all components are zero except for two components $p-1,\ p$ which correspond to the endpoints of some spring assumed to be non linear. As for free vibrations, we perform the change of function

$$\tilde{u} = \sum_{k=1}^{n} \tilde{y}_k \phi_k \tag{90}$$

with ϕ_k, the generalised eigenvectors of (66). However, the distribution of damping is almost always unknown and it is usually necessary to make an assumption about its distribution; a simple and widely used hypothesis is to choose a modal damping (hypothesis of Basile in french terminology):

$$C = \epsilon_M M + \epsilon_K K$$

Therefore

$$\ddot{\tilde{y}}_k + \epsilon\lambda_k \dot{\tilde{y}}_k + \omega_k^2 \tilde{y}_k + {}^t\phi_k \Phi(\sum_{i=1}^{n} \tilde{y}_i \phi_i, \epsilon) = \epsilon^2 f_k \cos\tilde{\omega}_\epsilon T_0, \qquad k = 1\ldots, n$$

with

$$\epsilon_M + \epsilon_K \omega_k^2 = \lambda_k \quad \mathit{and} \quad {}^t\phi_k F = f_k$$

As for the free vibration case, Φ has only 2 components which are not zero, so the system can be written:

$$\ddot{\tilde{y}}_k + \epsilon\lambda_k \dot{\tilde{y}}_k + \omega_k^2 \tilde{y}_k + (\phi_{k,p-1} - \phi_{k,p}) \left[c\left(\sum_{i=1}^{n} \tilde{y}_i(\phi_{i,p} - \phi_{i,p-1}) \right)^2 + \frac{d}{\epsilon} \left(\sum_{i=1}^{n} \tilde{y}_i(\phi_i - \phi_{i,p-1}) \right)^3 \right]$$
$$= \epsilon^2 f_k \cos\tilde{\omega}_\epsilon T_0, \quad \mathit{for} \quad k = 1\ldots, n \tag{91}$$

Remark 3.4. *As we intend to look for a small solution, we consider a change of function* $\boxed{\tilde{y}_k = \epsilon y_k}$ *and we obtain the transformed equation:*

$$\ddot{y}_k + \epsilon\lambda_k \dot{y}_k + \omega_k^2 y_k + (\phi_{k,p-1} - \phi_{k,p}) \left[\epsilon c\left(\sum_{i=1}^{n} y_i(\phi_{i,p} - \phi_{i,p-1}) \right)^2 + \epsilon d \left(\sum_{i=1}^{n} y_i(\phi_i - \phi_{i,p-1}) \right)^3 \right]$$
$$= \epsilon f_k \cos(\tilde{\omega}_\epsilon t) \quad \mathit{for} \quad k = 1\ldots, n \tag{92}$$

We will highlight a link between the frequency of the free solution of the preceding paragraph and the amplitude of the steady state forced solution; it is assumed that the excitation frequency is close to the natural frequency of the linear system

$$\tilde{\omega_\epsilon} = \omega_1 + \epsilon\sigma \tag{93}$$

As in the previous case, we look for a small solution with a triple scale expansion, more precisely, we look for **a periodic solution close to an eigenmode of the linear system**, for example, we consider mode y_1 (by permuting the indexes it could be any mode); we set:

$$T_0 = \tilde{\omega}_\epsilon t, \quad T_1 = \epsilon t, \quad T_2 = \epsilon^2 t \text{ hence } \quad D_0 y_k = \frac{\partial y_k}{\partial T_0}, \quad D_1 y_k = \frac{\partial y_k}{\partial T_1} \text{ and } D_2 y_k = \frac{\partial y_k}{\partial T_2}$$

Derivatives of y_k may be expanded:

$$\frac{dy_k}{dt} = \tilde{\omega}_\epsilon D_0 y_k + \epsilon D_1 y_k + \epsilon^2 D_2 y_k \tag{94}$$

and

$$\frac{d^2 y_k}{dt^2} = \tilde{\omega}_\epsilon^2 D_0^2 y_k + 2\epsilon\tilde{\omega}_\epsilon D_0 D_1 y_k + 2\epsilon^2 D_0 D_2 y_k + \epsilon^2 D_1^2 y_k + 2\epsilon^3 D_1 D_2 y_k + \epsilon^4 D_2^2 y_k \tag{95}$$

we use the *ansatz*

$$y_k(t) = y_k(T_0, T_1, T_2) = y_k^{(1)}(T_0, T_1, T_2) + \epsilon y_k^{(2)}(T_0, T_1, T_2) + \epsilon^2 r_k(T_0, T_1, T_2) \tag{96}$$

we get:

$$\begin{aligned}\frac{dy_k}{dt} &= \frac{dy_k^{(1)}}{dt} + \epsilon\frac{dy_k^{(2)}}{dt} + \epsilon^2\frac{dr_k}{dt} = \epsilon\frac{dy_k^{(1)}}{dt} + \epsilon^2\frac{dy_k^{(2)}}{dt} + \epsilon^2 D_0 r_k + \epsilon^2(\frac{dr_k}{dt} - D_0 r_k)\\ &= [\tilde{\omega}_\epsilon D_0 y_k^{(1)} + \epsilon D_1 y_k^{(1)} + \epsilon^2 D_2 y_k^{(1)}] + \epsilon[\tilde{\omega}_\epsilon D_0 y_k^{(2)} + \epsilon D_1 y_k^{(2)} + \epsilon^2 D_2 y_k^{(2)}]\\ &\qquad + \epsilon^2\tilde{\omega}_\epsilon D_0 r_k + \epsilon^2(\frac{dr_k}{dt} - \tilde{\omega}_\epsilon D_0 r_k)\end{aligned}$$

we note that $\frac{dr_k}{dt} - \tilde{\omega}_\epsilon D_0 r_k = \epsilon D_1 r_k + \epsilon^2 D_2 r_k$; it is of order 1 in ϵ. For the second derivative, as in the case of free vibration, we introduce:

$$\begin{aligned}\mathcal{D}_3 r_k &= \frac{1}{\epsilon}(\frac{d^2 r_k}{dt^2} - \tilde{\omega}_\epsilon^2 D_0^2 r_k)\\ &= 2\tilde{\omega}_\epsilon D_0 D_1 r_k + \epsilon\left[2\tilde{\omega}_\epsilon D_0 D_2 r_k + D_1^2 r_k + 2D_2 D_1 r_k\right] + \epsilon^3 D_2^2 r_k\end{aligned}$$

$$\begin{aligned}\frac{d^2 y_k}{dt^2} &= \frac{d^2 y_k^{(1)}}{dt^2} + \epsilon\frac{d^2 y_k^{(2)}}{dt^2} + \epsilon^2\frac{d^2 r_k}{dt^2} = \frac{d^2 y_k^{(1)}}{dt^2} + \epsilon^1\frac{d^2 y_k^{(2)}}{dt^2} + \epsilon^2\tilde{\omega}_\epsilon D_0^2 r_k + \epsilon^3\mathcal{D}_3 r_k\\ &= \tilde{\omega}_\epsilon^2 D_0^2 y_k^{(1)} + \epsilon\left[2\tilde{\omega}_\epsilon D_0 D_1 y_k^{(1)} + D_0^2 y_k^{(2)}\right]\\ &\qquad + \epsilon^2\left[2\tilde{\omega}_\epsilon D_0 D_2 y_k^{(1)} + D_1^2 y_k^{(1)} + 2\tilde{\omega}_\epsilon D_0 D_1 y_k^{(2)} + D_0^2 r_k\right]\\ &\qquad + \epsilon^3\left[2D_1 D_2 y_k^{(1)} + 2\tilde{\omega}_\epsilon D_0 D_2 y_k^{(2)} + D_1^2 y_k^{(2)} + \mathcal{D}_3 r_k\right]\\ &\qquad + \epsilon^4\left[D_2^2 y_k^{(1)} + 2D_1 D_2 y_k^{(2)} + \epsilon D_2^2 y_k^{(2)}\right]\end{aligned}$$

We plug previous expansions (94), (96) and (95) of y^k into (92); by identifying the coefficients of the powers of ϵ, we get:

$$\left\{\begin{array}{lll} \omega_1^2 D_0^2 y_k^{(1)} + \omega_k^2 y_k^{(1)} = 0 & , & k = 1\ldots, n\\ \omega_1^2 D_0^2 y_k^{(2)} + \omega_k^2 y_k^{(2)} = S_{2,k} & , & k = 1\ldots, n\\ \omega_1^2 D_0^2 r + \omega_k^2 r = S_{3,k} & , & k = 1\ldots, n \end{array}\right. \tag{97}$$

with

$$S_{2,k} = -c\delta\phi_{kp}\left(\sum_{l,m} y_l^{(1)}\delta\phi_{lp}y_m^{(1)}\delta\phi_{mp}\right) - d\delta\phi_{kp}\left(\sum_{g,n,o} y_g^{(1)}\delta\phi_{gp}y_n^{(1)}\delta\phi_{np}y_o^{(1)}\delta\phi_{op}\right)$$
$$- 2\omega_1 D_0 D_1 y_k^{(1)} - \lambda_k\omega_1 D_0 y_k^{(1)} - 2\omega\sigma D_0^2 y_k^{(1)} + f_k\cos(T_0),$$

$$S_{3,k} = -c\delta\phi_{kp}\left(\sum_{l,j} y_l^{(1)}y_j^{(2)}\delta\phi_{lp}\delta\phi_{jp}\right) - d\delta\phi_{kp}\left(\sum_{h,g,n} y_h^{(1)}y_g^{(1)}y_n^{(2)}\delta\phi_{hp}\delta\phi_{gp}\delta\phi_{np}\right)$$
$$- 2\omega_1 D_0 D_2 y_k^{(1)} - D_1^2 y_k^{(1)} - 2\omega_1 D_0 D_1 y_k^{(2)} - \sigma^2 D_0^2 y_k^{(1)} - 2\omega_1\sigma D_0^2 y_k^{(1)} - 2\sigma D_0 D_1 y_k^{(1)} - 2\omega_1\sigma D_0^2 y_k^{(2)}$$
$$- \lambda_k D_1 y_k^{(1)} - -\lambda_k\sigma D_0 y_k^{(1)} - \lambda_k\omega_1 D_0 y_k^{(2)} - \epsilon R_k(\epsilon, r_k, y_1^{(1)}, y_1^{(2)})$$

where $\delta\phi_{kp} = (\phi_{k,p} - \phi_{k,p-1})$ and with

$$R_k(\epsilon, r_k, y_k^{(1)}, y_k^{(2)}) = 2D_1 D_2 y_k^{(1)} + 2\omega_1 D_0 D_2 y_k^{(2)} + D_1^2 y_k^{(2)}$$
$$+ c\delta\phi_{kp}\left(\sum_{l,j} y_j^{(2)}y_j^{(2)}\delta\phi_{lp}\delta\phi_{jp}\right) + c\delta\phi_{kp}\left(\sum_{l,j} y_j^{(1)}r_l\delta\phi_{jp}\delta\phi_{lp}\right)$$
$$+ d\delta\phi_{kp}\left(\sum_{h,g,n} y_h^{(1)}y_g^{(2)}y_n^{(2)}\delta\phi_{hp}\delta\phi_{gp}\delta\phi_{np}\right) + d\delta\phi_{kp}\left(\sum_{h,g,n} y_h^{(1)}y_g^{(1)}r_n\delta\phi_{hp}\delta\phi_{gp}\delta\phi_{np}\right)$$
$$\lambda_k(\omega_1 D_0 r + D_2 y_k^{(1)} + D_1 y_k^{(2)} + \epsilon D_2 y_k^{(2)}) + \mathcal{D}_3 r$$
$$+ \epsilon\left(D_2^2 y_k^{(1)} + 2D_1 D_2 y_k^{(2)} + \epsilon D_2^2 y_k^{(2)}\right) + \lambda_k(\frac{dr}{dt} - \omega_1 D_0 r) + \epsilon\rho(y_k^{(1)}, y_k^{(2)}, r_k, \epsilon)$$

and the polynomial ρ displayed in (74).

We solve the first set of equations (97) imposing initial Cauchy data for $k \neq 1$ of order $\mathcal{O}(\epsilon^2)$ and $D_0 y_1^{(1)}(0) = 0$ we get:

$$\begin{cases} y_1^{(1)} = a_1(T_1, T_2)\ \cos(\theta) \\ y_k^{(1)} = 0, \quad k = 2, \dots, n \end{cases} \tag{98}$$

with $\theta(T_0, T_1, T_2) = T_0 + \beta(T_1, T_2)$ for which we have $D_0\theta = 1,\ D_1\theta = D_1\beta_1$; we put terms involving $y_k^1,\ k \geq 2$ into R_k; so we obtain:

$$S_{2,1} = -\delta\phi_{1p}\Big[\frac{ca_1^2}{2}(1 + \cos(2\theta))\delta\phi_{1p}^2 + \frac{da_1^3}{4}(\cos(3\theta) + 3\cos(\theta))\delta\phi_{1p}^3\Big]$$
$$+ 2\omega_1(D_1 a_1\sin(\theta) + a_1(D_1\beta_1 + \sigma)\cos(\theta)) + \lambda_1 a_1\omega_1\sin(\theta)$$
$$+ f_1(\cos(\theta)\cos(\beta_1) + \sin(\theta)\sin(\beta_1))$$

$$S_{2,k} = -\delta\phi_{kp}\Big[\frac{ca_1^2}{2}(1 + \cos(2\theta))\delta\phi_{1p}^2 + \frac{da_1^3}{4}(\cos(3\theta) + 3\cos(\theta))\delta\phi_{1p}^3\Big]$$
$$+ f_k(\cos(\theta)\cos(\beta_1) + \sin(\theta)\sin(\beta_1)), \quad k = 1, \dots, n.$$

We will eliminate the terms at angular frequency ω_1 hence the functions $a_1(T_1, T_2)$ and $\beta_1(T_1, T_2)$ satisfy:

$$\begin{cases} 2\omega_1 D_1 a_1 + \lambda_1 a_1 \omega_1 = -f_1 \sin(\beta_1) \\ 2\omega_1 a_1 D_1 \beta_1 + 2\omega_1 a\sigma - \frac{3d\delta\phi_{1p}^4 a_1^3}{4} = -f_1 \cos(\beta_1) \end{cases}$$

and the solution of the second equation of(97) is:

$$\begin{cases} y_1^{(2)} = \delta\phi_{1p}\Big[(\frac{-ca_1^2}{2\omega_1^2} + \frac{ca_1^2}{6\omega_1^2}\cos(2\theta))\delta\phi_{1p}^2 + \frac{da_1^3}{32\omega_1^2}\cos(3\theta)\delta\phi_{1p}^3\Big] \\ y_k^{(2)} = \delta\phi_{kp}\Big[-\frac{ca_1^2}{2(\omega_k^2-\omega_1^2)} + \frac{ca_1^2}{2(4\omega_1^2-2\omega_k^2)}\cos(2\theta))\delta\phi_{1p}^2 + \frac{da_1^3}{4(9\omega_1^2-\omega_k^2)}\cos(3\theta)\delta\phi_{1p}^3\Big] \end{cases} \tag{99}$$

where we have omitted the term at frequency ω_1 which is redundant with $y_1^{(1)}$

For the third equation of (97), the unknown is r_k; we do not solve it but we show that the solution is bounded on an interval dependent on ϵ. After some manipulations, the right hand side is:

$$\begin{aligned} S_{3,1} =& + \sin\theta\left[2\omega_1 D_2 a_1 + \lambda_1 a_1 D_1\beta_1 + 2D_1 a_1 D_1 \beta_1 + a_1 D_1^2\beta_1 + 2\sigma D_1 a_1 + \lambda_1 a_1 \sigma\right] \\ &+\cos\theta\left[2\omega_1 a_1 D_2\beta_1 - \lambda_1 D_1 a_1 - D_1^2 a_1 + a_1(D_1\beta_1)^2 + \sigma^2 a_1 + 2\sigma a_1 D_1\beta_1 + \frac{5c^2\delta\phi_{1p}^6 a_1^3}{6\omega_1} - \frac{3\delta\phi_{1p}^8 d^2 a_1^5}{128\omega_1}\right] \\ &+ S_3^\sharp - \epsilon R(\epsilon, r, u^{(1)}, u^{(2)}) \end{aligned}$$

where

$$\begin{aligned} S_{3,1}^\sharp =& \frac{5cd\delta\phi_{1p}^7 a_1^4}{8\omega_1^2} + \sin 2\theta\left[\frac{4c\delta\phi_{1p}^3 a_1}{3\omega_1}D_1 a_1 + \frac{\lambda_1 c\delta\phi_{1p}^3 a_1^2}{3\omega_1}\right] \\ &+ \cos 2\theta\left[\frac{4c\delta\phi_{1p}^3 a_1^2}{3\omega_1}D_1\beta_1 + \frac{15cd\delta\phi_{1p}^7 a_1^4}{32\omega_1^2}\right] \\ &+ \sin 3\theta\left[\frac{9d\delta\phi_{1p}^4 a_1^2}{16\omega_1}D_1 a_1 + \frac{3\lambda_1 d\delta\phi_{1p}^4 a_1^3}{16\omega_1}\right] + \cos 3\theta\left[\frac{9d\delta\phi_{1p}^4 a_1^3}{16\omega_1}D_1\beta_1 - \frac{c^2\delta\phi_{1p}^6 a_1^3}{6\omega_1^2} - \frac{3d^2\delta\phi_{1p}^8 a_1^4}{64\omega_1^2}\right] \\ &+ \cos 4\theta\left[-\frac{3cd\delta\phi_{1p}^7 a_1^4}{8\omega_1^2}\right] - \cos 5\theta\frac{3d^2\delta\phi_{1p}^8 a_1^5}{128\omega_1^2} \end{aligned}$$

and a similar expression for $S_{3,k}^\sharp$. To eliminate the secular terms, we impose,

$$\begin{cases} 2\omega_1 D_2 a_1 + \lambda_1 a_1 D_1\beta_1 + 2D_1 a_1 D_1\beta_1 + a_1 D_1^2\beta_1 + 2\sigma D_1 a_1 + \lambda_1 a_1 \sigma = 0 \\ 2\omega_1 a_1 D_2\beta_1 - \lambda_1 D_1 a_1 - D_1^2 a_1 + a_1(D_1\beta_1)^2 + \sigma^2 a_1 + 2\sigma a_1 D_1\beta_1 + \frac{5c^2\delta\phi_{1p}^6 a_1^3}{6\omega_1^2} - \frac{3\delta\phi_{1p}^8 d^2 a_1^5}{128\omega_1^2} = 0 \end{cases}$$

As a_1 and β_1 do not depend on T_0, the following relations hold:

$$\begin{cases} \dfrac{da_1}{dt} = \epsilon D_1 a_1 + \epsilon^2 D_2 a_1 + \mathcal{O}(\epsilon^3) \\ \dfrac{d\beta_1}{dt} = \epsilon D_1\beta_1 + \epsilon^2 D_2\beta_1 + \mathcal{O}(\epsilon^3) \end{cases} \tag{100}$$

On the other hand, we can determine the expression of D_2a_1 and $D_2\beta$, like for one degree of freedom:

$$\left\{\begin{array}{l} D_2a_1 = \frac{3d\lambda_1\delta\phi_{1p}^4a_1^3}{16\omega_1^2} + \frac{\sigma f_1 \sin\gamma}{4\omega_1^2} + \frac{\lambda_1 f_1\cos\gamma}{8\omega_1^2} + \frac{9d\delta\phi_{1p}^4a_1^2f_1\sin\gamma}{32\omega_1^3} \\ D_2\gamma = -\frac{\lambda_1^2}{8\omega_1} - \frac{15d^2\delta\phi_{1p}^8a_1^4}{256\omega_1^3} - \frac{5c^2\delta\phi_{1p}^6a_1^2}{12\omega_1^3} \\ \qquad\qquad + \frac{\sigma f_1\cos\gamma}{4\omega_1^2a_1} + \frac{3d\delta\phi_{1p}^4a_1f_1\cos\gamma}{32\omega_1^3} - \frac{\lambda_1 f_1\sin\gamma}{8\omega_1^2a_1} \end{array}\right. \tag{101}$$

now we return to (100) introducing (99) and (101), we obtain:

$$\begin{aligned} \frac{da_1}{dt} &= \epsilon\Big(-\frac{f_1\sin(\beta)}{2\omega_1} + \frac{\lambda_1a_1}{2}\Big) + \epsilon^2\left(\frac{3d\lambda_1\delta\phi_{1p}^4a_1^3}{16\omega_1^2} + \frac{\sigma f_1\sin\beta}{4\omega_1^2} + \frac{\lambda_1 f_1\cos\beta}{8\omega_1^2} + \frac{9d\delta\phi_{1p}^4a_1^2f_1\sin\beta}{32\omega_1^3}\right) + O(\epsilon^3) \\ \frac{d\beta}{dt} &= \epsilon\Big(-\sigma + \frac{3d\delta\phi_{1p}^4a_1^2}{8\omega_1} - \frac{f_1\cos(\beta)}{2\omega_1a_1}\Big) \\ &\quad + \epsilon^2\Big(-\frac{\lambda_1^2}{8\omega_1} - \frac{15d^2\delta\phi_{1p}^8a_1^4}{256\omega_1^3} - \frac{5c^2\delta\phi_{1p}^6a_1^2}{12\omega_1^3} + \frac{\sigma f_1\cos\beta}{4\omega_1^2a_1} + \frac{3d\delta\phi_{1p}^4a_1f_1\cos\beta}{32\omega_1^3} - \frac{\lambda_1f_1\sin\beta}{8\omega_1^2a_1}\Big) + O(\epsilon^3) \end{aligned} \tag{102}$$

Remark 3.5. *In this approach, like for one free degree of freedom, we are using the method of reconstitution. We notice these equations are similar to* (34)

Remark 3.6. $S_3^\sharp + R(\epsilon, r, u^{(1)}, u^{(2)})$ *has no term at frequency* ω_1 *or which goes to* ω_1 *.*
This will allow us to justify this expansion in certain conditions, before we consider the stationary solution of the system (102) *and the stability of the solution close to the stationary solution.*

3.2.2 Stationary solution and stability

Let us consider the stationary solution of (102), it satisfies:

$$\left\{\begin{array}{l} g_1(a_1, \beta_1, \sigma, \epsilon) = 0, \\ g_2(a_1, \beta_1, \sigma, \epsilon) = 0 \end{array}\right. \tag{103}$$

with

$$\left\{\begin{array}{l} g_1 = \epsilon(-\frac{f_1\sin(\beta)}{2\omega_1} + \frac{\lambda_1a_1}{2}) + \\ \qquad \epsilon^2(\frac{3d\lambda_1\delta\phi_{1p}^4a_1^3}{16\omega_1^2} + \frac{\sigma f_1\sin\beta}{4\omega_1^2} + \frac{\lambda_1f_1\cos\beta}{8\omega_1^2} + \frac{9d\delta\phi_{1p}^4a_1^2f_1\sin\beta}{32\omega_1^3}) + \mathcal{O}(\epsilon^3) \\ g_2 = \epsilon(-\sigma + \frac{3d\delta\phi_{1p}^4a_1^2}{8\omega_1} - \frac{f_1\cos(\beta)}{2\omega_1a_1}) \\ \qquad +\epsilon^2(-\frac{\lambda_1^2}{8\omega_1} - \frac{15d^2\delta\phi_{1p}^8a_1^4}{256\omega_1^3} - \frac{5c^2\delta\phi_{1p}^6a_1^2}{12\omega_1^3} + \frac{\sigma f_1\cos\beta}{4\omega_1^2a_1} - \frac{3d\delta\phi_{1p}^4a_1f_1\cos\beta}{32\omega_1^3} - \frac{\lambda_1f_1\sin\beta}{8\omega_1^2a_1}) + \mathcal{O}(\epsilon^3) \end{array}\right. \tag{104}$$

The situation is very close to the 1 d.o.f. case; except the replacement of c by $\check{c} = c\delta\phi_{1p}^3$ and d by of $\check{d} = d\delta\phi_{1p}^4$, the system (104) is the same as (36); the other components are zero. We state a similar proposition.

Proposition 3.2. *When*

$$\sigma \leq \frac{3\check{d}\bar{a}_1^2}{4\omega_1} - \frac{1}{2}\sqrt{\frac{9\check{d}^2\bar{a}_1^4}{16\omega_1^2} - \lambda_1^2}$$

and ϵ small enough, the stationary solution $(\bar{a}_1, \bar{\beta}_1)$ of (102) *is stable in the sense of Lyapunov (if the dynamic solution starts close to the stationary one, it remains close and converges to it); to the stationary case corresponds the approximate solution of* (92)

$$\tilde{y}_{1\,app} = \epsilon\bar{a}_1\cos(\tilde{\omega}_\epsilon t+\bar{\beta})+\epsilon^2\Big(\delta\phi_{1p}\Big[\Big(\frac{-c\bar{a}_1^2}{\omega_1^2}+\frac{c\bar{a}_1^2}{6\omega_1^2}\cos 2(\tilde{\omega}_\epsilon t+\bar{\beta})\Big)\delta\phi_{1p}^2+\frac{d\bar{a}_1^3}{32\omega_1^2}\cos(3(\tilde{\omega}_\epsilon t+\bar{\beta}))\delta\phi_{1p}^3\Big]\Big); \tag{105}$$

$$\tilde{y}_{k\,app} = \epsilon^2\Big(\delta\phi_{kp}\Big[\Big(\frac{-c\bar{a}_1^2}{2(\tilde{\omega}_k^2+\omega_1^2)} - \frac{c\bar{a}_1^2}{2(\omega_k^2-4\omega_1^2)}\cos 2(\tilde{\omega}_\epsilon t+\bar{\beta})\Big)\delta\phi_{1p}^2 - \frac{d\bar{a}_1^3}{4(\tilde{\omega}_k^2-9\omega_1^2)}\cos(3(\tilde{\omega}_\epsilon t+\bar{\beta}))\delta\phi_{1p}^3\Big]\Big) \tag{106}$$

it is periodic.

With this result of stability, we can state precisely the approximation of the solution of (89)

3.2.3 Convergence of the expansion

In order to prove that r_k is bounded, after eliminating terms at frequency ν_1, we go back to the variable t for the third set of equations of (97) .

$$\begin{aligned}
&\frac{d^2 r_k}{dt^2} + \omega_1^2 r_k = \tilde{S}_{3,k} \quad \text{for } k = 1, \dots n \quad \text{with}\\
&\tilde{S}_{3,1} = S^{\sharp}_{3,1}(t,\epsilon) - \epsilon\tilde{R}_1(y_1^{(1)}, y_1^{(2)}, r_1, \epsilon) \quad \text{and for } k \neq 1\\
&S_{3,k} = -2c\delta\phi_{kp}[y_1^{(1)}y_1^{(2)}\delta\phi_{1p}^2] - 3d\delta\phi_{kp}[y_1^{(1)2}y_1^{(2)}\delta\phi_{1p}^3] - \epsilon R_k(\epsilon, r_k, y_1^{(1)}, y_1^{(2)})
\end{aligned}$$

where

$$\tilde{R}_k(\epsilon, r_k, y_1^{(1)}, y_1^{(2)}) = R_k(\epsilon, r_k, y_1^{(1)}, y_1^{(2)}) - \mathcal{D}_2 r_k - \lambda_k\big(\frac{dr_k}{dt} - \omega_k D_0 r_k\big)$$

with all the terms expressed with the variable t.

Proposition 3.3. *Under the assumption that $\omega_k^2 \neq 4\omega_1^2$, $\omega_k^2 \neq 9\omega_1^2$ and ω_1^2 a simple eigenvalue (no internal resonance) for $k \neq 1$, there exists $\varsigma > 0$ such that for all $t \leq t_\epsilon = \frac{\varsigma}{\epsilon^2}$, the solution $\tilde{y} = \epsilon y$ of* (91) *with initial data*

$$\begin{aligned}
\tilde{y}_1(0) &= \epsilon a_1 + \epsilon^2\ \big(\frac{-\check{c}_1 a_{10}^2}{2\omega_1^2}\ +\ \frac{\check{c}_1 a_{10}^2}{6\omega_1^2}\cos(2\beta_0)\ +\frac{\check{d}_1 a_{10}^3}{32\omega_1^2}\cos(3\beta_0)\big) + \epsilon^3 r(0,\epsilon),\\
\tilde{y}_k(0) &= \epsilon^2\left(\frac{-\check{c}_1 a_{10}^2}{2(\omega_k^2-\omega_1^2)} + \frac{\check{c}_1 a_{10}^2}{2(4\omega_1^2-\omega_k^2)}\cos(2\beta_0) + \frac{\check{d}_1 a_{10}^3}{4(9\omega_1^2-\omega_k^2)}\cos(3(\beta_0))\right) + \epsilon^3 r(0,\epsilon),
\end{aligned}$$

with similar expressions for $\dot{y}_1(0), \dot{y}_k(0)$ and with (a_{10}, β_0) close to the stationary solution $(\bar{a}_1, \bar{\beta})$

$$|a_{10}-\bar{a}_1| \le \epsilon^2 C^1, \quad |\beta_0-\bar{\beta}| \le \epsilon^2 C^1$$

has the following expansion

$$\tilde{y}_1 = \epsilon a_1 \cos(\tilde{\omega}_\epsilon t+\beta(t))+\epsilon^2[((\frac{-\check{c}_1 a_1^2}{2\omega_1^2}+\frac{\check{c}_1 a_1^2}{6\omega_1^2}\cos(2(\tilde{\omega}_\epsilon t+\beta(t)))+\frac{\check{d}_1 a_1^3}{32\omega_1^2}\cos(3(\tilde{\omega}_\epsilon t+\beta(t))))]+\epsilon^3 r_1(t)$$

$$\tilde{y}_k = \epsilon^2\Big(\Big[(\frac{-\check{c}_k a_1^2}{2(\omega_k^2-\omega_1^2)}+\frac{\check{c}_k a_1^2}{2(4\omega_1^2-\omega_k^2)}\cos(2(\tilde{\omega}_\epsilon t+\beta(t)))+\frac{\check{d}_k a_1^3}{4(9\omega_1^2-\omega_k^2)}\cos(3(\tilde{\omega}_\epsilon t+\beta(t)))\Big]\Big)+\epsilon^3 r_k(t)$$

with a_1, β solution of (102) *and with r_k uniformly bounded in $\mathcal{C}^2(0,t_\epsilon)$ for $k=1,\dots n$ and ω_1, ϕ_1 are the eigenvalue and eigenvectors defined in* (66)*, with $\delta\phi_{1p}=(\phi_{1,p}-\phi_{1,p-1}), \delta\phi_{kp}=(\phi_{k,p}-\phi_{k,p-1})$, $\check{c}_1=c(\delta\phi_{1p})^3$, $\check{d}_1=d(\delta\phi_{1p})^4$ and $\check{c}_k=c(\delta\phi_{1p})^2\delta\phi_{kp}$, $\check{d}_k=d(\delta\phi_{1p})^3\delta\phi_{kp}$ as in proposition 3.1.*

Corollary 3.2. *The solution of* (89) *with*

$$\phi_1^T\tilde{u}(0)=\epsilon a_1+\epsilon^2\big(\frac{-\check{c}_1 a_{10}^2}{2\omega_{10}^2} + \frac{-\check{c}_1 a_{10}^2}{6\omega_1^2}\cos(2\gamma_0) + \frac{\check{d}_1 a_{10}^3}{32\omega_1^2}\cos(3\gamma_0)\big)+\epsilon^3 r_1(0,\epsilon),$$
$$\phi_k^T\tilde{u}(0)=\epsilon^2\left(\frac{-\check{c}_1 a_{10}^2}{2(\omega_k^2-\omega_1^2)}+\frac{\check{c}_1 a_{10}^2}{2(4\omega_1^2-\omega_k^2)}\cos(2\gamma_0)+\frac{\check{d}_1 a_{10}^3}{4(9\omega_1^2-\omega_k^2)}\cos(3(\gamma_0))\right)+\epsilon^3 r_k(0,\epsilon),$$

with similar expressions for $\phi_1^T\dot{\tilde{u}}(0), \phi_k^T\dot{\tilde{u}}(0)$ and with ω_k, ϕ_k the eigenvalues and eigenvectors defined in (66)*.*

$$\text{is } \tilde{u}(t)=\sum_{k=1}^n \tilde{y}_k(t)\phi_k \tag{107}$$

with the expansion of y_k of previous proposition.

Proof. We follow a similar route as for one degree of freedom, we use lemma 5.4. Set $S_1=S_{31}^\sharp$, $S_k=S_{3,k}$ for $k=1,\dots n$; as we have enforced (104), the functions S_k are not periodic but close to a periodic function, bounded and are orthogonal to $e^{\pm it}$, we have assumed that ω_k and ω_1 are $\mathbb{Z}$ independent for $k \ne 1$; so S satisfies the lemma hypothesis. Similarly, set $g=\tilde{R}$, it is a polynomial in r with coefficients which are bounded functions , so it is lipschitzian on the bounded subsets of $\mathbb{R}$, it satisfies the hypothesis of lemma 5.4 and so the proposition is proved. The corollary is an easy consequence of the proposition and the change of function (90) □

3.2.4 Maximum of the stationary solution

We can state results similar to the case of one degree of freedom.

Proposition 3.4. *The stationary solution of* (102) *satisfies*

$$\begin{cases} (-\frac{f_1\sin(\beta)}{2\omega_1}+\frac{\lambda_1 a_1}{2})+ \quad \epsilon A_1(a_1,\beta,\sigma)+\mathcal{O}(\epsilon^2)=0 \\ (-\sigma+\frac{3d\delta\phi_{1p}^4 a_1^2}{8\omega_1}-\frac{f_1\cos(\beta)}{2a_1\omega_1})+ \quad \epsilon A_2(a_1,\beta,\sigma)+\mathcal{O}(\epsilon^2)=0 \end{cases} \tag{108}$$

with

$$A_1(a,\beta,\sigma) = \frac{3d\delta\phi_{1p}^4\lambda_1 a_1^3}{16\omega_1^2} + \frac{\sigma f_1 \sin\beta}{4\omega_1^2} + \frac{\lambda f_1\cos\beta}{8\omega_1^2} + \frac{9d\delta\phi_{1p}^4 a_1^2 f_1\sin\beta}{32\omega_1^3}$$
$$A_2(a,\beta,\sigma) = -\frac{\lambda_1^2}{8\omega_1} - \frac{15d^2\delta\phi_{1p}^8 a_1^4}{256\omega_1^3} - \frac{5c^2\delta\phi_{1p}^6 a_1^2}{12\omega_1^3} + \frac{\sigma f_1\cos\beta}{4\omega_1^2 a_1} + \frac{3d\delta\phi_{1p}^4 a_1 f_1\cos\beta}{32\omega_1^3} - \frac{\lambda_1 f_1 \sin\beta}{8\omega_1^2 a_1}$$

this stationary solution reaches its maximum amplitude for $\sigma = \sigma_0^* + \epsilon\sigma_1^* + O(\epsilon^2)$ *with*

$$a_{1,0}^* = \frac{f_1}{\lambda_1\omega_1},\ \sigma_0^* = \frac{3\check{d}a_{1,0}^{*2}}{8\omega_1} = \frac{3\check{d}f_1^2}{8\lambda_1^2\omega_1^3}, \quad \beta_0^* = -\frac{\pi}{2} \tag{109}$$

and

$$\sigma_1^* = -\frac{87\check{d}^2 a_{1,0}^{*4}}{256\omega_1^3} - \frac{5\check{c}^2 a_{1,0}^{*2}}{12\omega_1^3} - \frac{\lambda_1^2}{4\omega_1}, \quad \beta_1^* = -\frac{\lambda_1}{2\omega_1}, \quad a_{1,1}^* = -\frac{a_{1,0}^*\sigma_0^*}{\omega_1}$$

the periodic forcing is at the angular frequency

$$\tilde{\omega}_\epsilon = \omega_1 + \epsilon\sigma_0^* + \epsilon^2\sigma_1^* + \mathcal{O}(\epsilon^2)$$

up to the term involving the damping ratio λ_1*, it is slightly different of the approximate angular frequency* ν_ϵ *of the undamped free periodic solution* (86)*; for this frequency, the approximation (of the solution* $\tilde{y} = \epsilon y$ *of* (91) *up to the order* ϵ^2*) is periodic:*

$$\begin{cases} \tilde{y}_1(t) = \epsilon\bar{a}_1^*\cos(\tilde{\omega}_\epsilon t + \bar{\beta}^*) + \epsilon^2[(\frac{-\check{c}_1\bar{a}_1^{*2}}{2\omega_1^2} + \frac{\check{c}_1\bar{a}_1^{*2}}{6\omega_1^2}\cos(2(\tilde{\omega}_\epsilon t + \bar{\beta}^*))) \\ \qquad\qquad + \frac{\check{d}_1\bar{a}_1^{*3}}{32\omega_1^2}\cos(3(\tilde{\omega}_\epsilon t + \bar{\beta}^*)] + \epsilon^3 r_1(\epsilon,t) \\ \tilde{y}_k(t) = \epsilon^2[(\frac{-\check{c}_k\bar{a}_1^{*2}}{2(\omega_k^2-\omega_1^2)} - \frac{\check{c}_k\bar{a}_1^{*2}}{2(\omega_k^2-4\omega_1^2)}\cos(2(\tilde{\omega}_\epsilon t + \bar{\beta}^*))) \\ \qquad\qquad - \frac{\check{d}_k\bar{a}_1^{*3}}{4(\omega_k^2-9\omega_1^2)}\cos(3(\tilde{\omega}_\epsilon t + \bar{\beta}^*)] + \epsilon^3 r_k(\epsilon,t) \end{cases} \tag{110}$$

and initial conditions like in proposition 3.1.

4 Conclusion

For some differential systems modelling spring-masses vibrations with non linear springs, we have derived and rigorously proved an asymptotic approximation of periodic solution of free vibrations (so called non linear normal modes); for damped vibrations with periodic forcing with frequency close (but different) to free vibration frequency (the so called primary resonance case), we have obtained an asymptotic expansion and derived that the amplitude is maximal close to the frequency of the non linear normal mode.

We emphasize that the use of three time scales provides a more accurate value of the link between frequency and amplitude (so called backbone) of a non linear mode but it yields also a new insight in the behavior of the solution which was not provided by a double-scale analysis: the influence of the ratio of c over d on the shape of the backbone and the amplitude of the forced response to an harmonic force as is clearly displayed in figure 2 and 3.

As an opening to a related problem, we can notice that such non linear vibrating systems linked to a bar generate acoustic waves; an analysis of the dilatation of a one-dimensional nonlinear crack impacted by a periodic elastic wave, with a smooth model of the crack may be carried over with a delay differential equation, [JL09].

Acknowledgment We thank S. Junca for his stimulating interest.

5 Appendix

5.1 Technical lemmas

All these lemmas are recalled here for convenience of the reader; they already have been proposed in [BR13].

Lemma 5.1. *Let w_ϵ be solution of*

$$\begin{aligned} w" + w = S(t, \epsilon) + \epsilon g(t, w, \epsilon) \\ w(0) = 0, \quad w'(0) = 0 \end{aligned} \tag{111}$$

If the right hand side satisfies the following conditions

1. *S is a sum of periodic bounded functions:*
   (a) *for all t and for all ϵ small enough, $S(t, \epsilon) \leq M$*
   (b) *$\int_0^{2\pi} e^{it} S(t, \epsilon)dt = 0, \quad \int_0^{2\pi} e^{-it} S(t, \epsilon)dt = 0$ uniformly for ϵ small enough*
2. *for all $R > 0$, there exists k_R such that for $|u| \leq R$ and $|v| \leq R$, the inequality $|g(t, u, \epsilon) - g(t, v, \epsilon)| \leq k_R|u - v|$ holds and $|g(t, 0, \epsilon)|$is bounded; in other words g is locally lipschitzian with respect to u.*

then, there exists $\gamma > 0$ such that for ϵ small enough, w_ϵ is uniformly bounded in $C^2(0, T_\epsilon)$ with $T_\epsilon = \frac{\gamma}{\epsilon}$

Proof. The proof is close to the proof of lemma 6.3 of [JR10]; but it is technically simpler since here we assume g to be locally lipschitzian with respect to u whereas it is only bounded in [JR10].

1. We first consider

$$\begin{aligned} w_1" + w_1 = S(t, \epsilon) \\ w_1(0) = 0, \quad w_1'(0) = 0 \end{aligned} \tag{112}$$

   as S is a sum of periodic functions which are uniformly orthogonal to e^{it} and e^{-it}, w_1 is bounded in $\mathcal{C}^2(0, +\infty)$

2. Then we perform a change of function: $w = w_1 + w_2$, the following equalities hold

$$\begin{aligned} w_2" + w_2 = \epsilon g_2(t, w_2, \epsilon) \\ w_2(0) = 0, \quad w_2'(0) = 0 \end{aligned} \tag{113}$$

with g_2 which satisfies the same hypothesis as g:

for all $R > 0$, there exists k_R such that for $|u| \leq R$ and $|v| \leq R$, the following inequality holds $|g_2(t,u,\epsilon) - g_2(t,v,\epsilon)| \leq k_R|u-v|$. Using Duhamel principle, the solution of this equation satisfies:

$$w_2 = \epsilon \int_0^t \sin(t-s) g_2(s, w_2(s), \epsilon) ds$$

from which

$$|w_2(t)| \leq \epsilon \int_0^t |g_2(s, w_2(s), \epsilon) - g_2(s, 0, \epsilon)| ds + \epsilon \int_0^t |g_2(s, 0, \epsilon)| ds$$

so if $|w| \leq R$, hypothesis of lemma imply

$$|w_2(t)| \leq \epsilon \int_0^t k_R |w_2| ds + \epsilon C t$$

A corollary of lemma of Bellman-Gronwall, see below, will enable to conclude. It yields

$$|w_2(t)| \leq \frac{C}{k_R} \left(\exp(\epsilon k_R t) - 1\right)$$

Now set $T_\epsilon = \sup\{t \,|\, |w| \leq R\}$, then we have

$$R \leq \frac{C}{k_R} \left(\exp(\epsilon k_R t) - 1\right)$$

this shows that there exists γ such that $|w_2| \leq R$ for $t \leq T_\epsilon$, which means that it is in $L^\infty(0, T_\epsilon)$ for $T_\epsilon = \frac{\gamma}{\epsilon}$; also, we have w in $\mathcal{C}(0, T_\epsilon)$ then as w is solution of (111), it is also bounded in $\mathcal{C}^2(0, T_\epsilon)$.

$\square$

Lemma 5.2. *(Bellman-Gronwall, [BG, Bel64]) Let u, ϵ, β be continuous functions with $\beta \geq 0$,*

$$u(t) \leq \epsilon(t) + \int_0^t \beta(s) u(s) ds \text{ for } 0 \leq t \leq T$$

then

$$u(t) \leq \epsilon(t) + \int_0^t \beta(s)\epsilon(s) \left[\exp(\int_s^t \beta(\tau) d\tau\right] ds$$

Lemma 5.3. *(a consequence of previous lemma, suited for expansions, see [SV85]) Let u be a positive function, $\delta_2 \geq 0$, $\delta_1 > 0$ and*

$$u(t) \leq \delta_2 t + \delta_1 \int_0^t u(s) ds$$

then

$$u(t) \leq \frac{\delta_2}{\delta_1} \left(exp(\delta_1 t) - 1\right)$$

Lemma 5.4. *Let $v_\epsilon = [v_1^\epsilon, \dots, v_N^\epsilon]^T$ be the solution of the following system:*

$$\omega^2 (v_k^\epsilon)" + \omega_k^2 v_k^\epsilon = S_k(t) + \epsilon g_k(t, v_\epsilon) \tag{114}$$

If ω and ω_k are $\mathbb{Z}$ independent for all $k = 2 \dots N$ and the right hand side satisfies the following conditions with $M > 0$, $C > 0$ prescribed constants:

1. *S_k is a sum of bounded periodic functions, $|S_k(t)| \le M$ which satisfy the non resonance conditions:*
2. *S_1 is orthogonal to $e^{\pm it}$, i.e. $\int_0^{2\pi} S_1(t) e^{\pm it} dt = 0$ uniformly for ϵ going to zero*
3. *for all $R > 0$ there exists k_R such that for $\|u\| \le R$, $\|v\| \le R$, the following inequality holds for $k = 1, \dots, N$:*
   $$|g_k(t, u, \epsilon) - g_k(t, v, \epsilon)| \le k_R \|u - v\|$$
   and $|g_k(t, 0, \epsilon)|$ is bounded

then there exists $\gamma > 0$ such that for ϵ small enough v_ϵ is bounded in $\mathcal{C}^2(0, T_\epsilon)$ with $T_\epsilon = \frac{\gamma}{\epsilon}$

Proof. 1. We first consider the linear system

$$\begin{gathered} \omega_1^2 (v_{k,1})" + \omega_k^2 v_{k,1} = S_k \\ v_{k,1}(0) = 0 \text{ and } (v_{k,1})' = 0 \end{gathered} \tag{115}$$

For $k = 1$, with hypothesis 1.a, S_1 is a sum of bounded periodic functions; it is orthogonal to $e^{\pm it}$, there is no resonance. For $k \ne 1$, there is no resonance as $\frac{\omega_k}{\omega_1} \notin \mathbb{Z}$ with hypothesis 1.b.

So $v_{k,1}$ belongs to $C^{(2)}$ for $k = 1, ..., n$

2. Then we perform a change of function

$$v_k^\epsilon = v_{k,1} + v_{k,2}^\epsilon$$

and $v_{k,2}^\epsilon$ are solutions of the following system :

$$\begin{aligned} \omega_1^2 (v_{k,2})" + \omega_k^2 v_{k,2} = \epsilon g_{k,2}(t, v_{k,2}, \epsilon),\ k = 1, \dots, N \\ v_{k,2}^\epsilon(0) = 0,\ (v_{k,2}^\epsilon)' = 0,\ k = 1, \dots, N \end{aligned} \tag{116}$$

with

$$g_{k,2}(t, ..., v_{k,2}^\epsilon, ...) = g_k(t, ..., v_{k,1} + v_{k,2}^\epsilon, ...)$$

where $g_{k,2}$ satisfies the same hypothesis as g_k:
for all $R > 0$ there exists k_R such that for $\| u_k \| \le R$, $\| v_k \| \le R$, the following inequality holds for $k = 1, \dots, N$:

$$\| g_{k,2}(t, u_k, \epsilon) - g_{k,2}(t, v_k, \epsilon) \| \le k_R \| u_k - v_k \| \tag{117}$$

Using Duhamel principle, the solution or the equation (116) satisfies:

$$v_{k,2}^\epsilon = \epsilon \int_0^t \sin(t - s) g_{k,2}(s, v_{k,2}^\epsilon(s), \epsilon) ds$$

so

$$\| v^{\epsilon}_{k,2}(t) \| \leq \epsilon \int_0^t \| g_{k,2}(s, v^{\epsilon}_{k,2}(s), \epsilon) - g_{k,2}(s, 0, \epsilon) \| ds +$$

$$\epsilon \int_0^t \| g_{k,2}(s, 0, \epsilon) \| ds$$

so with (117), we obtain

$$\| v^{\epsilon}_{k,2}(t) \| \leq \epsilon \int_0^t k \| v^{\epsilon}_{k,2}(t) \| ds + \epsilon C t$$

We shall conclude using Bellman-Gronwall lemma; we obtain

$$\| v_{k,2}(t) \| \leq \frac{C}{k_R}(exp(\epsilon k_R t) - 1)$$

this shows that there exists γ such that $|v^{\epsilon}_{k,2}| \leq R$ for $t \leq T_{\epsilon}$, which means that it is in $L^{\infty}(0, T_{\epsilon})$ for $T_{\epsilon} = \frac{\gamma}{\epsilon}$; also, we have v_k in $\mathcal{C}(0, T_{\epsilon})$ then as v_k is solution of (111), it is also bounded in $\mathcal{C}^2(0, T_{\epsilon})$.

□

Theorem 5.1. *(of Poincaré-Lyapunov, for example see [SV85]) Consider the equation*

$$\dot{x} = (A + B(t))x + g(t, x),\ x(t_0) = x_0,\ t \geq t_0$$

where $x, x_0 \in \mathbf{R}^n$, A is a constant matrix $n \times n$ with all its eigenvalues with negative real parts; $B(t)$ is a matrix which is continuous with the property $\lim_{t \to +\infty} \|B(t)\| = 0$. The vector field is continuous with respect to t and x is continuously differentiable with respect to x in a neighbourhood of $x = 0$; moreover

$$g(t, x) = o(\|x\|) \text{ when } \|x\| \to 0$$

uniformly in t. Then, there exists constants C, t_0, δ, μ such that if $\|x_0\| < \frac{\delta}{C}$

$$\|x\| \leq C\|x_0\|e^{-\mu(t-t_0)}, t \geq t_0$$

holds

5.2 Numerical computations of Fourier transform

Assuming a function f to be almost-periodic, the Fourier coefficients are :

$$\alpha_n = \lim_{T \to +\infty} \int_0^T f(t) e^{-i\lambda_n t} dt \tag{118}$$

where λ_n are countable Fourier exponents of f. (for example, see Fourier coefficients of an almost-periodic function in http://www.encyclopediaofmath.org/). For numerical purposes, we chose T large enough and with a fast Fourier transform, we compute numerically the Fourier coefficients of a function of period T equal to f in this interval.

5.3 Another way of computing the maximum amplitude

This is another way of computing some results of § 2.3.4. Eliminating β at first order in (61) , we get that a is solution of $f(a,\beta,\sigma,\epsilon)=0$ with

$$f=\frac{-F_m^2}{4\omega^2}+(-\frac{\lambda a}{2}+\epsilon A_1)^2+\left(\frac{3da^3}{8\omega}-\sigma a+\epsilon a A_2\right)^2+\mathcal{O}(\epsilon^2)$$

We look for a maximum with respect to σ; it will be reached at a value denoted σ^* which depends on ϵ. By differentiating, we get that

$$\frac{\partial a}{\partial\sigma}=-\frac{\frac{\partial f}{\partial\sigma}+\frac{\partial f}{\partial\beta}\frac{\partial\beta}{\partial\sigma}}{\frac{\partial f}{\partial a}}$$

So σ^* is solution of

$$\frac{\partial f}{\partial\sigma}+\frac{\partial f}{\partial\beta}\frac{\partial\beta}{\partial\sigma}=0 \;\text{ with }\; \frac{\partial f}{\partial a}\neq 0; \tag{119}$$

we compute the terms involved in the previous equation;

$$\frac{\partial f}{\partial\sigma}=2\epsilon(-\frac{\lambda a^*}{2}+\epsilon A_1)\frac{\partial A_1}{\partial\sigma}+2\left(\frac{3da^{*3}}{8\omega}-\sigma^*a^*+\epsilon a^*A_2\right)\left(-a^*+\epsilon a^*\frac{\partial A_2}{\partial\sigma}\right)+\mathcal{O}(\epsilon^2)$$

or

$$\frac{\partial f}{\partial\sigma}=-2\epsilon(\frac{\lambda a^*}{2})\frac{\partial A_1}{\partial\sigma}-2a^*(\frac{3da^{*3}}{8\omega}-\sigma^*a^*)-2\epsilon a^{*2}A_2+2\epsilon(\frac{3da^{*3}}{8\omega}-\sigma^*a^*)a^*\frac{\partial A_2}{\partial\sigma}+\mathcal{O}(\epsilon^2) \tag{120}$$

we simplify for $a=a_0^*+\mathcal{O}(\epsilon),\ \sigma=\sigma_0^*+\mathcal{O}(\epsilon)$

$$\begin{aligned}\frac{\partial f}{\partial\sigma}=&-2a_0^*\left(\frac{3da_0^{*3}}{8\omega}-\sigma_0^*a_0^*\right)-2\epsilon a_1^*\left(\frac{3da_0^{*3}}{8\omega}-\sigma_0^*a_0^*\right)-2\epsilon a_0^*\left(\frac{9da_0^{*2}a_1^*}{8\omega}-a_0^*\sigma_1^*-\sigma_0^*a_1^*\right)\\&\epsilon\lambda a_0^*\frac{\partial A_{1,0}^*}{\partial\sigma}-2\epsilon\left(\frac{3da_0^{*3}}{8\omega}-\sigma_0^*a_0^*\right)a_0\frac{\partial A_{2,0}^*}{\partial\sigma}-2\epsilon a_0^{*2}A_{2,0}^*+{\prime}\epsilon^2)\end{aligned}$$

We use (47) and the lower order term cancels;

$$\begin{aligned}\frac{\partial f}{\partial\sigma}&=-2\epsilon a_0^*\left(\frac{9da_0^{*2}a_1^*}{8\omega}-a_0^*\sigma_1^*-\sigma_0^*a_1^*\right)+\frac{\epsilon\lambda^2a_0^{*2}}{4\omega}-2\epsilon a_0^{*2}A_{2,0}^*+\mathcal{O}(\epsilon^2)\\&=-2\epsilon a_0^*\left(3\sigma_0^*a_1-a_0^*\sigma_1^*-\sigma_0^*a_1^*\right)+\frac{\epsilon\lambda^2a_0^{*2}}{4\omega}-2\epsilon a_0^{*2}A_{2,0}^*+\mathcal{O}(\epsilon^2)\\&=\epsilon a_0^*\Big[-2\left(2\sigma_0^*a_1-a_0^*\sigma_1^*\right)+\frac{\lambda^2a_0}{4\omega}-2a_0A_{2,0}^*\Big]+\mathcal{O}(\epsilon^2)\\&=\epsilon a_0^*\Big[2a_0^*\sigma_1^*-4\sigma_0^*a_1+\frac{\lambda^2a_0}{4\omega}-2a_0A_{2,0}^*\Big]+\mathcal{O}(\epsilon^2).\end{aligned}$$

We compute the derivative with respect to β;

$$\frac{\partial f}{\partial\beta}=2\epsilon\Big[(-\frac{\lambda a}{2}+\epsilon A_1)\frac{\partial A_1}{\partial\beta}-\left(\frac{3da^3}{8\omega}-\sigma a-\epsilon aA_2\right)a\frac{\partial A_2}{\partial\beta}\Big]+\mathcal{O}(\epsilon^2) \tag{121}$$

and for $a_0^*,\ \beta_0^*$

$$\frac{\partial f}{\partial \beta} = -\epsilon\lambda a_0^* \frac{\partial A_1^*}{\partial \beta} + \mathcal{O}(\epsilon^2) \tag{122}$$

the partial derivatives of $A_1,\ A_2$ are computed at $a = a_0^*,\ \beta = \beta_0^*$, we get:

$$\frac{\partial f}{\partial \beta} = -\epsilon \frac{\lambda^2 a_0^* F_m}{8\omega^2} = -\epsilon \frac{\lambda^3 a_0^{*2}}{8\omega} \tag{123}$$

and

$$\frac{\partial f}{\partial \beta}\frac{\partial \beta}{\partial \sigma} = \epsilon \frac{\lambda^2 a_0^{*2}}{4\omega}. \tag{124}$$

We use (120), (124) in (119); this last equation defines implicitly σ^* as a function of ϵ; we use the expansions (48), and we get

$$\begin{aligned}
\frac{\partial f}{\partial \sigma} + \frac{\partial f}{\partial \beta}\frac{\partial \beta}{\partial \sigma} &= \epsilon a_0^* \Big[2a_0^*\sigma_1^* - 4\sigma_0^* a_1 + \frac{\lambda^2 a_0}{4\omega} - 2a_0 A_{2,0}^*\Big] + \epsilon \frac{\lambda^2 a_0^{*2}}{4\omega} + \mathcal{O}(\epsilon^2) \\
&= \epsilon a_0^* \Big[2a_0^*\sigma_1^* - 4\sigma_0^* a_1 + \frac{\lambda^2 a_0}{2\omega} - 2a_0 A_{2,0}^*\Big] + \mathcal{O}(\epsilon^2) \\
&= 2\epsilon a_0^* \Big[a_0^*\sigma_1^* - 2\sigma_0^* a_1 + \frac{\lambda^2 a_0}{4\omega} - a_0 A_{2,0}^*\Big] + \mathcal{O}(\epsilon^2) \\
&= 2\epsilon a_0^{*2} \Big[\sigma_1^* - 2\frac{\sigma_0^* a_1}{a_0} + \frac{\lambda^2}{4\omega} - A_{2,0}^*\Big] + \mathcal{O}(\epsilon^2).
\end{aligned}$$

So we obtain

$$\begin{aligned}
\sigma_1^* &= 2\frac{\sigma_0^* a_1}{a_0} - \frac{\lambda^2}{4\omega} + A_{2,0}^* && (125) \\
&= 2\frac{\sigma_0^* a_1}{a_0} - \frac{\lambda^2}{4\omega} - \frac{5\sigma_0^2}{12\omega} - \frac{5c^2 a_0^{*2}}{12\omega^3} && (126)
\end{aligned}$$

and we obtain with (53):

$$\begin{aligned}
\sigma_1^* &= 2\frac{\sigma_0^*}{a_0}\left(\frac{-a_0^*\sigma_0^*}{\omega}\right) - \frac{\lambda^2}{4\omega} - \frac{5\sigma_0^{;*2}}{12\omega} - \frac{5c^2 a_0^{*2}}{12\omega^3} && (127) \\
&= -\frac{29\sigma_0^2}{12\omega} - \frac{\lambda^2}{4\omega} - \frac{5c^2 a_0^{*2}}{12\omega^3} && (128)
\end{aligned}$$

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