Perfect simulation for the Feynman-Kac law on the path space.

Christophe Andrieu, Nicolas Chopin, Arnaud Doucet, Sylvain Rubenthaler

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Christophe Andrieu*, Nicolas Chopin †Arnaud Doucet‡, Sylvain Rubenthaler§

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Abstract

This paper describes an algorithm of interest. This is a preliminary version and we intend on writing a better descripition of it and getting bounds for its complexity.

1 Introduction

We are given a transition kernel M (on a space E), M_1 a probability measure on E and potentials $(G_k)_{k\geq 1}$ ($G_k : E \to \mathbb{R}_+$). All densities and kernels are supposed to have a density with respect to some reference measure μ on E. We want to draw samples according to the law (on paths of length P)

$$\pi(f) = \frac{\mathbb{E}(f(X_1,\dots,X_P)\prod_{i=1}^{P-1} G_i(X_i)}{\mathbb{E}(\prod_{i=1}^{P-1} G_i(X_i)}$$

where (X_k) is Markov with initial law M_1 and transition M. For all $n \in \mathbb{N}^*$, we note $[n] = \{1,\dots,n\}$.

2 Densities of branching processes

2.1 Description of a branching system

We start with n_1 particles (i.i.d. with law M_1, n_1 is a fixed number). We then proceed recursively through time. If we have N_i particules at time k, the system evolves in the following manner:

- The number of childern of X_k^i (the i-th particle at time k) is a random variable A_{k+1}^i with law f_{k+1} such that : $\mathbb{P}(A_{k+1}^i = j) = f_{k+1}(G_k(X_k^i), j)$ (here, f_k is a law with a parameter $G_k(X_k^i)$, we will define this law later). The variables A_{k+1}^i $(1 \leq i \leq N_k)$ are independent. We then have $N_{k+1} = \sum_{i=1}^{N_k} A_{k+1}^i$
- If $N_{k+1} \neq 0$, we draw σ_{k+1} uniformly in $\mathcal{S}_{N_{k+1}}$ (the N_{k+1}-th symmetric group). If $N_{k+1} = 0$, we use the convention $\mathcal{S}_{N_{k+1}} = \emptyset$, we do not draw σ_{k+1} and the system stops there.
- We set $\forall i \in [N_k]$, $C_{k+1}^i = \{A_{k+1}^1 + \dots + A_{k+1}^{i-1}, \dots, A_{k+1}^1 + \dots + A_{k+1}^{i-1} + A_{k+1}^i\}$. If $j \in \sigma_{k+1}(C_{k+1}^i)$, we draw $X_{k+1}^j \sim M(X_k^i, .)$. We say that i at time k is the father of j at time $k+1$. We will denote this relation by the symbol $(i,k) \rightsquigarrow (j,k+1)$.

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Definition 2.1. If we have integers $k < k'$ and a sequence $(i_k, i_{k+1}, \dots, i_{k'})$ such that $(i_k, k) \rightsquigarrow (i_{k+1}, k+1) \rightsquigarrow \cdots \rightsquigarrow (i_{k'}, k')$, we will say that the particle i_k at time k is ancestor of $i_{k'}$ at time k' and we will write $(i_k, k) \rightsquigarrow (i_{k'}, k')$. We define $P' = \sup\{1 \le k \le P, N_k \neq 0\}$ (it is a random variable).

Such a system has a density on the space

$$\{(n_2, \dots, n_p, x_k^i, a_k^i, s_k) : n_2, \dots, n_P \in \mathbb{N}, \\ x_k^i \in E(1 \le k \le P, 1 \le i \le n_k), a_k^i \in \mathbb{N}(2 \le k \le P, 1 \le i \le n_k), s_k \in \mathcal{S}_{N_k}(2 \le k \le P)\}\,.$$

This density is equal to :

$$q_0(n_2, \dots, n_P, (a_k^i)_{2\le k\le p', 1\le i\le N_k}, (x_k^i)_{1\le k\le p', 1\le i\le n_k}, (s_n)_{2\le n\le p'}) \\ = \prod_{i=1}^{N_1} M_1(x_1^i) \prod_{k=2}^{P} \prod_{i=1}^{N_{k-1}} f_k(G_{k-1}(x_{k-1}^i), a_k^i) \frac{1}{n_k!} \prod_{j\in s_k(B_k^i)} M(x_{k-1}^i, x_k^j)\,,$$

with the convention $\prod_{i\in\emptyset} \cdots = 1$. In all this article, we use the letters X, A, B, N, P', σ for random variables (respectively in E, $\mathbb{N}$, $\mathbb{N}$, $\mathbb{N}$, $\mathbb{N}$, some permutation set) and we will use the letters x, a, b, n, p', s when describing densities for these variables.

Remark 2.2. The random permutations σ_k ease the writing of the formulas but have no deep signification. Their technical purpose is to ensure the densities q, $\widehat{\pi}$ defined below are mutually absolutly continuous on some set of non-zero measure (for q and $\widehat{\pi}$).

2.2 Proposal density

We take the above branching system and we draw a path by drawing a number i uniformly in $\{1, \dots, N_P\}$. We take a random number $B_{P'}$ uniformly in $[N_{P'}]$. We then look at the ancestral path of $X_{N_{P'}}^{B_{P'}}$, meaning we build recursively backwards $(B_k)_{1\le k\le P'}$ by taking for all k in $\{1, \dots, P'\}$, B_k such that $B_{k+1} \in \sigma_{k+1}(C_{k+1}^{B_k})$. We obtain a branching system containing one special trajectory $(X_1^{B_1}, \dots, X_{P'}^{B_{P'}})$. This random variable lives in the following space (with $p' = \sup\{1 \le k \le P : N_k \neq 0\}$).

$$\{(n_2, \dots, n_P, x_k^i, a_k^i, s_k, b_i) : n_2, \dots, n_P \in \mathbb{N}, \\ x_k^i \in E(1 \le k \le p', 1 \le i \le n_k), a_k^i \in \mathbb{N}(2 \le k \le p', 1 \le i \le n_i), \\ s_k \in \mathcal{S}_{n_k}(2 \le k \le p'), b_i \in [n_i](1 \le i \le p')\}\,, \quad (2.1)$$

and have a density q satisfying:

$$q(n_2, \dots, n_P, (a_k^i)_{2\le k\le P, 1\le i\le n_k}, (x_k^i)_{1\le k\le P, 1\le i\le n_k}, (s_k)_{2\le k\le P}, (b_k)_{1\le k\le P}) \times 1_{p'=P} \\ = \frac{1}{n_P} q_0(n_2, \dots, n_P, (a_k^i)_{2\le k\le P, 1\le i\le n_k}, (x_k^i)_{1\le k\le P, 1\le i\le n_k}, (s_k)_{2\le k\le P}) \times 1_{p'=P}\,.$$

What happens precisely outside the set $\{p' = P\}$ is not useful to us. We define ancestry relationships in this system as in Definition 2.1.

2.3 Target law

We draw a trajectory $(Y_1, \dots, Y_P)$ with the law π then a branching system conditioned on containing the trajectory $(Y_1, \dots, Y_P)$. The order of operations is as followed

- Draw $(Y_1, \dots, Y_P)$ with law $\pi(.)$.

- We draw B_1 uniformly in $[N_1]$, we set $X_1^{B_1} = Y_1$. We draw $(X_1^i)_{1\leq i\leq n_1, i\neq b_1}$ i.i.d. variables of law M_1.
- If we have the $(k-1)$-th generation, we draw $A_k^{B_{k-1}}$ with law $f_k(G_{k-1}(X_{k-1}^{B_{k-1}}),.)$ conditioned to be in $\mathbb{N}^*$ (we call this law $\widehat{f}(G_{k-1}(X_{k-1}^{B_{k-1}}),.)$). For $i \in N_{k-1}$, $i \neq B_{k-1}$, we draw $A_k^i \sim f_k(G_{n-1}(x_{k-1}^{B_{k-1}}),.)$. We set $N_k = \sum_{i=1}^{N_{k-1}} A_k^i$. We draw σ_k uniformly in $\mathcal{S}_{N_k}$. We set $B_k = \sigma_k(A_k^1 + \cdots + A_k^{b_{k-1}-1} + 1)$, $X_k^{B_k} = Y_k$. For $j \in [N_k]$, if $j \neq B_k$ and $j \in \sigma_k(C_k^i)$ ($C_k^i = \{A_k^1 + \cdots + A_k^{i-1} + 1, \ldots, A_k^1 + \cdots + A_k^i\}$), we draw $X_k^j \sim M(X_{k-1}^i,.)$.

We get a variable in the following space

$$\{(n_2,\ldots,n_P,x_k^i,a_k^i,\sigma_k,b_i) : n_2,\ldots,n_P \in \mathbb{N}^*, x_k^i \in E(1\leq k\leq P, 1\leq i\leq n_k),\\ A_k^i \in \mathbb{N}(1\leq k\leq P, 1\leq i\leq n_k), s_k \in \mathcal{S}_{N_n}(2\leq n\leq P), b_i \in [N_i](1\leq i\leq n)\},$$

with the following density:

$$\widehat{\pi}(n_2,\ldots,n_P,(a_k^i)_{2\leq k\leq P,1\leq i\leq N_k},(x_k^i)_{1\leq k\leq P,1\leq i\leq N_k},(s_k)_{2\leq k\leq P},(b_k)_{1\leq k\leq P}))\\ = \pi(x_1^{b_1},\ldots,x_P^{b_P})\frac{1}{N_1}\prod_{1\leq i\leq N_1, i\neq b_1} M_1(x_1^i)\\ \prod_{k=2}^{P}\left(\widehat{f}_k(G_{k-1}(x_{k-1}^{b_{k-1}}),a_k^{b_{k-1}})\prod_{1\leq i\leq N_{k-1},i\neq b_{k-1}} f_k(G_{k-1}(x_{k-1}^i),a_k^i)\right.\\ \left.\times\frac{1}{n_k!}\prod_{1\leq i\leq N_{k-1}}\prod_{j\in s_k(B_k^i),,j\neq b_k} M(x_{k-1}^i,x_k^j)\right). \quad (2.2)$$

Notice that: $(\forall z,j)\ \widehat{f}_k(g,j) = \frac{f_k(g,j)}{1-f_k(g,0)}$ ($x_{k-1}^{b_{k-1}}$ is conditioned on having at least one children). We define ancestry relationships in this system as in Definition 2.1.

2.4 Ratio of the densities

We write the ratio $\widehat{\pi}/q$ and we get:

$$\frac{\widehat{\pi}(N_2,\ldots,N_P,(A_k^i)_{1\leq k\leq P-1,1\leq i\leq N_k},(x_k^i)_{1\leq k\leq P,1\leq i\leq N_k},(s_k)_{2\leq k\leq P},(b_k)_{1\leq k\leq P}))}{q(N_2,\ldots,N_P,(A_k^i)_{1\leq k\leq P-1,1\leq i\leq N_k},(x_k^i)_{1\leq k\leq P,1\leq i\leq N_k},(s_k)_{2\leq k\leq P},(b_k)_{1\leq k\leq P}))}\\ = \pi(x_1^{b_1},\ldots,x_P^{b_P})\times\frac{N_P}{N_1}\times\frac{1}{M_1(x_1^{b_1})\prod_{k=2}^P M(x_{k-1}^{b_{k-1}},x_k^{b_k})}\times\prod_{k=2}^{P}\frac{\widehat{f}_k(G_{k-1}(x_{k-1}^{b_{k-1}}),A_k^{b_{k-1}})}{f_k(G_{k-1}(x_{k-1}^{b_{k-1}}),A_k^{b_{k-1}})}.$$

Let us take f_k such that for all g,i ($i\neq 0$), $\frac{\widehat{f}_k(g,i)}{f_k(g,i)} = \frac{\|G_{k-1}\|_\infty}{g}$. This means that $1-f_k(g,0) = \frac{g}{\|G_{k-1}\|_\infty}$. We then get:

$$\frac{\widehat{\pi}(\ldots)}{q(\ldots)} = \frac{N_P\prod_{i=2}^P\|G_{i-1}\|_\infty}{N_1 Z}, \quad (2.3)$$

with $Z = \mathbb{E}(\prod_{k=1}^{P-1}G_k(X_k))$ ($(X_k)_{k\geq 1}$ is a Markov chain with initial law M_1 and kernel transition M).

3 Perfect simulation algorithm

3.1 Stability of the branching process

We want the branching process to be stable. So we need that

$$\frac{1}{N_{k-1}} \sum_{i=1}^{N_{k-1}} \sum_{j=1}^{+\infty} j f_k(G_{k-1}(X_{k-1}^i), j) \text{ be of order 1 } (\forall \text{k}). \tag{3.1}$$

Let us take (for some q_k),

$$f_k(g,0) = 1 - \frac{\|G_k\|_\infty}{\beta_k}\,,\ f_k(g,i) = \frac{\|G_k\|_\infty}{q_k \beta_k} \text{ for } 1 \le i \le q_k\,.$$

We then get $\sum_{i=1}^{q_k} i \times f_k(g,i) = \frac{(q_k+1)g}{2\beta_k}$. So it is sensible to fix q_k such that

$$\|G_k\|_\infty = \frac{q_k+1}{2} \times \frac{1}{N} \sum_{i=1}^{N} G_{k-1}(\overline{X}_{k-1}^i) \tag{3.2}$$

where $(\overline{X}_{k-1}^i)$ is a sequential Monte-Carlo system with N particles, this has to be computed beforehand. Simulations show that this procedure indeed gives you stable branching processes.

3.2 Markovian transition

Let us now describe a Markovian transition in E^P. The integer N_1 is fixed. We start in $(z_1, \dots, z_P) \in E^P$.

1. We draw $N_2, \dots, N_P$, $(X_k^i)_{1\le k\le P, 1\le i\le N_k, i\neq B_k}$, $(A_k^i)_{1\le k\le P, 1\le i\le n_k}$, $(S_k \in \mathcal{S}_{N_k})_{2\le k\le P}$, $(B_k)_{1\le k\le P}$ with the density
$$\frac{\widehat{\pi}(\dots, z_1, \dots, z_P, \dots)}{\pi(z_1, \dots, z_P)} \tag{3.3}$$
($z_1, \dots z_P$ in place of $x_1^{b_1}, \dots, x_P^{b_P}$ in equation (2.2)). This amounts to drawing a genealogy conditionned to contain $(z_1, \dots, z_P)$. Let us set $\forall k \in \{1, \dots, P\}$, $X_k^{B_k} = z_k$. Let $\mathcal{X}$ be the variable containing all the N_k, X_k^i, A_k^i, S_k, B_k. We say that χ is a conditionnal forest.

2. We draw $\overline{N}_2, \dots, \overline{N}_P$ $(\overline{X}_k^i)_{1\le k\le P, 1\le i\le \overline{N}_k}$, $(\overline{A}_k^i)_{1\le k\le P, 1\le i\le \overline{N}_k}$, $(\overline{S}_k \in \mathcal{S}_{N_k})_{2\le k\le P}$, $(\overline{B}_k)_{1\le k\le P}$ with density $q(.)$. We denote by $\overline{\mathcal{X}}$ the corresponding variable. We say that $\overline{\chi}$ is a proposal.

3. We draw V uniform on $[0,1]$. If $V \le \alpha(\chi, \overline{\chi}) := \inf\left(1, \frac{\widehat{\pi}(\overline{\mathcal{X}})q(\mathcal{X})}{\widehat{\pi}(\mathcal{X})q(\overline{\mathcal{X}})}\right)$, we set $(\overline{Z}_1, \dots, \overline{Z}_P) = (\overline{X}_1^{B_1}, \dots, \overline{X}_P{}^{B_P})$, and if not, we set $(\overline{Z}_1, \dots, \overline{Z}_P) = (z_1, \dots, z_P)$.

Remark 3.1. In the case $\overline{N}_P = 0$, we then have $q(\overline{\chi}) = 0$, $\widehat{\pi}(\chi) \neq 0$ and so $\alpha(\chi, \overline{\chi}) = 0$.

Lemma 3.2. *The transformation of $(z_1, \dots, z_P)$ into $(\overline{Z}_1, \dots, \overline{Z}_P)$ is a Metropolis Markov kernel (on E^P) for which π is invariant (much in the spirit of [ADH10]).*

Proof. Suppose that we draw $(Z_1, \dots, Z_P) \sim \pi$ and we use it as a starting point in the transition described above. Then $\chi \sim \widehat{\pi}$. Wet set $\overline{\overline{\chi}} = \overline{\chi} 1_{V\le\alpha(\chi,\overline{\chi})} + \chi 1_{V>\alpha(\chi,\overline{\chi})}$. By Metropolis, we have that $\overline{\overline{\chi}} \sim \widehat{\pi}$. And so $(\overline{Z}_1, \dots, \overline{Z}_P) \sim \widehat{\pi}$. □

Due to (2.3), we have

$$\frac{\widehat{\pi}(\overline{\mathcal{X}})q(\mathcal{X})}{\widehat{\pi}(\mathcal{X})q(\overline{\mathcal{X}})} = \frac{\overline{N}_P}{N_P}\,. \tag{3.4}$$

3.3 Backward coupling

We know want to use a backward coupling algorithm (as in [FT98, PW96]). Any U_{-i} is sufficient to make a simulation of the Markovian transition above. We can always parametrize the Markov transition above on the following way

$$(\overline{Z}_1, \dots, \overline{Z}_P) = F_U(z_1, \dots, z_P)\,,\, U \sim \mu\,, \tag{3.5}$$

with some law μ on a big space E' and F above is a function of U and $(z_1, \dots, z_P)$ (U is written as an index because it will be more convenient later). We will write $\chi(z_1, \dots, z_P, U)$, $\overline{\chi}(U)$, $V(U)$ for, respectively, the conditionnal forest, the proposal and the uniform variable appearing in the building of $(\overline{Z}_1, \dots, \overline{Z}_P)$. We remark here that $\overline{\chi}(U)$ and $V(U)$ do not depend on $(z_1, \dots, z_P)$. We will write $N_P(z_1, \dots, z_P, U)$ for the size of the population in $\chi(z_1, \dots, z_P, U)$ at time P and $\overline{N}_P$ for the size of the population in $\overline{\chi}(U)$ at time P.

By Theorem 3.1 of [FT98], if T is a stopping time, relatively to the filtration $(\sigma(U_0, \dots, U_{-i}))_{i\geq 0}$, such that $\forall (z_1^{(1)}, \dots, z_P^{(1)}), (z_1^{(2)}, \dots, z_P^{(2)}) \in E^P$, $F_{U_0} \circ \dots \circ F_{U_{-T}}(z_1^{(1)}, \dots, z_P^{(1)}) = F_{U_0} \circ \dots \circ F_{U_{-T}}(z_1^{(2)}, \dots, z_P^{(2)})$, then $F_{U_0} \circ \dots \circ F_{-T}(z_1^{(1)}, \dots, z_P^{(1)})$ is exactly of law π. So we have the following lemma.

Lemma 3.3. *Suppose we have a bound of the form $N_P(z_1, \dots, z_P, u) \leq M(u)$, for all $(z_1, \dots, z_P)$ in $\mathcal{E}_P := \{(X_1(\omega), \dots, X_P(\omega)),\, \omega \in \Omega\}$, $\forall u \in E'$. Let*

$$T = \inf\left\{ i : V(U_{-i}) \leq \frac{\overline{N}_P(U_{-i})}{M(U_{-i})} \right\}.$$

Then, for all $(z_1, \dots, z_P) \in \mathcal{E}_P$, $F_{U_0} \circ \dots \circ F_{U_{-T}}(z_1, \dots, z_P) \sim \pi$.

Under the assumption of the above lemma, we can implement the following algorithm.

Algorithm 1 Perfect simulation algorithm

i=-1
Repeat
  i=i+1
  draw U_{-i}
  compute $R_{-i} = \overline{N}_P(U_{-i})/M(U_{-i})$
Until $(V(U_{-i}) \leq R_{-i})$
Take any $(z_1, \dots, z_P) \in \mathcal{E}_P$
Display $F_{U_0} \circ \dots \circ F_{U_{-i}}(z_1, \dots, z_P)$

4 Applications

4.1 Random directed polymers

This kind of model is described in [dH09] (chapter 12), [BTV08]. Let $(X_k)_{k\geq 1}$ be a symmetric simple random walk in $\mathbb{Z}$ with $X_1 = 0$. We are given i.i.d. variables $(\xi_{k,i})_{k\geq 1, i\in\mathbb{Z}}$ with Bernoulli law of parameter $p > 0$. These variables are called "random environment". We set (random) potentials : $V_k(i) = \exp(-\beta\xi_{k,i})$ ($\beta > 0$) and we are interested in the following quenched law (quenched means the $\xi_{k,i}$ are fixed, the $\mathbb{E}_X$ below means we are taking expectation on the X_i's only and not on the $\xi_{n,i}$'s) :

$$\pi_{1:P}(f) = \frac{\mathbb{E}_X(f(X_{1:P}) \prod_{i=1}^{P-1} V_i(X_i))}{\mathbb{E}_X(\prod_{i=1}^{P-1} V_i(X_i))}\,.$$

For a trajectory $(z_1, \dots, z_P) \in \mathcal{E}_P$ and a random variable U we draw a conditionnal forest $\chi(z_1, \dots, z_P, U)$ as in 3.2. We denote $(\forall i \in [P-1])$ by $N_P^{(i)}(z_i, U)$ the number of particles at

time P different from the particle B_P and having (B_i, i) as an ancestor. This is indeed a function independant of $z_1, \dots, z_{i-1}, z_{i+1}, \dots, z_n$. The trajectory of a simple random walk in $\mathbb{Z}$ is such that

$$M_P^{(i)}(U) := \sup\{N_P^{(i)}(z_i, U), (z_1, \dots, z_P) \in \mathcal{E}_P\} = \sup\{N_P^{(i)}(z_i, U), z_i \in \{-(i-1), -(i-1)+2, \dots, i-1\} < \infty\,.$$

We denote by $N_P^{(c)}(U)$ the number of particles at time P different from the particle B_P and not having any (B_i, i) as an ancestor. This is indeed a function independant of $(z_1, \dots, z_P)$. We have $N_P = 1 + \sum_{i=1}^{P-1} N_P^{(i)}(z_i, U) + N_P^{(c)}(U)$. And so we can bound

$$N_P \leq 1 + \sum_{i=1}^{P-1} M_P^{(i)}(U) + N_P^{(c)}(U)\,.$$

Figures 4.1 and 4.1 show two numerical experiments, in two different random environments. They illustrate in particular the weak disorder behavior explained in [dH09], chapter 12.

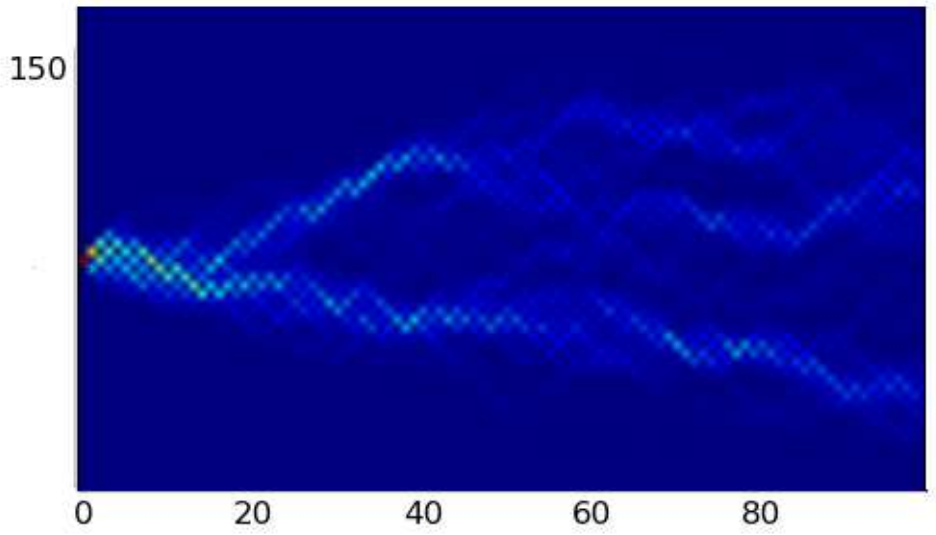

Figure 4.1: Random directed polymer, 100 trajectories.

If we take the expectation over all the variable: $\mathbb{E}(\text{max de la traj. sous } \pi_{1:n})$ behaves as n^ζ with $\zeta \neq 1/2$.Using our algorithm, we can simulate trajectories under the law $\pi_{1:P}$ (for fixed ξ, $P \in \mathbb{N}^*$). The research of the ancestors having the biggest number of descendants at time P makes that the computational cost is P^2. Here is the drawing of $\mathbb{E}(\max \dots)$ as a function of n in a log-log scale (the blue line has gradient $2/3$, the green line has gradient

4.2 Continuous state space

Hypothesis 1. *There exists a function $f : \mathbb{R} \to \mathbb{R}$ such that: for all $x_1 \in E$, $i \in \mathbb{N}$, U_{-i} fixed, $(x_1, \dots, x_P)$ trajectory drawn with transitions M using the variables U_{-i} (which we will denote by $x_{j+1} = M_{U_{-i}}(x_j)$, $\forall j \in \{2, \dots, P\}$), for all $S\epsilon > 0$, $\forall j \in \{2, \dots, P\}$, $diam(M_{U_{-i}}^{\circ(j-1)}(B_\epsilon(x_1))) \leq f^{\circ(j-1)}(\epsilon)$.*

Example 4.1. *If the transition M is (for some constants a, b) :*

$$M(x, dy) = \frac{1}{\sqrt{2\pi b^2}} \exp\left(-\frac{(y-ax)^2}{2b^2}\right)\,,$$

then we can take $f(x) = ax$.

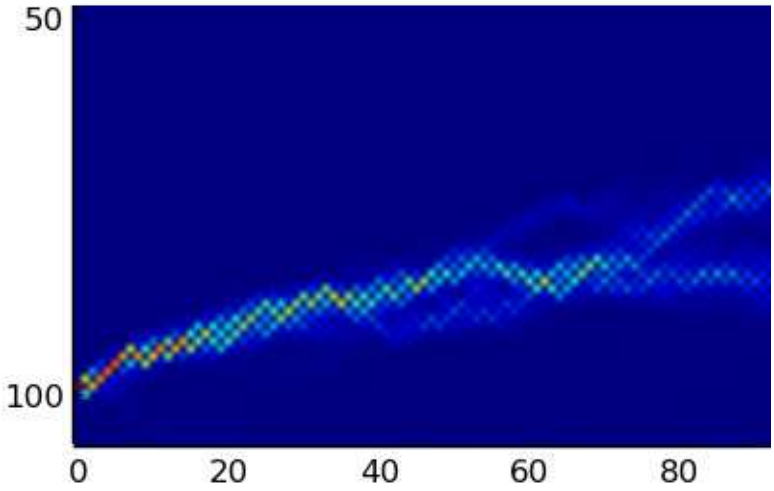

Figure 4.2: Random directed polymer, 500 trajectories

We now want to bound the number of descendants generated by the trajectory $(z_1, \dots, z_P)$ during the conditional drawing using the variables U_{-i}. Let us precise how we do this conditional drawing $(z_1, \dots, z_P)$. We fix $\forall n$, $\beta_k = \|G_k\|_\infty$ and q_k satisfying (3.2). For $g \in [0; \|G\|_\infty]$, we set $u \mapsto F^{-1}_{k,g}(u)$ to be pseudo-inverse of the cumulative distribution function of the law $f_k(g, .)$ and we set $u \mapsto \widehat{F}^{-1}_{k,g}(u)$ to be the pseudo-inverse of the cumulative distribution function of the law $\widehat{f_k}(g, .)$. We are given a family $(V_{\mathbf{u}}, W_{\mathbf{u}})_{\mathbf{u} \in (\mathbb{N}^*)^{[k]}, k \geq 1}$ (random variables indexed by infinite sequences of $\mathbb{N}^*$) of independent variables of law $\mathcal{U}([0; 1])$. We are given $(\sigma_{k,N})_{k,N \geq 1}$ independent variables such that $\forall k, N$, $\sigma_{k,N}$ is uniform in $\mathcal{S}_N$. Suppose there exists $M' : E \times [0; 1] \to E$ such that if $U \sim \mathcal{U}([0; 1])$, $x \in E$ then $M'(x, U) \sim M(x, dy)$. Suppose there exists $M'_1 : [0; 1] \to \mathbb{R}$ tsuch that if $U \sim \mathcal{U}([0; 1])$, then $M'_1(U) \sim M_1(dx)$. The simulation goes as follows.

- We set $X^i_1 = M'_1(V_{(i)})$ ((i) is a sequence of length 1 taking value i) for all $i \in [N_1] \backslash \{1\}$, and $X^1_1 = z_1$. We define $\Psi_1 : [N_1] \to (\mathbb{N}^*)^{[1]}$ by $\Psi_1(i) = (i)$.
- Suppose we have made the simulation up to time $k < P$ and we have a function $\Psi_k : [N_k] \to (\mathbb{N}^*)^{[N_k]}$ (describing the genealogy of the particles, $\Psi_k(i)$ is the complete ancestral line of particle i).
  - For $i \in [N_k] \backslash \{1\}$, we set $A^i_{k+1} = F^{-1}_{k, G_k(X^k_i)}(W_{\Psi_k(i)})$;
  - and if $i = 1$, then $X^1_k = z_k$, and we set $A^i_{k+1} = \widehat{F}^{-1}_{k+1, G_k(z_k)}(W_{\Psi_k(i)})$.

  We set $N_{k+1} = \sum_{i=1}^{N_k} A^i_{k+1}$.
  - For $j \in [N_{k+1}] \backslash \{1\}$, if $A^1_{k+1} + \dots + A^{i-1}_{k+1} < j \leq A^1_{k+1} + \dots + A^i_{k+1}$, we set $\Psi_{k+1}(j) = (\Psi_k(i), j - (A^1_{k+1} + \dots + A^{i-1}_{k+1}))$, $X^j_{k+1} = M'(X^i_k, V_{\Psi_k(j)})$,
  - and if $j = 1$, we set $X^j_{k+1} = z_{k+1}$, $\Psi_{k+1}(j) = (1, 1, \dots, 1)$.
- We then set $\overline{X}^i_1 = X^{\sigma_{1,N_1}(i)}_1$ $(1 \leq i \leq N_1)$, $B_1 = \sigma^{-1}_{1,N_1}(1)$. We then proceed by recurrence. If we have $(\overline{X}^i_j)_{1 \leq j \leq k, 1 \leq i \leq N_k}$, $(\overline{A}^i_j)_{2 \leq j \leq k, 1 \leq i \leq N_{j-1}}$, $(\sigma_j)_{2 \leq j \leq k}$, $B_1, \dots, B_n$ with $\overline{X}^i_j = X^{\sigma_{j,N_j}(i)}_j$ $(\forall j \in [k],\ i \in [N_j])$ then:

We set $\overline{A}^i_{k+1} = A^{\sigma_{k,N_k}(i)}_{k+1}$, $B^i_{k+1} = \{A^1_{k+1} + \cdots + A^{i-1}_{k+1} + 1, \ldots, A^1_{k+1} + \cdots + A^i_{k+1}\}$, $\sigma_{k+1} = \sigma^{-1}_{k+1,N_{k+1}}$, $\overline{B}^i_{k+1} = \sigma_{k+1}(B^{\sigma_{k,N_k}(i)}_{k+1})$, $\overline{X}^i_{k+1} = X^{\sigma_{k+1,N_{k+1}}(i)}_{k+1}$, $(\forall i \ldots)$. We have

- if $i \in \overline{B}^q_k = \sigma^{-1}_{k+1,N_{k+1}}(B^{\sigma_{k,N_k}(q)}_{k+1})$ and $i \neq B_{k+1} := \sigma^{-1}_{k+1,N_{k+1}}(1)$, $\sigma_{k+1,N_{k+1}}(i) \in B^{\sigma_{k,N_k}(q)}_{k+1}$, $X^{\sigma_{k+1,N_{k+1}}(i)}_{k+1} = M'(X^{\sigma_{k,N_k}(q)}_k, V_{\Psi_k(\sigma_{k,N_k}(q))})$, then $\overline{X}^i_{k+1} = M'(\overline{X}^q_k, V_{\Psi_n(\sigma_{k,N_k}(q))})$
- and in the case $i = B_{k+1}$, $\overline{X}^{B_{k+1}}_{k+1} = X^1_{k+1} = z_{k+1}$.

And we have

- if $B_{k+1} \notin \overline{B}^i_{k+1}$, then $\#\overline{B}^i_{k+1} = \#B^{\sigma_{k,N_k}(i)}_{k+1} = A^{\sigma_{k,N_k}(i)}_{k+1} = F^{-1}_{k,G_k(\overline{X}^i_k)}(W_{\Psi_k(\sigma_{k,N_k(i)})})$,
- if $B_{k+1} \in \overline{B}^i_{k+1}$,then $\#\overline{B}^i_{k+1} = \widehat{F}^{-1}_{k,G_k(\overline{X}^i_k)}(W_{\Psi_k(\sigma_{k,N_k(i)})})$.

This procedure draw $(\overline{X}^i_k, \overline{A}^i_k, B_k, \sigma_k)$ with the density (3.3) (in practice, one can get rid of the simulation of the permutations since they have no influence on the trajectories we are interested in). We will note $(X^i_k, A^i_k, B_k, \sigma_k, k \in \ldots) = \Phi((z_i)_{i\in[P]}, (V_{\mathbf{u}}, W_{\mathbf{u}})_{\mathbf{u}\in(\mathbb{N}^*)^{[k]}, k\geq 1}, (G_k)_{1\leq k\leq P})$.

Lemma 4.2. *If in the procedure above, we replace $A^i_{k+1} = \widehat{F}^{-1}_{k+1,G_k(z_k)}(W_{\Psi_k(i)})$ (in the case $\Psi_k(i) = (N_1, 1, \ldots, 1)$) by $\widetilde{A}^i_{k+1} = \widehat{F}^{-1}_{k+1,H_k(z_k)}(W_{\Psi_k(i)})$ for some function $H_k \geq G_k$, then we get a different system, which we will note with $\widetilde{\ }$,*

$$(\widetilde{X}^i_k, \widetilde{A}^i_k, \widetilde{B}_k, \widetilde{\sigma}_k, k \in \ldots) = \Phi((z_i)_{i\in[P]}, (V_{\mathbf{u}}, W_{\mathbf{u}})_{\mathbf{u}\in(\mathbb{N}^*)^{[k]}, k\geq 1}, (H_k)_{1\leq k\leq P}))\,,$$

such that $\forall k$, $\{X^i_k, 1 \leq i \leq N_k\} \subset \{\widetilde{X}^i_k, 1 \leq i \leq \widetilde{N}_k\}$. moreover, the descendants of $z_1, \ldots, z_P$ at time P are independent variables.

Let $\delta > 0$. For all $k \in [P]$, let us take $H_1 = G_1, \ldots, H_{k-1} = G_{k-1}$, and for $j \geq k$,

$$H_j(x) = \begin{cases} sup_{|y-z_j|<f^{\circ(j-k)}(\delta)} G_k(y) & \text{if } |x - z_j| \leq f^{\circ(j-k)}(\delta) \\ G_k(y) & \text{otherwise}\,, \end{cases}$$

and let us note with $\widetilde{\ }$ the corresponding system,

$$\text{meaning } (\widetilde{X}^i_k, \widetilde{A}^i_k, \widetilde{B}_k, \widetilde{\sigma}_k, k \in \ldots) = \Phi((z_i)_{i\in[P]}, (V_{\mathbf{u}}, W_{\mathbf{u}})_{\mathbf{u}\in(\mathbb{N}^*)^{[k]}, k\geq 1}, (H_k)_{1\leq k\leq P}))\,.$$

Let $z'_1, \ldots, z'_P$ be such that $z'_i \in B_\delta(z_i)$, $\forall i$. We have

$$(X^i_k, A^i_k, B_k, \sigma_k, k \in \ldots) = \Phi((z'_i)_{i\in[P]}, (V_{\mathbf{u}}, W_{\mathbf{u}})_{\mathbf{u}\in(\mathbb{N}^*)^{[k]}, k\geq 1}, (G_k)_{1\leq k\leq P})\,.$$

Using the Lemma above and Hypothesis 1, we have $N_P \leq \widetilde{N}_P$. Let Φ' be such that

$$N_P = \Phi'((z_i)_{i\in[P]}, (V_{\mathbf{u}}, W_{\mathbf{u}})_{\mathbf{u}\in(\mathbb{N}^*)^{[k]}, k\geq 1}, (H_k)_{1\leq k\leq P}))\,.$$

4.3 Examples

4.3.1 Gaussian example

We draw sequences $(X_k)_{k\in[P]}$, $(Y_k)_{k\in[P]}$ such that: $X_1 \sim \mathcal{N}(0,1)$, $X_{k+1} = aX_k + bV_{k+1} (a \in]0;1[)$, $Y_k = X_k + cW_k$ with i.i.d. variables V_k, W_k of law $\mathcal{N}(0,1)$. We set

$$G_k(x) = \frac{1}{\sqrt{2\pi c^2}} \exp\left(-\frac{1}{2c^2}(x - Y_k)^2\right),$$

$M_1(dx) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$, $M(x, dy) = \frac{1}{\sqrt{2\pi b^2}} \exp\left(-\frac{(y-ax)^2}{2b^2}\right) dy$. We want to bound. at time P, the particles descending from a fixed trajectory. The descendants of different z_k are independant so we look, for all k, at which is the z_k spawning the most descendants at time P. Using the result above, we slice E in balls of size δ. If z'_k is in a ball of size δ containing z_k, the number of descendants of z'_k at time P computed with potentials $G_.$ is bounded by the number of descendants of z_k at time P computed with potentials $H_.$. The potentials G_k going to 0 at $\pm\infty$, we do not have to explore the whole of $\mathbb{R}$, as soon as z_k is far enough from Y_k so that it has 0 children under potential H_k, we can stop the exploration.

Remark 4.1. With $\delta = 0$ (or δ very small), if we look at the number of descendants at time P of an individual at time k and we maximise in the position of the individual, we will finite some finite quantity (not exploding when $P - k \to +\infty$. For the maximisation step, we have to take $\delta > 0$ and then this maximum explodes (slowly). So, there a balance to find between δ small (maximisation step takes a lot of time) and δ big (explosion in the number of particles). A rule of thumb, coming from the experience, is that the population do not explode as long as the number of children per individual is of order $2, 3$.

4.3.2 Directed polymers

Let $(X_k)_{k\geq 1}$ be a symmetric simple random walk in $\mathbb{Z}$ with $X_1 = 0$. We are given i.i.d. variables $(\xi_{k,i})_{k\geq 1, i\in\mathbb{Z}}$ with Bernoulli law of parameter $p > 0$. We set (random) potentials : $V_k(i) = \exp(-\beta\xi_{k,i})$ ($\beta > 0$) and we are interested in the following law (quenched, meaning the $\xi_{k,i}$ are fixed) :

$$\pi_{1:k}(f) = \frac{\mathbb{E}_\xi(f(X_{1:k}) \prod_{i=1}^k V_i(X_i))}{\mathbb{E}_\xi(\prod_{i=1}^k V_i(X_i))}.$$

This kind of model is described in [BTV08]. If we take the expectation over all the variable: $\mathbb{E}(\max$ de la traj. sous $\pi_{1:n})$ behaves as n^ζ with $\zeta \neq 1/2$.

Using our algorithm, we can simulate trajectories under the law $\pi_{1:P}$ (for fixed ξ, $P \in \mathbb{N}^*$). The research of the ancestors having the biggest number of descendants at time P makes that the computational cost is P^2. Here is the drawing of $\mathbb{E}(\max \dots)$ as a function of n in a log-log scale (the blue line has gradient $2/3$, the green line has gradient $1/2$):

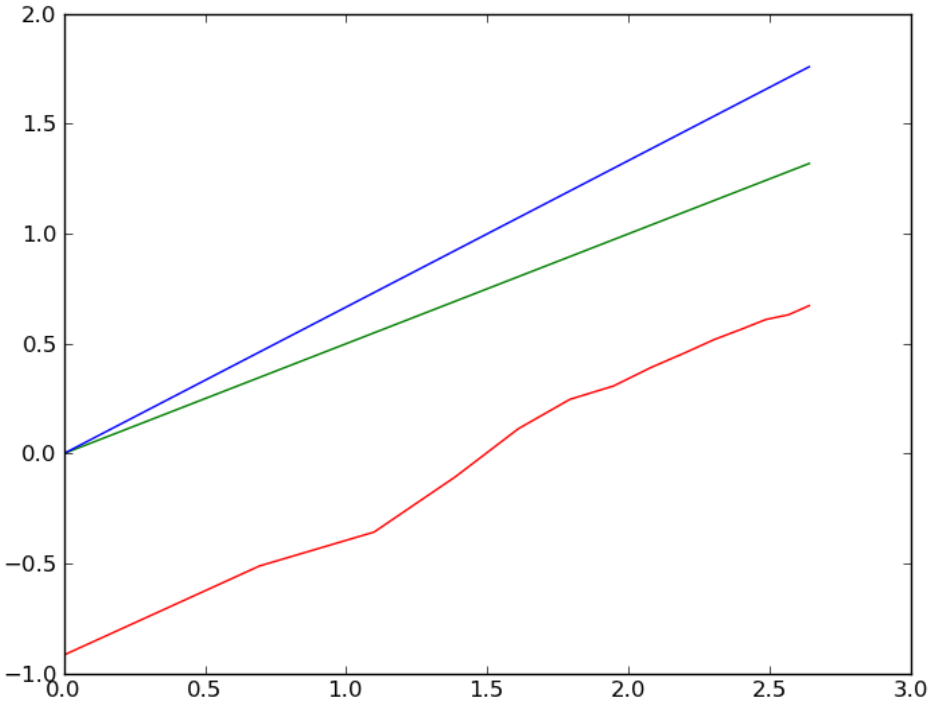

Figure 4.3: gradient estimation (least square)=0,63

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