Perfect simulation for the Feynman-Kac law on the path space.

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Christophe Andrieu[*], Nicolas Chopin [†], Arnaud Doucet[‡], Sylvain Rubenthaler[§]

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Abstract

This paper describes an algorithm of interest. This is a preliminary version and we intend on writing a better descripition of it and getting bounds for its complexity.

1 Introduction

We are given a transition kernel M (on a space E), M_1 a probability measure on E and potentials $(G_k)_{k\geq 1}$ $(G_k : E \to \mathbb{R}_+)$. We want to draw samples according to the law (on paths of length P)

$$\pi(f) = \frac{\mathbb{E}(f(X_1, \dots, X_P)\prod_{i=1}^{P-1} G_i(X_i))}{\mathbb{E}(\prod_{i=1}^{P-1} G_i(X_i))}$$

where (X_k) is Markov with initial law M_1 and transition M. For all $n \in \mathbb{N}$, we note $[n] = \{1, \dots, n\}$.

2 Densities of branching processes

2.1 Description of a branching system

We start with N_1 particles (i.i.d. with law M_1, N_1 is a fixed number). If we have N_i particules at time n, the system evolves in the following manner:

- The number of childern of X_n^i (the i-th particle at time n) is a random variable A_{n+1}^i with law f_{n+1} such that : $\mathbb{P}(A_{n+1}^i = j) = f_{n+1}(G_n(X_n^i), j)$ (here, f_n is a law with a parameter $G_n(X_n^i)$, we will define this law later). The variables A_{n+1}^i $(1 \leq i \leq N_n)$ are independent. We than have $N_{n+1} = \sum_{i=1}^{N_n} A_{n+1}^i$
- We draw σ_{n+1} uniformly in $\mathcal{S}_{N_{n+1}}$ (the N_{n+1}-th symmetric group).
- We set $\forall j \in [N_n]$, $B_{n+1}^j = \{A_{n+1}^1 + \cdots + A_{n+1}^{j-1}, \dots, A_{n+1}^1 + \cdots + A_{n+1}^{j-1} + A_{n+1}^j\}$. If $i \in \sigma_{n+1}(B_{n+1}^j)$, we draw $X_{n+1}^i \sim M(X_n^j, .)$.

Such a system has a density on the space

$$\{(n_2, \dots, n_p, x_n^i, A_n^i, \sigma_n) : n_2, \dots, n_P \in \mathbb{N},$$

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$$x_n^i \in E(1 \le n \le P, 1 \le i \le n_n), A_n^i \in \mathbb{N}(2 \le n \le P, 1 \le i \le n_n), \sigma_n \in \mathcal{S}_{N_n}(2 \le n \le P)\} \,.$$

This density is equal to :

$$\begin{aligned} & q_0(N_2, \dots, N_P, (A_n^i)_{2 \le n \le P, 1 \le i \le N_n}, (x_n^i)_{1 \le n \le P, 1 \le i \le N_n}, (\sigma_n)_{2 \le n \le P}) \\ & \qquad = \prod_{i=1}^{N_1} M_1(x_1^i) \prod_{n=2}^{P} \prod_{i=1}^{N_{n-1}} f_n(G_{n-1}(x_{n-1}^i), A_n^i) \frac{1}{N_n!} \prod_{j \in \sigma_n(B_n^i)} M(x_{n-1}^i, x_n^j) \,. \end{aligned}$$

The random permutations σ_N ease the writing of the formulas but have no deep signification.

2.2 Proposal density

We take the above branching system and we draw a path by drawing a number i uniformly in $\{1, \dots, N_P\}$ and taking the path of the ancestors of X_P^i. The branching system plus this trajectory live in the following space

$$\begin{aligned} & \{(n_2, \dots, n_P, x_n^i, A_n^i, \sigma_n, b_i) : n_2, \dots, n_P \in \mathbb{N}, \\ & \qquad x_n^i \in E(1 \le n \le P, 1 \le i \le n_n), A_n^i \in \mathbb{N}(2 \le n \le P, 1 \le i \le n_i), \\ & \qquad\qquad \sigma_n \in \mathcal{S}_{N_n}(2 \le n \le P), b_i \in [N_i](1 \le i \le n)\} \,, \end{aligned} \tag{2.1}$$

and have the following density :

$$\begin{aligned} & q(N_2, \dots, N_P, (A_n^i)_{2 \le n \le P, 1 \le i \le N_n}, (x_n^i)_{1 \le n \le P, 1 \le i \le N_n}, (\sigma_n)_{2 \le n \le P}, (b_k)_{1 \le k \le P}) \\ & \qquad = \frac{1}{N_P} q_0(N_2, \dots, N_P, (A_n^i)_{2 \le n \le P, 1 \le i \le N_n}, (x_n^i)_{1 \le n \le P, 1 \le i \le N_n}, (\sigma_n)_{2 \le n \le P}) \,. \end{aligned}$$

2.3 Target law

We draw a trajectory $(y_1, \dots, y_P)$ with the law π then a branching system conditioned on containing the trajectory $(y_1, \dots, y_P)$. The order of operations is as followed

- Draw $(y_1, \dots, y_P)$ with law $\pi(.)$.
- We draw b_1 uniformly in $[N_1]$, we set $x_1^{b_1} = y_1$. We draw $(x_1^i)_{1 \le i \le N_1, i \ne b_1}$ i.i.d. variables of law M_1.
- If we have the $(n-1)$-th generation, we draw $A_n^{b_{n-1}}$ with law $f(G_{n-1}(x_{n-1}^{b_{n-1}}), .)$ conditioned to be in $\mathbb{N}^*$ (we call this law $\widehat{f}(G_{n-1}(x_{n-1}^{b_{n-1}}), .)$). For $i \in N_{n-1}$, $i \ne b_{n-1}$, we draw $A_n^i \sim f_n(G_{n-1}(x_{n-1}^{b_{n-1}}), .)$. We set $N_n = \sum_{i=1}^{N_{n-1}} A_n^i$. Weaw σ_n uniformly in $\mathcal{S}_{N_n}$. We set $b_n = \sigma_n(A_n^1 + \dots + A_n^{b_{n-1}-1} + 1)$, $x_n^{b_n} = y_n$. For $j \in [N_n]$, if $j \ne b_n$ and $j \in \sigma_n(B_n^i)$ $(B_n^i = \{A_n^1 + \dots + A_n^{i-1} + 1, \dots, A_n^1 + \dots + A_n^i\})$, we draw $x_n^j \sim M(x_{n-1}^i, .)$.

We get a variable in the following space

$$\begin{aligned} & \{(n_2, \dots, n_P, x_n^i, A_n^i, \sigma_n, b_i) : n_2, \dots, n_P \in \mathbb{N}^*, x_n^i \in E(1 \le n \le P, 1 \le i \le n_n), \\ & \qquad A_n^i \in \mathbb{N}(1 \le n \le P, 1 \le i \le n_n), \sigma_n \in \mathcal{S}_{N_n}(2 \le n \le P), b_i \in [N_i](1 \le i \le n)\} \,, \end{aligned}$$

with the following density:

$$\begin{aligned} & \widehat{\pi}(N_2, \dots, N_P, (A_n^i)_{2 \le n \le P, 1 \le i \le N_n}, (x_n^i)_{1 \le n \le P, 1 \le i \le N_n}, (\sigma_n)_{2 \le n \le P}, (b_k)_{1 \le k \le P})) \\ & \qquad = \pi(x_1^{b_1}, \dots, x_P^{b_P}) \frac{1}{N_1} \prod_{1 \le i \le N_1, i \ne b_1} M_1(x_1^i) \end{aligned}$$

$$\prod_{n=2}^{P}\left(\widehat{f}_n(G_{n-1}(x_{n-1}^{b_{n-1}}),A_n^{b_{n-1}})\prod_{1\leq i\leq N_{n-1},i\neq b_{n-1}} f_n(G_{n-1}(x_{n-1}^i),A_n^i)\right.$$
$$\left.\times\frac{1}{N_n!}\prod_{1\leq i\leq N_{n-1}}\prod_{j\in\sigma_n(B_n^i),,j\neq b_n} M(x_{n-1}^i,x_n^j)\right). \quad (2.2)$$

Notice that: $(\forall z,k)$ $\widehat{f}_n(g,k)=\frac{f_n(g,k)}{1-f_n(g,0)}$ ($x_{n-1}^{b_{n-1}}$ is conditioned on having at least one children).

2.4 Ratio of the densities

We write the ratio $\widehat{\pi}/q$ and we get:

$$\frac{\widehat{\pi}(N_2,\dots,N_P,(A_n^i)_{1\leq n\leq P-1,1\leq i\leq N_n},(x_n^i)_{1\leq n\leq P,1\leq i\leq N_n},(\sigma_n)_{2\leq n\leq P},(b_k)_{1\leq k\leq P}))}{q(N_2,\dots,N_P,(A_n^i)_{1\leq n\leq P-1,1\leq i\leq N_n},(x_n^i)_{1\leq n\leq P,1\leq i\leq N_n},(\sigma_n)_{2\leq n\leq P},(b_k)_{1\leq k\leq P}))}$$
$$=\pi(x_1^{b_1},\dots,x_P^{b_P})\times\frac{N_P}{N_1}\times\frac{1}{M_1(x_1^{b_1})\prod_{n=2}^{P}M(x_{n-1}^{b_{n-1}},x_n^{b_n})}\times\prod_{n=2}^{P}\frac{\widehat{f}_n(G_{n-1}(x_{n-1}^{b_{n-1}}),A_n^{b_{n-1}})}{f_n(G_{n-1}(x_{n-1}^{b_{n-1}}),A_n^{b_{n-1}})}.$$

Let us take f_n such that for all g,i $(i\neq 0)$, $\frac{\widehat{f}_n(g,i)}{f_n(g,i)}=\frac{\beta_n}{g}$ for some comstant β_n. This means that $1-f_n(g,0)=\frac{g}{\beta_n}$. We then get:

$$\frac{\widehat{\pi}(\dots)}{q(\dots)}=\frac{N_P\prod_{i=2}^{P}\beta_i}{N_1 Z},$$

with $Z=\mathbb{E}(\prod_{n=1}^{P-1}G_n(X_n))$ ($(X_n)_{n\geq 1}$ is a Markov chain with initial law M_1 and kernel transition M).

3 Perfect simulation algorithm

3.1 Stability of the branching process

We want the branchin process to be stable. So we need that

$$\frac{1}{N_{n-1}}\sum_{i=1}^{N_{n-1}}\sum_{j=1}^{+\infty} jf_n(G_{n-1}(x_{n-1}^i),j)\text{ be of order 1 }(\forall\text{n}). \quad (3.1)$$

Let us take: $\beta_n\geq\|G_n\|_\infty$ $(\forall n)$, and (for some k_n), $f_n(g,0)=1-\frac{g}{\beta_n}$, $f_n(g,i)=\frac{g}{k_n\beta_n}$ pour $1\leq i\leq k_n$. We then get $\sum_{i=1}^{k_n} i\times f_n(g,i)=\frac{(k_n+1)g}{2\beta_n}$. So it is sensible to fix k_n such that

$$\beta_n=\frac{k_n+1}{2}\times\frac{1}{N}\sum_{i=1}^{N}G_{n-1}(\overline{x}_{n-1}^i) \quad (3.2)$$

where $(\overline{x}_{n-1}^i)$ is a sequential Monte-Carlo system with N particles, this has to be computed beforehand. Simulations show that this procedure indeed gives you stable branching processes.

3.2 Markovian transition

We know want to use a backward coupling algorithm (as in [FT98, PW96]). The integer N_1 is fixed. We take $(z_1,\dots,z_P)\in E^P$.

- We draw $N_2,\dots,N_P$, $(X_n^i)_{1\le n\le P,1\le i\le N_n,i\neq B_n}$, $(A_n^i)_{1\le n\le P,1\le i\le n_n}$, $(S_n\in\mathcal{S}_{N_n})_{2\le n\le P}$, $(B_k)_{1\le k\le P}$ with the density
$$\frac{\widehat{\pi}(\dots,z_1,\dots,z_P,\dots)}{\pi(z_1,\dots,z_P)} \tag{3.3}$$
($z_1,\dots z_P$ in place of $x_1^{b_1},\dots,x_P^{b_P}$ in equation (2.2)). This amounts to drawing a genealogy conditionned to contain $(z_1,\dots,z_P)$. Let us set $\forall n\in\{1,\dots,P\}$, $X_n^{B_n}=z_n$. Let $\mathcal{X}$ be the variable containing all the N_n, X_n^i, A_n^i, S_n, B_n.

- We draw $\overline{N}_2,\dots,\overline{N}_P$ $(\overline{X}_n^i)_{1\le n\le P,1\le i\le\overline{N}_n}$, $(\overline{A}_n^i)_{1\le n\le P,1\le i\le\overline{N}_n}$, $(\overline{S}_n\in\mathcal{S}_{N_n})_{2\le n\le P}$, $(\overline{B}_k)_{1\le k\le P}$ with density $q(.)$. We denote by $\overline{\mathcal{X}}$ the corresponding variable.

- With probability $\inf\left(1,\frac{\widehat{\pi}(\overline{\mathcal{X}})q(\mathcal{X})}{\widehat{\pi}(\mathcal{X})q(\overline{\mathcal{X}})}\right)$, we set $(\overline{Z}_1,\dots,\overline{Z}_P)=(\overline{X}_1^{B_1},\dots,\overline{X}_P{}^{B_P})$, and with the complementary probability, we set $(\overline{Z}_1,\dots,\overline{Z}_P)=(z_1,\dots,z_P)$.

The transformation of $(z_1,\dots,z_P)$ into $(\overline{Z}_1,\dots,\overline{Z}_P)$ is a Metropolis Markov kernel (on E^P) for which π is invariant (much in the spirit of [ADH10]). Recall that
$$\frac{\widehat{\pi}(\overline{\mathcal{X}})q(\mathcal{X})}{\widehat{\pi}(\mathcal{X})q(\overline{\mathcal{X}})}=\frac{\overline{N}_P}{N_P}\,. \tag{3.4}$$

3.3 Backward coupling

We are given i.i.d. variables $(U_0,U_{-1},U_{-2},\dots)$. Any U_{-i} is sufficient to make a simulation of the Markovian transition above. We introduce a function F parametrizing this transition (we can write the transition in the following manner: $(\overline{Z}_1,\dots,\overline{Z}_P)=F_U(z_1,\dots,z_P)$). By Theorem 3.1 of [FT98], if T is a stopping time, relatively to the filtration $(\sigma(U_0,\dots,U_{-i}))_{i\ge0}$, such that $\forall(z_1^{(1)},\dots,z_P^{(1)}),(z_1^{(2)},\dots,z_P^{(2)})\in E^P$, $F_{U_{-T}}\circ\dots\circ F_{U_0}(z_1^{(1)},\dots,z_P^{(1)})=F_{U_{-T}}\circ\dots\circ F_{U_0}(z_1^{(2)},\dots,z_P^{(2)})$, then $F_{U_{-T}}\circ\dots\circ F_{U_0}(z_1^{(1)},\dots,z_P^{(1)})$ is exactly of law π.

We now look for a lower bound of (3.4) for a trajectory $(z_1,\dots,z_P)\in E^P$ and $i\in\mathbb{N}$, U_{-i} fixed. We add the following hypothesis.

Hypothesis 1. There exists a function $f:\mathbb{R}\to\mathbb{R}$ such that: for all $x_1\in E$, $i\in\mathbb{N}$, U_{-i} fixed, $(x_1,\dots,x_P)$ trajectory drawn with transitions M using the variables U_{-i} (which we will denote by $x_{j+1}=M_{U_{-i}}(x_j)$, $\forall j\in\{2,\dots,P\}$), for all $S\epsilon>0$, $\forall j\in\{2,\dots,P\}$, $\operatorname{diam}(M_{U_{-i}}^{\circ(j-1)}(B_\epsilon(x_1)))\le f^{\circ(j-1)}(\epsilon)$.

Example 3.1. If the transition M is (for some constants a,b) :
$$M(x,dy)=\frac{1}{\sqrt{2\pi b^2}}\exp\left(-\frac{(y-ax)^2}{2b^2}\right)\,,$$
then we can take $f(x)=ax$.

We now want to bound the number of descendants generated by the trajectory $(z_1,\dots,z_P)$ during the conditional drawing using the variables U_{-i}. Let us precise how we do this conditional drawing $(z_1,\dots,z_P)$. We fix $\forall n$, $\beta_n=\|G_n\|_\infty$ and k_n satisfying (3.2). For $g\in[0;\|G\|_\infty]$, we set $u\mapsto F_{n,g}^{-1}(u)$ to be pseudo-inverse of the cumulative distribution function of the law $f_n(g,.)$ and we set $u\mapsto\widehat{F}_{n,g}^{-1}(u)$ to be the pseudo-inverse of the cumulative distribution function of the law $\widehat{f}_n(g,.)$. We are given a family $(V_{\mathbf{u}},W_{\mathbf{u}})_{\mathbf{u}\in(\mathbb{N}^*)^{[n]},n\ge1}$ (random variables indexed by infinite sequences of $\mathbb{N}^*$) of independent variables of law $\mathcal{U}([0;1])$. We are given $(\sigma_{n,N})_{n,N\ge1}$ independent variables such that $\forall n,N$, $\sigma_{n,N}$ is uniform in $\mathcal{S}_N$. Suppose there exists $M':E\times[0;1]\to E$ such that if $U\sim\mathcal{U}([0;1])$, $x\in E$ then $M'(x,U)\sim M(x,dy)$. Suppose there exists $M_1':[0;1]\to\mathbb{R}$ tsuch that if $U\sim\mathcal{U}([0;1])$, then $M_1'(U)\sim M_1(dx)$. The simulation goes as follows.

- We set $X_1^i=M_1'(V_{(i)})$ ((i) is a sequence of length 1 taking value i) for all $i\in[N_1]\backslash\{1\}$, and $X_1^1=z_1$. We define $\Psi_1:[N_1]\to(\mathbb{N}^*)^{[1]}$ by $\Psi_1(i)=(i)$.

- Suppose we have made the simulation up to time $n < P$ and we have a function $\Psi_n : [N_n] \rightarrow (\mathbb{N}^*)^{[N_n]}$ (describing the genealogy of the particles, $\Psi_n(i)$ is the complete ancestral line of particle i).
  - For $i \in [N_n]\backslash\{1\}$, we set $A^i_{n+1} = F^{-1}_{n,G_n(X^n_i)}(W_{\Psi_n(i)})$;
  - and if $i = 1$, then $X^1_n = z_n$,

  and we set $A^i_{n+1} = \widehat{F}^{-1}_{n+1,G_n(z_n)}(W_{\Psi_n(i)})$. We set $N_{n+1} = \sum_{i=1}^{N_n} A^i_{n+1}$.
  - For $j \in [N_{n+1}]\backslash\{1\}$, if $A^1_{n+1} + \dots + A^{i-1}_{n+1} < j \leq A^1_{n+1} + \dots + A^i_{n+1}$, we set $\Psi_{n+1}(j) = (\Psi_n(i), j - (A^1_{n+1} + \dots + A^{i-1}_{n+1}))$, $X^j_{n+1} = M'(X^i_n, V_{\Psi_n(j)})$,
  - and if $j = 1$, we set $X^j_{n+1} = z_{n+1}$, $\Psi_{n+1}(j) = (1, 1, \dots, 1)$.

- We then set $\overline{X}^i_1 = X_1^{\sigma_{1,N_1}(i)}$ $(1 \leq i \leq N_1)$, $B_1 = \sigma^{-1}_{1,N_1}(1)$ (beware, B_i and B^i_j have different meanings). We then proceed by recurrence. If we have $(\overline{X}^i_j)_{1\leq j\leq n, 1\leq i\leq N_n}$, $(\overline{A}^i_j)_{2\leq j\leq n, 1\leq i\leq N_{j-1}}$, $(\sigma_j)_{2\leq j\leq n}$, $B_1, \dots, B_n$ with $\overline{X}^i_j = X_j^{\sigma_{j,N_j}(i)}$ $(\forall j \in [n],\, i \in [N_j])$ then:
  We set $\overline{A}^i_{n+1} = A^{\sigma_{n,N_n}(i)}_{n+1}$, $B^i_{n+1} = \{A^1_{n+1} + \dots + A^{i-1}_{n+1} + 1, \dots, A^1_{n+1} + \dots + A^i_{n+1}\}$, $\sigma_{n+1} = \sigma^{-1}_{n+1,N_{n+1}}$, $\overline{B}^i_{n+1} = \sigma_{n+1}(B^{\sigma_{n,N_n}(i)}_{n+1})$, $\overline{X}^i_{n+1} = X^{\sigma_{n+1,N_{n+1}}(i)}_{n+1}$, $(\forall i \dots)$. We have
  - if $i \in \overline{B}^q_n = \sigma^{-1}_{n+1,N_{n+1}}(B^{\sigma_{n,N_n}(q)}_{n+1})$ and $i \neq B_{n+1} := \sigma^{-1}_{n+1,N_{n+1}}(1)$, $\sigma_{n+1,N_{n+1}}(i) \in B^{\sigma_{n,N_n}(q)}_{n+1}$, $X^{\sigma_{n+1,N_{n+1}}(i)}_{n+1} = M'(X^{\sigma_{n,N_n}(q)}_n, V_{\Psi_n(\sigma_{n,N_n}(q))})$, then $\overline{X}^i_{n+1} = M'(\overline{X}^q_n, V_{\Psi_n(\sigma_{n,N_n}(q))})$
  - and in the case $i = B_{n+1}$, $\overline{X}^{B_{n+1}}_{n+1} = X^1_{n+1} = z_{n+1}$.

  And we have
  - if $B_{n+1} \notin \overline{B}^i_{n+1}$, then $\#\overline{B}^i_{n+1} = \#B^{\sigma_{n,N_n}(i)}_{n+1} = A^{\sigma_{n,N_n}(i)}_{n+1} = F^{-1}_{n,G_n(\overline{X}^i_n)}(W_{\Psi_n(\sigma_{n,N_n}(i))})$,
  - if $B_{n+1} \in \overline{B}^i_{n+1}$,then $\#\overline{B}^i_{n+1} = \widehat{F}^{-1}_{n,G_n(\overline{X}^i_n)}(W_{\Psi_n(\sigma_{n,N_n}(i))})$.

This procedure draw $(\overline{X}^i_n, \overline{A}^i_n, B_n, \sigma_n)$ with the density (3.3) (in pratice, one can get rid of the simulation of the permutations since they have no influence on the trajectories we are interested in). We will note $(X^i_n, A^i_n, B_n, \sigma_n, n \in \dots) = \Phi((z_i)_{i\in[P]}, (V_\mathbf{u}, W_\mathbf{u})_{\mathbf{u}\in(\mathbb{N}^*)^{[n]}, n\geq 1}, (G_n)_{1\leq n\leq P})$.

Lemma 3.2. If in the procedure above, we replace $A^i_{n+1} = \widehat{F}^{-1}_{n+1,G_n(z_n)}(W_{\Psi_n(i)})$ (in the case $\Psi_n(i) = (N_1, 1, \dots, 1)$) by $\widetilde{A}^i_{n+1} = \widehat{F}^{-1}_{n+1,H_n(z_n)}(W_{\Psi_n(i)})$ for some function $H_n \geq G_n$, then we get a different system, which we will note with $\widetilde{\ }$,

$$(\widetilde{X}^i_n, \widetilde{A}^i_n, \widetilde{B}_n, \widetilde{\sigma}_n, n \in \dots) = \Phi((z_i)_{i\in[P]}, (V_\mathbf{u}, W_\mathbf{u})_{\mathbf{u}\in(\mathbb{N}^*)^{[n]}, n\geq 1}, (H_n)_{1\leq n\leq P}),$$

such that $\forall n$, $\{X^i_n, 1 \leq n \leq N_n\} \subset \{\widetilde{X}^i_n, 1 \leq n \leq \widetilde{N}_n\}$. moreover, the descendants of $z_1, \dots, z_P$ at time P are independent variables.

Let $\delta > 0$. For all $n \in [P]$, let us take $H_1 = G_1, \dots, H_{n-1} = G_{n-1}$, and for $k \geq n$,

$$H_k(x) = \begin{cases} sup_{|y-z_k|<f^{\circ(k-n)}(\delta)} G_n(y) & \text{if } |x - z_k| \leq f^{\circ(k-n)}(\delta) \\ G_n(y) & \text{otherwise }, \end{cases}$$

and let us note with $\widetilde{\ }$ the corresponding system,

$$\text{meaning } (\widetilde{X}^i_n, \widetilde{A}^i_n, \widetilde{B}_n, \widetilde{\sigma}_n, n \in \dots) = \Phi((z_i)_{i\in[P]}, (V_\mathbf{u}, W_\mathbf{u})_{\mathbf{u}\in(\mathbb{N}^*)^{[n]}, n\geq 1}, (H_n)_{1\leq n\leq P}).$$

Let $z'_1, \dots, z'_P$ be such that $z'_i \in B_\delta(z_i)$, $\forall i$. We have

$$(X^i_n, A^i_n, B_n, \sigma_n, n \in \dots) = \Phi((z'_i)_{i\in[P]}, (V_{\mathbf{u}}, W_{\mathbf{u}})_{\mathbf{u}\in(\mathbb{N}^*)^{[n]}, n\geq 1}, (G_n)_{1\leq n\leq P}) .$$

Using the Lemma above and Hypothesis 1, we have $N_P \leq \widetilde{N}_P$. Let Φ' be such that

$$N_P = \Phi'((z_i)_{i\in[P]}, (V_{\mathbf{u}}, W_{\mathbf{u}})_{\mathbf{u}\in(\mathbb{N}^*)^{[n]}, n\geq 1}, (H_n)_{1\leq n\leq P})) .$$

3.4 Examples

3.4.1 Gaussian example

We draw sequences $(X_n)_{n\in[P]}$, $(Y_n)_{n\in[P]}$ such that: $X_1 \sim \mathcal{N}(0,1)$, $X_{n+1} = aX_n + bV_{n+1} (a \in]0;1[)$, $Y_n = X_n + cW_n$ with i.i.d. variables V_n, W_n of law $\mathcal{N}(0,1)$. We set

$$G_n(x) = \frac{1}{\sqrt{2\pi c^2}} \exp\left(-\frac{1}{2c^2}(x - Y_n)^2\right) ,$$

$M_1(dx) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$, $M(x, dy) = \frac{1}{\sqrt{2\pi b^2}} \exp\left(-\frac{(y-ax)^2}{2b^2}\right) dy$. We want to bound. at time P, the particles descending from a fixed trajectory. The descendants of different z_n are independant so we look, for all n, at which is the z_n spawning the most descendants at time P. Using the result above, we slice E in balls of size δ. If z'_n is in a ball of size δ containing z_n, the number of descendants of z'_n at time P computed with potentials $G_.$ is bounded by the number of descendants of z_n at time P computed with potentials $H_.$. The potentials G_n going to 0 at $\pm\infty$, we do not have to explore the whole of $\mathbb{R}$, as soon as z_n is far enough from Y_n so that it has 0 children under potential H_n, we can stop the exploration.

Remark 3.1. With $\delta = 0$ (or δ very small), if we look at the number of descendants at time P of an individual at time n and we maximise in the position of the individual, we will finite some finite quantity (not exploding when $P - n \to +\infty$. For the maximisation step, we have to take $\delta > 0$ and then this maximum explodes (slowly). So, there a balance to find between δ small (maximisation step takes a lot of time) and δ big (explosion in the number of particles). A rule of thumb, coming from the experience, is that the population do not explode as long as the number of children per individual is of order $2, 3$.

3.4.2 Directed polymers

Let $(X_n)_{n\geq 1}$ be a symmetric simple random walk in $\mathbb{Z}$ with $X_1 = 0$. We are given i.i.d. variables $(\xi_{n,i})_{n\geq 1, i\in\mathbb{Z}}$ with Bernoulli law of parameter $p > 0$. We set (random) potentials : $V_n(i) = \exp(-\beta\xi_{n,i})$ $(\beta > 0)$ and we are interested in the following law (quenched, meaning the $\xi_{n,i}$ are fixed) :

$$\pi_{1:n}(f) = \frac{\mathbb{E}_\xi(f(X_{1:n}) \prod_{k=1}^n V_k(X_k))}{\mathbb{E}_\xi(\prod_{k=1}^n V_k(X_k))} .$$

This kind of model is described in [BTV08]. If we take the expectation over all the variable: $\mathbb{E}(\text{max de la traj. sous } \pi_{1:n})$ behaves as n^ζ with $\zeta \neq 1/2$.

Using our algorithm, we can simulate trajectories under the law $\pi_{1:P}$ (for fixed ξ, $P \in \mathbb{N}^*$). The research of the ancestors having the biggest number of descendants at time P makes that the computational cost is P^2. Here is the drawing of $\mathbb{E}(\max \dots)$ as a function of n in a log-log scale (the blue line has gradient $2/3$, the green line has gradient $1/2$):

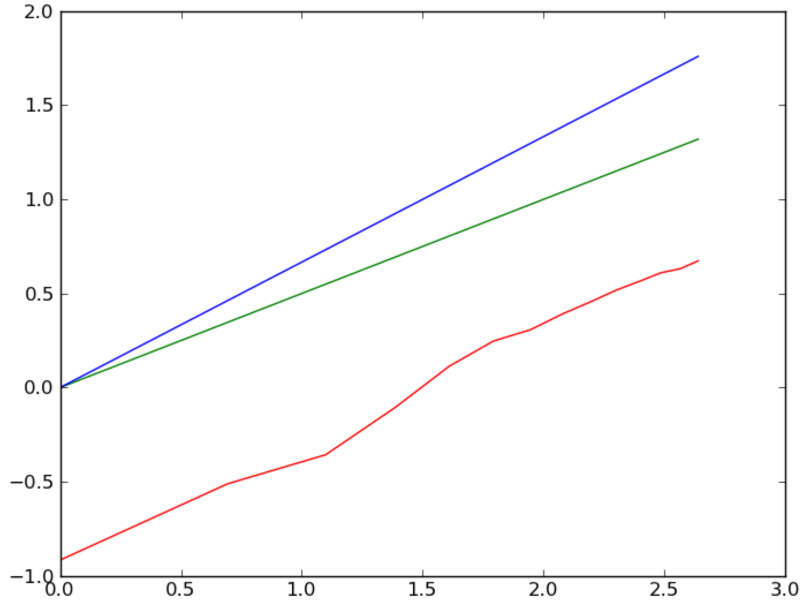

Figure 3.1: gradient estimation (least square)=0,63

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