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Reflected diffusions in polyhedral domains are commonly used as approximate models for stochastic processing networks in heavy traffic. Stationary distributions of such models give useful information on the steady state performance of the corresponding stochastic networks and thus it is important to develop reliable and efficient algorithms for numerical computation of such distributions. In this work we propose and analyze a Monte-Carlo scheme based on an Euler type discretization of the reflected stochastic differential equation using a single sequence of time discretization steps which decrease to zero as time approaches infinity. Appropriately weighted empirical measures constructed from the simulated discretized reflected diffusion are proposed as approximations for the invariant probability measure of the true diffusion model. Almost sure consistency results are established that in particular show that weighted averages of polynomially growing continuous functionals evaluated on the discretized simulated system converge a.s. to the corresponding integrals with respect to the invariant measure. Proofs rely on constructing suitable Lyapunov functions for tightness and uniform integrability and characterizing almost sure limit points through an extension of Echeverria's criteria for reflected diffusions. Regularity properties of the underlying Skorohod problems play a key role in the proofs. Rates of convergence for suitable families of test functions are also obtained. A key advantage of Monte-Carlo methods is the ease of implementation, particularly for high dimensional problems. A numerical example of a eight dimensional Skorohod problem is presented to illustrate the applicability of the approach.

Key words: Reflected Diffusions, Heavy Traffic Theory, Stochastic Networks, Skorohod Problem, Invariant Measures, Stochastic Algorithms.

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1. Introduction Reflected diffusion processes in polyhedral domains have been proposed as approximate models for critically loaded stochastic processing networks. Starting with the influential paper of Reiman[29], there have been many works[28, 12, 26, 36, 24, 35] that justify approximations via reflected diffusions rigorously by establishing a limit theorem under appropriate heavy traffic assumptions. Many performance measures for stochastic networks are formulated to capture the long term behavior of the system and a key object involved in the computation of such measures is the corresponding steady state distribution. Although classical heavy traffic limit theorems only justify approximations of the network behavior through the associated diffusion limit over any fixed finite time horizon, there are now several results[18, 8, 9] that prove, for certain generalized Jackson network models, the convergence of steady state distributions of stochastic networks to those of the associated limit diffusions. Such limit theorems then lead to the important question: How does one compute the stationary distributions of reflected diffusions? Indeed, one of the main motivations for introducing diffusion approximations in the study of stochastic processing systems is the expectation that diffusion models are easier to analyze than their stochastic network counterparts. Classical results of Harrison and Williams [22] show that under certain geometric conditions on the underlying problem data, stationary densities of reflected Brownian motions have explicit product form expressions. However, once one moves away from this special family of models there are no explicit formulas and thus one needs to use numerical procedures.

The objective of the current work is to propose and study the performance of one such numerical procedure for computing stationary distributions of reflected diffusions in polyhedral domains. For diffusions in $\mathbb{R}^m$ there are two basic approaches for computation of invariant distributions: PDE methods and Monte-Carlo methods. PDE approaches are based on the well known basic property that invariant densities of diffusions can be characterized as solutions of certain stationary Fokker-Planck equations. For reflected Brownian motions in polyhedral domains the papers[13, 23, 11] develop similar characterization results. The characterization in this case is formulated for the invariant density together with certain boundary densities and is given in terms of the second order differential operator describing the underlying unconstrained dynamics and a collection of first order operators corresponding to the boundary reflections. Using this characterization as a starting point Dai and Harrison[11] develop an approximation scheme for the stationary density by constructing projections on to certain finite dimensional Hilbert spaces that are described in terms of the above collection of differential operators. Although PDE methods such as above are quite efficient for settings where the state dimension m is small, one finds that Monte-Carlo methods, based on the use of the ergodic theorem, have advantages in higher dimensions. With this in mind, we will propose and study here a Monte-Carlo method for the computation of stationary distributions. Approximations of invariant distributions of diffusions in $\mathbb{R}^m$ using simulation of paths have been studied in several works [2, 27, 32, 31, 25]. One of the key difficulties in using simulation methods is that paths of diffusions cannot be simulated exactly and so one has to contend with two sources of errors: Discretization of the SDE and finite time empirical average approximation for the steady state behavior. In particular, the long term behavior of the discretized SDE could, in general, be quite different from that of the original system and thus a performance analysis of such Monte-Carlo schemes requires a careful understanding of the stability properties of the underlying systems.

The Monte-Carlo approach studied in the current work is inspired by the papers [2], [27], [25] which have analyzed the properties of weighted empirical measures constructed from a Euler scheme, based on a single sequence of time discretization steps decreasing to zero, for diffusions in $\mathbb{R}^m$. For multi-dimensional diffusions with reflection one first needs to describe a suitable analog of an 'Euler discretization step'. In order to do so, we begin with a precise description of the stochastic dynamical system of interest.

Let $G \subset \mathbb{R}^m$ be the convex polyhedral cone in $\mathbb{R}^m$ with the vertex at origin given as the intersection of half spaces G_i, $i = 1, \ldots, N$. Let n_i be the unit vector associated with G_i via the relation

$$G_i = \{x \in \mathbb{R}^m : \langle x, n_i \rangle \geq 0\}.$$

Denote the boundary of a set $S \subset \mathbb{R}^m$ by ∂S. We will denote the set $\{x \in \partial G : \langle x, n_i \rangle = 0\}$ by F_i. For $x \in \partial G$, define the set, $n(x)$, of unit inward normals to G at x by

$$n(x) \doteq \{r : |r| = 1, \langle r, x - y \rangle \leq 0, \forall y \in G\}.$$

With each face F_i we associate a unit vector d_i such that $\langle d_i, n_i \rangle > 0$. This vector defines the direction of constraint associated with the face F_i. For $x \in \partial G$ define

$$d(x) \doteq \left\{ d \in \mathbb{R}^m : d = \sum_{i \in \mathrm{In}(x)} \alpha_i d_i; \alpha_i \geq 0; |d| = 1 \right\},$$

where

$$\mathrm{In}(x) \doteq \{i \in \{1, 2, \cdots, N\} : \langle x, n_i \rangle = 0\}.$$

Roughly speaking, the set $d(x)$ represents the set of permissible directions of constraint available at a point $x \in \partial G$. In a typical stochastic network setting this set valued function is determined from the routing structure of the network and governs the precise constraining mechanism that is used. This mechanism specifies how a RCLL trajectory ψ with values in $\mathbb{R}^m$ is constrained to form a new trajectory with values in G, through the associated Skorohod problem, which is defined as follows.

Let $D([0, \infty) : \mathbb{R}^m)$ denote the set of functions mapping $[0, \infty)$ to $\mathbb{R}^m$ that are right continuous and have left limits. We endow $D([0, \infty) : \mathbb{R}^m)$ with the usual Skorokhod topology. Let

$$D_G([0, \infty) : \mathbb{R}^m) \doteq \{\psi \in D([0, \infty) : \mathbb{R}^m) : \psi(0) \in G\}.$$

For $\eta \in D([0, \infty) : \mathbb{R}^m)$ let $|\eta|(T)$ denote the total variation of η on $[0, T]$ with respect to the Euclidean norm on $\mathbb{R}^m$.

Definition 1.1 *Let $\psi \in D_G([0, \infty) : \mathbb{R}^m)$ be given. Then the pair $(\phi, \eta) \in D([0, \infty) : \mathbb{R}^m) \times D([0, \infty) : \mathbb{R}^m)$ solves the Skorokhod problem (SP) for ψ with respect to G and d if and only if $\phi(0) = \psi(0)$, and for all $t \in [0, \infty)$*

(i) $\phi(t) = \psi(t) + \eta(t)$;

(ii) $\phi(t) \in G$;

(iii) $|\eta|(t) < \infty$;

(iv) $|\eta|(t) = \int_{[0,t]} I_{\{\phi(s)\in\partial G\}} d|\eta|(s)$;

(v) *There exists Borel measurable* $\gamma : [0,\infty) \to \mathbb{R}^m$ *such that* $\gamma(t) \in d(\phi(t))$, $d|\eta|$*-almost everywhere and*

$$\eta(t) = \int_{[0,t]} \gamma(s) d|\eta|(s).$$

In the above definition ϕ represents the constrained version of ψ and η describes the correction applied to ψ in order to produce ϕ. On the domain $D \subset D_G([0,\infty) : \mathbb{R}^m)$ on which there is a unique solutions to the Skorokhod problem we define the Skorokhod map (SM) Γ as $\Gamma(\psi) \doteq \phi$, if $(\phi, \psi - \phi)$ is the unique solution of the Skorokhod problem posed by ψ. We will make the following assumption on the regularity of the Skorokhod map defined by the data $\{(d_i, n_i); i = 1, 2, \cdots, N\}$.

Condition 1.1 *The Skorokhod map is well defined on all of* $D_G([0,\infty) : \mathbb{R}^m)$, *that is,* $D = D_G([0,\infty) : \mathbb{R}^m)$ *and the SM is Lipschitz continuous in the following sense. There exists a* $K < \infty$ *such that for all* $\phi_1, \phi_2 \in D_G([0,\infty) : \mathbb{R}^m)$,

$$\sup_{0\le t<\infty} |\Gamma(\phi_1)(t) - \Gamma(\phi_2)(t)| < K \sup_{0\le t<\infty} |\phi_1(t) - \phi_2(t)|.$$

We will also make the following assumption on the problem data.

Condition 1.2 *For every* $x \in \partial G$, *there is a* $n \in n(x)$ *such that* $\langle d, n\rangle > 0$ *for all* $d \in d(x)$.

The above condition is equivalent to the assumption that the $N \times N$ matrix with $(i,j)^{th}$ entry $\langle d_i, n_j\rangle$ is complete-S (see [15, 30]). When $G = \mathbb{R}^m_+$ and $N = m$, it is known that Condition 1.1 implies Condition 1.2 (see [33]). An important consequence of Condition 1.2 that will be used in our work is the following result from [3] (see also [14]).

Lemma 1.1 *Suppose that Condition 1.2 holds. Then there exists a* $g \in C^2_b(\mathbb{R}^m)$ *such that*

$$\langle \nabla g(x), d_i\rangle \ge 1 \quad \forall x \in F_i, \quad i \in \{1, ..., N\}. \tag{1}$$

We remark here that the function constructed in [3] is defined only on G, however a minor modification of the construction there gives a C^2 extension to all of $\mathbb{R}^m$.

We refer the reader to [20, 15, 16] for sufficient conditions under which Condition 1.1 and Condition 1.2 hold. For example, the paper [16] shows that if $G = \mathbb{R}^m_+$, $N = m$ and the square matrix $D = [d_1, ..., d_m]$ is of the form $D = M(I - V)$, where M is a diagonal matrix with positive diagonal entries, V is off diagonal and the spectral radius of $|V|$ is less than 1, then both Conditions 1.1 and 1.2 hold. Here $|V|$ represents the matrix with entries $(|V_{ij}|)$, where V_{ij} is the (i,j)-th entry of V.

We now describe the constrained diffusion process that will be studied in this paper. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a complete probability space on which is given a filtration $\{\mathcal{F}_t\}_{t\ge0}$ satisfying the usual hypotheses. Let $(W(t), \mathcal{F}_t)$ be a m-dimensional standard Wiener process on the above probability space. For $x \in G$, denote by X^x the unique solution to the following stochastic integral equation,

$$X^x(t) = \Gamma\left(x + \int_0^{\cdot} \sigma(X^x(s))dW(s) + \int_0^{\cdot} b(X^x(s))ds\right)(t), \tag{2}$$

where $\sigma : G \to \mathbb{R}^{m\times m}$ and $b : G \to \mathbb{R}^m$ are maps satisfying the following condition.

Condition 1.3 *There exists* $a_1 \in (0,\infty)$ *such that*

$$|\sigma(x) - \sigma(y)| + |b(x) - b(y)| \le a_1|x - y| \quad \forall x, y \in G \tag{3}$$

and

$$|\sigma(x)| \le a_1, \quad |b(x)| \le a_1, \quad \forall x \in G. \tag{4}$$

Unique solvability of (2) can be shown using the above condition and the regularity assumption on the Skorokhod map. In fact, the classical method of Picard iteration gives the following:

THEOREM 1.1 *For each $x \in G$ there exists a unique pair of continuous $\{\mathcal{F}_t\}$ adapted process $(X^x(t), k(t))_{t\geq 0}$ and a progressively measurable process $(\gamma(t))_{t\geq 0}$ such that the following hold:*

(i) $X^x(t) \in G$, *for all* $t \geq 0$, *a.s.*

(ii) *For all* $t \geq 0$,

$$X^x(t) = x + \int_0^t \sigma(X^x(s))dW(s) + \int_0^t b(X^x(s))ds + k(t), \tag{5}$$

a.s.

(iii) *For all* $T \in [0, \infty)$,

$$|k|(T) < \infty \quad a.s.$$

(iv) *Almost surely, for every* $t \geq 0$,

$$|k|(t) = \int_0^t I_{\{X^x(s)\in\partial G\}} d|k|(s),$$

$k(t) = \int_0^t \gamma(s)d|k|(s)$, *and* $\gamma(s) \in d(X^x(s))$ *a.e,* $[d|k|]$.

In this work we are interested in the invariant distributions of the strong Markov process $\{X^x\}$. One of the basic results due to Harrison and Williams[21] (see also [6]) on invariant distributions of such Markov processes says that if b and σ are constants and σ is invertible, then X^x has a unique invariant probability measure if $b \in \mathcal{C}^o$ (the interior of $\mathcal{C}$), where

$$\mathcal{C} \doteq \left\{ -\sum_{i=1}^N \alpha_i d_i : \alpha_i \geq 0; i \in \{1, \cdots, N\} \right\}.$$

This result was extended to a setting with state dependent coefficients in [1] as follows. We introduce the following two additional assumptions. For $\delta \in (0, \infty)$, define

$$\mathcal{C}(\delta) \doteq \{v \in \mathcal{C} : \mathrm{dist}(v, \partial\mathcal{C}) \geq \delta\}.$$

Condition 1.4 *There exists a $\delta \in (0, \infty)$ such that for all $x \in G$, $b(x) \in \mathcal{C}(\delta)$.*

Condition 1.5 *There exists $\underline{\sigma} \in (0, \infty)$ such that for all $x \in G$ and $\alpha \in \mathbb{R}^m$,*

$$\alpha'(\sigma(x)\sigma'(x))\alpha \geq \underline{\sigma}\alpha'\alpha.$$

The following is the main result of [1].

THEOREM 1.2 *Assume that Conditions 1.1-1.5 hold. Then the strong Markov process $\{X^x(\cdot); x \in G\}$ is positive recurrent and has a unique invariant probability measure.*

We remark that in [1] a somewhat weaker assumption than Condition 1.4 is used, which says that $b(x) \in \mathcal{C}(\delta)$ for all x outside a bounded set. In the current work, for simplicity we will use the stronger form as in Condition 1.4. Conditions 1.1-1.5 will be assumed to hold for the rest of this work and will not be explicitly noted in the statements of various results.

We now summarize some of the notation that will be used in this work. For a Polish space S, $\mathcal{P}(S)$ will denote the space of probability measures, and $\mathcal{M}_F(S)$ the space of finite measures on S endowed with the usual topology of weak convergence. For a closed set $G \subset \mathbb{R}^m$, we say $f \in C_b^2(G)$, [respectively $f \in C_c^2(G)$] if f is defined on some open set $O \supset G$ and f is a twice continuously differentiable on O with bounded first two derivatives [respectively compact support]. For $\nu \in \mathcal{P}(S)$ and a ν-integrable $f : S \to \mathbb{R}$, we write $\int_S f d\nu$ as $\langle f, \nu\rangle$ or $\nu(f)$ interchangeably. We will use the symbol "$\Rightarrow$" or "$\xrightarrow{\mathcal{L}}$" to denote convergence in distribution. Let $\mathbb{R}^m$ denote the set of m-dimensional real vectors. Euclidean norm will be denoted by $|\cdot|$ and the corresponding inner product by $\langle\cdot,\cdot\rangle$. The symbols, $\xrightarrow{\mathbf{P}}$, $\xrightarrow{L^p}$ denote convergence in probability and L^p respectively. Denote by $||\cdot||_\infty$ the supremum norm. A vector $v \in \mathbb{R}^m$ is said to be nonnegative (and we write $v \geq 0$) if it is componentwise nonnegative.

1.1 Numerical Scheme and Main Results Throughout this work, the unique invariant measure for the Markov process $\{X^x\}$ will be denoted by ν. The goal of this work is to develop a convergent numerical procedure for approximating ν. We now describe this procedure.

Let $\{\lambda_k\}_{k\geq 1}$ be a sequence of positive real numbers such that

$$\lambda_k \to 0, \text{ as } k \to \infty \text{ and letting } \Lambda_n := \sum_{k=1}^{n} \lambda_k, \; \Lambda_n \to \infty \text{ as } n \to \infty. \tag{6}$$

Note the condition is satisfied if $\lambda_n = \frac{1}{n^\theta}$ with $\theta \in (0,1]$. Define the map $\mathcal{S} : G \times \mathbb{R}^m \to G$ by the relation

$$\mathcal{S}(x,v) = \Gamma(x + vi)(1), \tag{7}$$

where $i : [0,\infty) \to [0,\infty)$ is the identity map. The map $\mathcal{S}$ will be used to construct an Euler discretization of the stochastic dynamical system described by (5). We now introduce the noise sequence that will be used in the Euler discretization of (5).

Let $\{U_{k,j}; k \in \mathbb{N}, j = 1,...,m\}$ be an array of mutually independent $\mathbb{R}$ valued random variables, given on some probability space $(\Omega, \mathcal{F}, \mathbf{P})$, such that $\mathbf{E}U_{k,j} = 0$ and $\mathbf{E}U_{k,j}^2 = 1$, for all $k \in \mathbb{N}, j = 1,...,m$. We denote the $\mathbb{R}^m$ valued random variable $(U_{k,1},...,U_{k,m})'$ by U_k. We will make the following assumption on the array $\{U_{k,j}\}$.

Condition 1.6 *For some $\alpha \in (0,\infty)$,*

$$\mathbf{E}e^{\lambda U_{k,j}} \leq e^{\alpha\lambda^2} \textit{ for all } k \in \mathbb{N}, \; j = 1,...,m, \; \lambda \in \mathbb{R}.$$

The above condition is clearly satisfied when $U_{k,j} \sim N(0,1)$. Also, using well known concentration inequalities it can be checked that the condition also holds if $\text{supp}(U_{k,j})$ is uniformly bounded (see Appendix for a proof of the latter statement). Condition 1.6 will be assumed to hold throughout this work.

The Euler scheme is given as follows. Define iteratively, sequences $\{X_k\}_{k\in\mathbb{N}_0}$, $\{Y_k\}_{k\in\mathbb{N}_0}$ of G and $\mathbb{R}^m$ valued random variables, respectively, as follows. Fix $x_0 \in G$.

$$\begin{cases} X_0 = x_0, \\ Y_{k+1} = X_k + b(X_k)\lambda_{k+1} + \sigma(X_k)\sqrt{\lambda_{k+1}}U_{k+1}, \\ X_{k+1} = \mathcal{S}(X_k, Y_{k+1} - X_k)\,. \end{cases} \tag{8}$$

Note that $\{X_k\}$ is a sequence of G valued random variables. The last equation of the above display describes a projection for the Euler step that is consistent with the Skorohod problem associated with the problem data.

Define a sequence of $\mathcal{P}(G)$ valued random variables as

$$\nu_n = \frac{1}{\Lambda_n}\sum_{k=1}^{n} \lambda_k \delta_{X_{k-1}}, \quad n \in \mathbb{N}.$$

The above random measures define our basic sequence of approximations for the invariant measure ν. In particular, they yield an approximation for any integral of the form $\int_G f(x)d\nu(x)$ through the corresponding weighted averages:

$$\frac{1}{\Lambda_n}\sum_{k=1}^{n} \lambda_k f(X_{k-1}). \tag{9}$$

The following is the first main result of this work.

Theorem 1.3 *As $n \to \infty$, ν_n converges weakly to ν, almost surely.*

The above result ensures that (9) gives an almost surely consistent approximation for $\nu(f)$ for any bounded and continuous f. In fact we have a substantially stronger statement as follows:

Theorem 1.4 *There exists a $\zeta \in (0,\infty)$ such that for all continuous $f : G \to \mathbb{R}$ satisfying $\limsup_{x\to\infty} e^{-\zeta|x|}|f(x)| = 0$, we have $\nu_n(f) \to \nu(f)$, a.s.*

The key ingredient in the proof of the above almost sure limit theorems is a certain Lyapunov function that was introduced in [7] to study geometric ergodicity properties of reflected diffusions. Using this Lyapunov function we establish a.s. bounds on exponential moments of ν_n that are uniform in n. These bounds in particular guarantee tightness of $\{\nu_n(\omega), n \geq 1\}$, for a.e. ω. Then the remaining work, for proving the above theorems, lies in the characterization of the limit points of $\nu_n(\omega)$. For this we use an extension of the well known Echeverria criterion for invariant distributions of Markov processes that has been developed in [13, 23] (see also [3]). Verification of this criteria (stated as Theorem 2.1 in the current work) for a typical limit point ν_0 of $\{\nu_n\}$ requires showing that, ν_0 along with a certain collection $\{\mu_0^i, i = 1, \cdots N\}$ of finite measures supported on various parts of the boundary of G satisfy a relation of the form in (27). The measures $\{\mu_0^i\}$ are obtained by taking weak limits of certain finite measures constructed from the Euler scheme. Although these pre-limit measures may place positive mass away from the boundary of the domain, we argue using the regularity properties of the Skorohod map (a key ingredient here is Lemma 1.1), that in the limit these finite measures are supported on the correct parts of the boundary.

Under additional assumptions, one can obtain rates of convergence as follows. For $\alpha > 0$, set

$$\Lambda_n^{(\alpha)} = \lambda_1^\alpha + ... + \lambda_n^\alpha.$$

Denote the normal distribution with mean a and variance b^2 by $\mathcal{N}(a, b^2)$. For $\phi \in C^3(G)$ (space of three times continuously differentiable functions on G) and $v \in \mathbb{R}^m$, let $D^3\phi(x)(v)^{\otimes 3} = \sum_{i,j,k} D^3_{i,j,k}\phi(x)v_iv_jv_k$.

For $f \in C_c^2(G)$, define $\mathcal{A}f : G \to \mathbb{R}$ and $D_if : G \to \mathbb{R}$; $i = 1, ..., N$ as

$$\mathcal{A}f(x) = b(x)\cdot\nabla f(x) + \frac{1}{2}\sigma'(x)D^2f(x)\sigma(x), \quad x \in G,$$

$$D_if(x) = d_i \cdot \nabla f(x), \quad x \in G,$$

where ∇ is the gradient operator and D^2 is the $m \times m$ Hessian matrix.

THEOREM 1.5 *Assume that U_i's are i.i.d with common distribution μ. There exists a $\zeta \in (0, \infty)$ such that whenever $\phi \in C^2(G)$ satisfies $\lim_{|x|\to\infty} e^{-\zeta|x|}|\nabla\phi(x)|^2 = 0$, we have the following:*

(a) Fast-decreasing step. Suppose $\lim_{n\to\infty} \frac{\Lambda_n^{(3/2)}}{\sqrt{\Lambda_n}} = 0$, $D^2\phi$ is bounded and Lipschitz, and

$$\begin{cases} \langle\nabla\phi(x), d_i\rangle = 0, & \forall x \in F_i, \forall i; \\ D^2\phi(x)d_i = \mathbf{0}, & \forall x \in F_i, \forall i. \end{cases} \tag{10}$$

Then the following CLT holds:

$$\sqrt{\Lambda_n}\nu_n(\mathcal{A}\phi) \xrightarrow{\mathcal{L}} \mathcal{N}\left(0, \int_G |\sigma^T\nabla\phi|^2 d\nu\right).$$

(b) Slowly decreasing step. Suppose that $\lim_{n\to\infty}(1/\sqrt{\Lambda_n})\Lambda_n^{(3/2)} = \tilde{\lambda} \in (0, +\infty]$, $\phi \in C^3(G)$ and $D^3\phi$ is bounded and Lipschitz. Further suppose that

$$\begin{cases} \langle\nabla\phi(x), d_i\rangle = 0, & \forall x \in F_i, \forall i; \\ D^2\phi(x)d_i = \mathbf{0}, & \forall x \in F_i, \forall i; \\ D^3_{\cdot jk}\phi(x)\cdot d_i = 0, & \forall x \in F_i, \forall i, j, k. \end{cases} \tag{11}$$

Then we have

$$\sqrt{\Lambda_n}\nu_n(\mathcal{A}\phi) \xrightarrow{\mathcal{L}} \mathcal{N}\left(\tilde{\lambda}\tilde{m}, \int_G |\sigma^T\nabla\phi|^2 d\nu\right) \qquad \text{if } \tilde{\lambda} < \infty, \tag{12}$$

$$\frac{\Lambda_n}{\Lambda_n^{(3/2)}}\nu_n(\mathcal{A}\phi) \xrightarrow{\mathbf{P}} \tilde{m} \qquad \text{if } \tilde{\lambda} = +\infty, \tag{13}$$

where

$$\tilde{m} = -\frac{1}{6}\int_G\int_{\mathbb{R}^m} D^3\phi(x)(\sigma(x)u)^{\otimes 3}\mu(du)\nu(dx).$$

Note that when $\lambda_k = \frac{1}{k^\alpha}$, $\Lambda_n^{(3/2)}/\sqrt{\Lambda_n}$ converges to 0 [resp. ∞, $\tilde{\lambda} \in (0,+\infty)$], if $\alpha > 1/2$ [resp. $\alpha < 1/2$, $\alpha = 1/2$]. Also note that if ϕ is a smooth function supported in the interior of G then it automatically satisfies (10) and (11).

Proof of Theorem 1.5 is quite similar to that of Theorem 9 in [25], the main difference is in the treatment of the reflection terms for which once more we appeal to regularity properties of the Skorohod map and an estimate based on Lemma 1.1 (see proof of (33) which is crucially used in proofs of Section 3).

A key step in the implementation of the Euler scheme in (8) is the evaluation of the one time step Skorohod map $\mathcal{S}(x, v)$. In Section 4.1 we describe one possible approach to this evaluation that uses relationships between Skorohod problems and Linear Complementarity problems(LCPs). There are many well developed numerical codes for solving LCPs (for example in MATLAB) and we will describe in Section 4.2 some results from numerical experiments that use a quadratic programming algorithm for LCPs (cf. [10]) in implementing the scheme in (8). As remarked earlier, one of the advantages of Monte-Carlo methods is the ease of implementation, particularly for high dimensional problems. To illustrate this, in Section 4.2 we present numerical results for a eight dimensional Skorohod problem.

The paper is organized as follows. In Section 2 we prove Theorem 1.3 and 1.4. Theorem 1.3 is proved in two steps. Section 2.1 shows the tightness of the random measures $\{\nu_n\}$, and Section 2.2 characterizes the limit of the measures $\{\nu_n\}$ as the invariant measure of the constrained diffusion in (2). Section 2.3 gives the proof of Theorem 1.4. Rate of convergence theorem (Theorem 1.5) is proved in Section 3. Finally we conclude by describing some numerical results in Section 4.

2. Proofs of Theorems 1.3 and 1.4 The proof of Theorem 1.3 proceeds by showing that for a.e. ω, the sequence of random probability measures $\{\nu_n(\omega)\}_{n\geq 1}$ is tight and then characterizing the limit points of the sequence using a generalization of Echeverria's criteria. Tightness is argued in Section 2.1 while the limit points are characterized in Section 2.2. Finally in Section 2.3, we give the proof of Theorem 1.4.

2.1 Tightness We begin by presenting a Lyapunov function introduced in [1] that plays a key role in the stability analysis of constrained diffusion processes of the form studied here (see [1, 7, 8, 9, 3, 5, 4]).

Throughout this work we will fix a $\delta > 0$ as in Condition 1.4.

For $x \in G$, let $\mathcal{A}(x)$ be the collection of all absolutely continuous functions $z : [0,\infty) \to \mathbb{R}^m$ defined via

$$z(t) \doteq \Gamma\left(x + \int_0^{\cdot} v(s)ds\right)(t), \quad t \in [0,\infty), \tag{14}$$

for some $v : [0,\infty) \to \mathcal{C}(\delta)$ which satisfies

$$\int_0^t |v(s)|ds < \infty, \quad \text{for all } t \in [0,\infty). \tag{15}$$

Define $T : G \to [0,\infty)$ by the relation

$$T(x) \doteq \sup_{z\in\mathcal{A}(x)} \inf\{t \in [0,\infty) : z(t) = 0\}, \quad x \in G. \tag{16}$$

The function T has the following properties (see [1]).

Lemma 2.1 *There exist constants $c, C \in (0,\infty)$ such that the following hold:*

(i) For all $x, y \in G$,

$$|T(x) - T(y)| \leq C|x - y|.$$

(ii) For all $x \in G$, $T(x) \geq c|x|$. Thus, in particular, for all $M \in (0,\infty)$ the set $\{x \in G : T(x) \leq M\}$ is compact.

(iii) Fix $x \in G$ and let $z \in \mathcal{A}(x)$. Then for all $t > 0$,

$$T(z(t)) \leq (T(x) - t)^+.$$

We next present an elementary lemma that will be used in obtaining moment estimates. For $k \in \mathbb{N}$, let $\mathcal{F}_k = \sigma(U_1, ..., U_k)$. Set $\mathcal{F}_0 = \{\emptyset, \Omega\}$.

LEMMA 2.2 *There exist $c_1, c_2 \in (1, \infty)$ for which the following holds. Let $\{v_i\}_{i\in\mathbb{N}}$ be a sequence of $\mathbb{R}^m$ valued random variables such that v_i is $\mathcal{F}_{i-1}$ measurable for all $i \geq 1$ and*

$$\operatorname*{ess\,sup}_{\omega} |v_i(\omega)| \equiv |v_i|_\infty < \infty.$$

Let $S_n = \sum_{i=1}^n v_i \cdot U_i$, $n \in \mathbb{N}$. Then for every $r \geq 0$ and $n \geq 1$,

$$\mathbf{E} \max_{1\leq i\leq n} e^{r|S_i|} \leq c_1 e^{c_2 r^2 \sum_{i=1}^n |v_i|_\infty^2}.$$

PROOF. We will only give the proof for the case $m = 1$. The general case is treated similarly.

From Doob's maximal inequalities for submartingales, we have

$$\begin{aligned}\mathbf{E} \max_{1\leq i\leq n} e^{r|S_i|} &\leq 4\mathbf{E}e^{r|S_n|} \\ &\leq 4\left(\mathbf{E}e^{rS_n} + \mathbf{E}e^{-rS_n}\right)\end{aligned}$$

From Condition 1.6, it follows that for every $r \in \mathbb{R}$,

$$\begin{aligned}\mathbf{E}\left(e^{rS_n}|\mathcal{F}_{n-1}\right) &\leq e^{rS_{n-1}} e^{\alpha r^2 v_n^2} \\ &\leq e^{rS_{n-1}} e^{\alpha r^2 |v_n|_\infty^2}\end{aligned}$$

The result now follows by a successive conditioning argument. □

Define $\lambda : [0, \infty) \to [0, \infty)$ and $j : [0, \infty) \to \mathbb{N}_0$ as

$$\lambda(s) = \Lambda_k; \quad j(s) = k, \text{ if } \Lambda_k \leq s < \Lambda_{k+1},\ k \in \mathbb{N}_0;$$

where we define $\Lambda_0 = 0$. Define piecewise linear $\mathbb{R}^m$ valued stochastic process as follows,

$$\hat{W}(t) = \sum_{i\leq j(t)} \sqrt{\lambda_i} U_i + \frac{t - \lambda(t)}{\sqrt{\lambda_{j(t)+1}}} U_{j(t)+1}, \quad t \geq 0.$$

Let $\hat{X}(t)$ be the solution of the following integral equation

$$\hat{X}(t) = \Gamma\left(x_0 + \int_0^\cdot b(\hat{X}(\lambda(s)))ds + \int_0^\cdot \sigma(\hat{X}(\lambda(s)))d\hat{W}(s)\right)(t), \quad t \geq 0.$$

Clearly, $\hat{X}(\lambda(t)) = X_{j(t)}$ for all $t \geq 0$.

Fix $\rho \in (0, 1]$. Define

$$\varpi = \frac{1}{2(1+\rho)L}, \quad \Delta = 4\lambda_0 + 16L\ln(c_1), \tag{17}$$

where $L = c_2 a_1^2 C^2 K^2$ and $\lambda_0 = \sup_{i\geq 1} \lambda_i$. Let $V : G \to \mathbb{R}_+$ be defined as

$$V(x) = e^{\varpi T(x)}, \quad x \in G. \tag{18}$$

LEMMA 2.3 *There exist $\beta \in (0, 1)$ and $\phi \in [0, \infty)$ such that for each $\zeta \in [0, \rho]$ and for all $t \geq 0$,*

$$\mathbf{E}(V^{1+\zeta}(X_{j(t+\Delta)})|\mathcal{F}_{j(t)}) \leq (1-\beta)V^{1+\zeta}(X_{j(t)}) + \phi \tag{19}$$

PROOF. Fix $t \geq 0$ and $\zeta \in [0, \rho]$. Define $\xi : [\lambda(t), \infty) \to G$ as

$$\xi(s) = \Gamma\left(X_{j(t)} + \int_{\lambda(t)}^{\lambda(t)+\cdot} b(X_{j(u)})du\right)(s - \lambda(t)), \quad s \geq \lambda(t).$$

Using the Lipschitz property of the Skorokhod map (Condition 1.1), we have

$$\begin{aligned}\sup_{\lambda(t)\leq s\leq\lambda(t)+\Delta+\lambda_0} |\hat{X}(s) - \xi(s)| &\leq K \sup_{\lambda(t)\leq s\leq\lambda(t)+\Delta+\lambda_0} \left|\int_{\lambda(t)}^s \sigma(\hat{X}(\lambda(u)))d\hat{W}(u)\right| \\ &=: K\bar{\nu}(t, \Delta).\end{aligned}$$

Note that
$$\Delta - \lambda_0 \leq \lambda(t+\Delta) - \lambda(t) \leq \Delta + \lambda_0.$$
Using this observation along with Lemma 2.1 (i) and (iii),
$$\begin{aligned} T(\hat{X}(\lambda(t+\Delta))) \leq & T(\xi(\lambda(t+\Delta))) + CK\bar{\nu}(t,\Delta) \\ \leq & (T(\hat{X}(\lambda(t))) - (\lambda(t+\Delta)-\lambda(t)))^+ + CK\bar{\nu}(t,\Delta) \\ \leq & (T(\hat{X}(\lambda(t))) - (\Delta-\lambda_0))^+ + CK\bar{\nu}(t,\Delta). \end{aligned}$$
From the above estimate and the definition of $V(x)$, we now have
$$\begin{aligned} \frac{\mathbf{E}(V(\hat{X}(\lambda(t+\Delta)))^{1+\zeta}|\mathcal{F}_{j(t)})}{V(\hat{X}(\lambda(t)))^{1+\zeta}} \leq & \mathbf{E}\left(\exp(\varpi(1+\zeta)\left((T(\hat{X}(\lambda(t))) - (\Delta-\lambda_0))^+ + CK\bar{\nu}(t,\Delta))\right)|\mathcal{F}_{j(t)}\right) \\ & \times \exp(-\varpi(1+\zeta))T(\hat{X}(\lambda(t))). \end{aligned} \tag{20}$$

Letting, for $q \in \mathbb{N}_0$, $\sigma_q = \sigma(X_q)$, we have, for any $s \in [\lambda(t), \lambda(t)+\Delta+\lambda_0]$,
$$\int_{\lambda(t)}^{s} \sigma(\hat{X}(\lambda(u)))d\hat{W}(u) \leq \begin{cases} \sum_{q=j(t)}^{j(s)} \sigma_q\sqrt{\lambda_{q+1}}U_{q+1}, & \text{if } \sigma_{j(s)}U_{j(s)} \geq 0 \\ \\ \sum_{q=j(t)}^{j(s)-1} \sigma_q\sqrt{\lambda_{q+1}}U_{q+1}, & \text{if } \sigma_{j(s)}U_{j(s)} < 0 \end{cases},$$
which can be bounded by
$$\max_{j(t)\leq j\leq j(s)} \sum_{q=j(t)}^{j} \sigma_q\sqrt{\lambda_{q+1}}U_{q+1}.$$
Similarly,
$$-\int_{\lambda(t)}^{s} \sigma(\hat{X}(\lambda(u)))d\hat{W}(u) \leq \max_{j(t)\leq j\leq j(s)} -\sum_{q=j(t)}^{j} \sigma_q\sqrt{\lambda_{q+1}}U_{q+1}.$$
And therefore
$$\bar{\nu}(t,\Delta) = \sup_{\lambda(t)\leq s\leq \lambda(t)+\Delta+\lambda_0} \left|\int_{\lambda(t)}^{s} \sigma(\hat{X}(\lambda(u)))d\hat{W}(u)\right| \leq \max_{j(t)\leq j\leq j_t^*} \left|\sum_{q=j(t)}^{j} \sigma_q\sqrt{\lambda_{q+1}}U_{q+1}\right|,$$
where $j_t^* = j(\lambda(t)+\Delta+\lambda_0)$.

Using Lemma 2.2, we now have that, with $m_0 = \varpi(1+\zeta)CK$,
$$\mathbf{E}\left[e^{m_0\bar{\nu}(t,\Delta)}\,|\mathcal{F}_{j(t)}\right] \leq c_1 e^{c_2 m_0^2 a_1^2 \sum_{q=j(t)}^{j_t^*}\lambda_{q+1}} \leq c_1 e^{c_2 m_0^2 a_1^2(\Delta+2\lambda_0)}. \tag{21}$$

In the case $T(\hat{X}(\lambda(t))) \geq \Delta - \lambda_0$, we have from (20) and (21) that
$$\mathbf{E}(V(\hat{X}(\lambda(t+\Delta)))^{1+\zeta}|\mathcal{F}_{j(t)}) \leq V(\hat{X}(\lambda(t)))^{1+\zeta}e^{-\varpi(1+\zeta)(\Delta-\lambda_0)} \times c_1 e^{c_2 m_0^2 a_1^2(\Delta+2\lambda_0)}.$$
Recalling the choice of ϖ and Δ, we now see that
$$\mathbf{E}(V(\hat{X}(\lambda(t+\Delta)))^{1+\zeta}|\mathcal{F}_{j(t)}) \leq (1-\beta)V(\hat{X}(\lambda(t)))^{1+\zeta},$$
where $\beta = 1 - e^{-3\ln c_1}$.

In the case $T(\hat{X}(\lambda(t))) < \Delta - \lambda_0$, we have
$$\mathbf{E}(V(\hat{X}(\lambda(t+\Delta)))^{1+\zeta}|\mathcal{F}_{j(t)}) \leq \mathbf{E}\left(e^{m_0\bar{\nu}(t,\Delta)}|\mathcal{F}_{j(t)}\right) \leq c_1 e^{c_2 m_0^2 a_1^2(\Delta+2\lambda_0)} \leq c_1 e^{\frac{1}{4L}(\Delta+2\lambda_0)} \equiv \phi.$$

Combining the two cases, we have (19). □

The following lemma follows from Lemma 2.3 through a recursive argument.

Lemma 2.4 *There exists $a_2 \in (0,\infty)$ such that*
$$\sup_t \mathbf{E}(V(\hat{X}(\lambda(t)))^{1+\rho}) \leq a_2. \tag{22}$$

PROOF.

For any $t \in (\Delta, \infty)$, we can find $t' \in (0, \Delta]$ and $j \in \mathbb{N}$ such that $t = t' + j\Delta$. By a recursive argument using (19), we then have

$$\mathbf{E}(V(\hat{X}(\lambda(t)))^{1+\rho}) \leq \mathbf{E}(V(\hat{X}(\lambda(t')))^{1+\rho}) + \frac{\phi}{\beta}.$$

Thus

$$\sup_t \mathbf{E}(V(\hat{X}(\lambda(t)))^{1+\rho}) \leq \sup_{0 \leq t < \Delta} \mathbf{E}(V(\hat{X}(\lambda(t)))^{1+\rho}) + \frac{\phi}{\beta}.$$

The supremum on the right side is bounded by $\max_{j \leq j(\Delta+\lambda_0)} \mathbf{E}(V(X_j)^{1+\rho})$, which is finite using Condition 1.6, boundedness of b, σ and the Lipschitz property of Γ. □

Now we can prove the following lemma.

LEMMA 2.5 *For a.e. ω, $\sup_n \langle V, \nu_n(\omega)\rangle < \infty$. Consequently, the sequence $\{\nu_n(\omega)\}_{n\geq 1}$ is tight for a.e. ω.*

PROOF. Let n_0 be such that $\Lambda_{n_0} > \Delta$. Then it suffices to consider the supremum in the above display over all $n \geq n_0$. For $i \in \mathbb{N}_0$, define $s(i) = \inf\{j \in \mathbb{N}_0 : \Lambda_j \geq i\Delta\}$. Then $s(\lfloor \Lambda_n/\Delta \rfloor) \leq n$ and therefore, for $n \geq n_0$,

$$\begin{aligned}\nu_n(V) =& \frac{1}{\Lambda_n} \sum_{k=1}^{n} \lambda_k V(X_{k-1}) = \frac{1}{\Lambda_n} \int_0^{\Lambda_n} V(\hat{X}(\lambda(t)))dt \\ \leq& \frac{1}{\Lambda_{s(\lfloor \Lambda_n/\Delta \rfloor)}} \int_0^{(\lfloor \Lambda_n/\Delta \rfloor + 1)\Delta} V(\hat{X}(\lambda(t)))dt.\end{aligned}$$

Using Lemma 2.3 with $\zeta = 0$, we have

$$\begin{aligned}&\frac{1}{\Lambda_{s(\lfloor \Lambda_n/\Delta \rfloor)}} \int_0^{(\lfloor \Lambda_n/\Delta \rfloor + 1)\Delta} V(\hat{X}(\lambda(t)))dt \\ &\quad \leq \frac{1}{\Lambda_{s(\lfloor \Lambda_n/\Delta \rfloor)}} \int_0^{(\lfloor \Lambda_n/\Delta \rfloor + 1)\Delta} [V(\hat{X}(\lambda(t))) - \mathbf{E}(V(\hat{X}(\lambda(t+\Delta)))|\mathcal{F}_{j(t)})]dt \\ &\quad + \frac{1}{\Lambda_{s(\lfloor \Lambda_n/\Delta \rfloor)}} \int_0^{(\lfloor \Lambda_n/\Delta \rfloor + 1)\Delta} [V(\hat{X}(\lambda(t)))(1-\beta) + \phi]dt.\end{aligned}$$

Thus, rearranging terms,

$$\begin{aligned}&\frac{\beta}{\Lambda_{s(\lfloor \Lambda_n/\Delta \rfloor)}} \int_0^{(\lfloor \Lambda_n/\Delta \rfloor + 1)\Delta} V(\hat{X}(\lambda(t)))dt \\ &\quad \leq \frac{1}{\Lambda_{s(\lfloor \Lambda_n/\Delta \rfloor)}} \int_0^{(\lfloor \Lambda_n/\Delta \rfloor + 1)\Delta} [V(\hat{X}(\lambda(t))) - \mathbf{E}(V(\hat{X}(\lambda(t+\Delta)))|\mathcal{F}_{j(t)})]dt \\ &\quad + \phi \frac{(\lfloor \Lambda_n/\Delta \rfloor + 1)\Delta}{\Lambda_{s(\lfloor \Lambda_n/\Delta \rfloor)}}.\end{aligned}$$

Next note that $\Lambda_{s(\lfloor \Lambda_n/\Delta \rfloor)} \geq \lfloor \Lambda_n/\Delta \rfloor \Delta$, and, for $n \geq n_0$,

$$\Lambda_{s(\lfloor \Lambda_n/\Delta \rfloor)} \geq \Lambda_{s(1)} \geq \lambda_1.$$

Thus

$$\sup_{n \geq n_0} \phi \frac{(\lfloor \Lambda_n/\Delta \rfloor + 1)\Delta}{\Lambda_{s(\lfloor \Lambda_n/\Delta \rfloor)}} \leq \phi(1 + \frac{\Delta}{\lambda_1}) < \infty.$$

To prove the lemma, it is now enough to show

$$\sup_{n \geq n_0} \frac{1}{\Lambda_{s(\lfloor \Lambda_n/\Delta \rfloor)}} \int_0^{(\lfloor \Lambda_n/\Delta \rfloor + 1)\Delta} [V(\hat{X}(\lambda(t))) - \mathbf{E}(V(\hat{X}(\lambda(t+\Delta)))|\mathcal{F}_{j(t)})]dt < \infty, \quad \text{a.e. } \omega. \tag{23}$$

The above expression can be split into two terms:

$$\begin{aligned}\frac{1}{\Lambda_{s(\lfloor\Lambda_n/\Delta\rfloor)}}\int_0^{(\lfloor\Lambda_n/\Delta\rfloor+1)\Delta}&[V(\hat{X}(\lambda(t)))-\mathbf{E}(V(\hat{X}(\lambda(t+\Delta)))|\mathcal{F}_{j(t)})]dt\\
&=\frac{1}{\Lambda_{s(\lfloor\Lambda_n/\Delta\rfloor)}}\int_0^{(\lfloor\Lambda_n/\Delta\rfloor+1)\Delta}[V(\hat{X}(\lambda(t)))-V(\hat{X}(\lambda(t+\Delta)))]dt\\
&+\frac{1}{\Lambda_{s(\lfloor\Lambda_n/\Delta\rfloor)}}\int_0^{(\lfloor\Lambda_n/\Delta\rfloor+1)\Delta}[V(\hat{X}(\lambda(t+\Delta)))-\mathbf{E}(V(\hat{X}(\lambda(t+\Delta)))|\mathcal{F}_{j(t)})]dt\\
&\equiv T_1+T_2\end{aligned}$$

Consider the first term:

$$\begin{aligned}T_1=&\frac{1}{\Lambda_{s(\lfloor\Lambda_n/\Delta\rfloor)}}\int_0^{(\lfloor\Lambda_n/\Delta\rfloor+1)\Delta}V(\hat{X}(\lambda(t)))dt-\frac{1}{\Lambda_{s(\lfloor\Lambda_n/\Delta\rfloor)}}\int_0^{(\lfloor\Lambda_n/\Delta\rfloor+1)\Delta}V(\hat{X}(\lambda(t+\Delta)))dt\\
=&\frac{1}{\Lambda_{s(\lfloor\Lambda_n/\Delta\rfloor)}}\int_0^{(\lfloor\Lambda_n/\Delta\rfloor+1)\Delta}V(\hat{X}(\lambda(t)))dt-\frac{1}{\Lambda_{s(\lfloor\Lambda_n/\Delta\rfloor)}}\int_\Delta^{(\lfloor\Lambda_n/\Delta\rfloor+1)\Delta+\Delta}V(\hat{X}(\lambda(t)))dt\\
=&\frac{1}{\Lambda_{s(\lfloor\Lambda_n/\Delta\rfloor)}}\int_0^{\Delta}V(\hat{X}(\lambda(t)))dt-\frac{1}{\Lambda_{s(\lfloor\Lambda_n/\Delta\rfloor)}}\int_{(\lfloor\Lambda_n/\Delta\rfloor+1)\Delta}^{(\lfloor\Lambda_n/\Delta\rfloor+1)\Delta+\Delta}V(\hat{X}(\lambda(t)))dt\\
\leq&\frac{1}{\Lambda_{s(\lfloor\Lambda_n/\Delta\rfloor)}}\sum_{k:\Lambda_{k-1}\leq\Delta}\lambda_kV(X_{k-1}).\end{aligned}$$

Let $Z=\sum_{k:\Lambda_{k-1}\leq\Delta}\lambda_kV(X_{k-1})$. Then from (22), we have $\mathbf{E}Z\leq a_2(\Delta+\lambda_0)$. Combining this with the fact that for $n\geq n_0$, $\Lambda_{s(\lfloor\Lambda_n/\Delta\rfloor)}\geq\lambda_1$, we have that

$$\sup_{n\geq n_0}T_1(\omega)<\infty,\quad \text{a.e. } \omega. \tag{24}$$

Next, consider T_2:

$$\begin{aligned}T_2=&\frac{1}{\Lambda_{s(\lfloor\Lambda_n/\Delta\rfloor)}}\int_0^{(\lfloor\Lambda_n/\Delta\rfloor+1)\Delta}[V(\hat{X}(\lambda(t+\Delta)))-\mathbf{E}(V(\hat{X}(\lambda(t+\Delta)))|\mathcal{F}_{j(t)})]dt\\
=&\frac{1}{\Lambda_{s(\lfloor\Lambda_n/\Delta\rfloor)}}\sum_{i=0}^{\lfloor\Lambda_n/\Delta\rfloor}\int_{i\Delta}^{(i+1)\Delta}[V(\hat{X}(\lambda(t+\Delta)))-\mathbf{E}(V(\hat{X}(\lambda(t+\Delta)))|\mathcal{F}_{j(t)})]dt.\end{aligned}$$

From Kronecker's Lemma (see page 63 of [17]), the last sum is bounded in n a.s. (in fact converges to 0) if the following series is summable a.s.

$$\sum_{i=1}^{\infty}\frac{1}{\Lambda_{s(i)}}\int_{i\Delta}^{(i+1)\Delta}[V(\hat{X}(\lambda(t+\Delta)))-\mathbf{E}(V(\hat{X}(\lambda(t+\Delta)))|\mathcal{F}_{j(t)})]dt.$$

Consider the sum over even and odd terms separately. For even terms, the sum can be written as

$$\sum_{k=1}^{\infty}\frac{1}{\Lambda_{s(2k)}}\int_{2k\Delta}^{(2k+1)\Delta}[V(\hat{X}(\lambda(t+\Delta)))-\mathbf{E}(V(\hat{X}(\lambda(t+\Delta)))|\mathcal{F}_{j(t)})]dt. \tag{25}$$

Let

$$\xi_{k+1}=\frac{1}{\Lambda_{s(2k)}}\int_{2k\Delta}^{(2k+1)\Delta}[V(\hat{X}(\lambda(t+\Delta)))-\mathbf{E}(V(\hat{X}(\lambda(t+\Delta)))|\mathcal{F}_{j(t)})]dt$$

and $\mathcal{G}_k=\mathcal{F}_{j(2k\Delta)}$, then we have $\mathbf{E}(\xi_{i+1}|\mathcal{G}_i)=0$. Also note that ξ_{i+1} is $\mathcal{G}_{i+1}$ measurable. Thus $S_n=\sum_{i=1}^n\xi_i$ is a martingale with respect to the filtration $\{\mathcal{G}_n\}$. Consequently, by Chow's Theorem (see Theorem 2.17 of [19]), the series in (25) is a.s. summable if $\sum_{k=1}^{\infty}\mathbf{E}(|\xi_k|^{1+\rho})<\infty$. Now note that

$$\begin{aligned}\mathbf{E}|\xi_k|^{1+\rho}=&\mathbf{E}\left(\left|\frac{1}{\Lambda_{s(2k)}}\int_{2k\Delta}^{(2k+1)\Delta}[V(\hat{X}(\lambda(t+\Delta)))-\mathbf{E}(V(\hat{X}(\lambda(t+\Delta)))|\mathcal{F}_{j(t)})]dt\right|^{1+\rho}\right)\\
\leq&\frac{2^{1+\rho}\Delta^{1+\rho}}{\Lambda_{s(2k)}^{1+\rho}}\sup_t\mathbf{E}(V(\hat{X}(t))^{1+\rho})\leq\frac{2^{1+\rho}\Delta^{1+\rho}a_2}{\Lambda_{s(2k)}^{1+\rho}},\end{aligned}$$

where the last inequality follows form Lemma 2.4. Since $\Lambda_{s(k)} \geq k\Delta$, we have that

$$\sum_{k=1}^{\infty} \frac{1}{\Lambda_{s(k)}^{1+\rho}} \leq \frac{1}{\Delta^{1+\rho}} \sum_{k=1}^{\infty} \frac{1}{k^{1+\rho}} < \infty.$$

This proves that the series in (25) is summable. The odd terms are treated in a similar manner. Thus we have proved

$$\sup_{n \geq n_0} T_2(\omega) < \infty, \quad \text{a.e. } \omega. \tag{26}$$

Now (23) is an immediate consequence of (24) and (26), which proves the lemma. □

2.2 Identification of the limit In this section we will complete the proof of Theorem 1.3 by arguing that for a.e. ω, every weak limit point of $\nu_n(\omega)$ equals ν. For this we will use the following extension of the Echeverria Criteria (see [23, 34], see also Theorem 5.7 of [3]).

THEOREM 2.1 *Let $\nu_0 \in \mathcal{P}(G)$ and $\mu_0^i \in \mathcal{M}_F(F_i)$, $i = 1, ..., N$ be such that for all $f \in C_c^2(G)$,*

$$\nu_0(\mathcal{A}f) + \sum_{i=1}^{N} \mu_0^i(D_i f) = 0. \tag{27}$$

Then $\nu_0 = \nu$.

In order to apply the above theorem to show convergence of ν_n to ν, we will consider a sequence of finite measure $\{\mu_n^i\}_{n\in\mathbb{N}}$; $i = 1, ..., N$, which, roughly speaking, correspond to the prelimit versions of the measures $\{\mu_0^i\}$ that appear in the theorem above. We now describe this sequence.

For $u \in \mathbb{R}^m,\ v \in G,\ r \in (0, \infty)$, define, for $t \in [0, 1]$,

$$\begin{aligned}
\mathbf{z}(u, v, r|t) &\equiv \mathbf{z}(t) = v + (b(v)r + \sigma(v)\sqrt{r}u)t, \\
\mathbf{x}(u, v, r|t) &\equiv \mathbf{x}(t) = \Gamma(\mathbf{z})(t), \\
\mathbf{y}(u, v, r|t) &\equiv \mathbf{y}(t) = \mathbf{x}(t) - \mathbf{z}(t).
\end{aligned}$$

Then, one can represent the trajectory $\mathbf{y}$ as

$$\mathbf{y}(t) = \sum_{i=1}^{N} d_i \int_0^t \alpha_i(s) d|\mathbf{y}|(s); \quad t \in [0, 1], \tag{28}$$

where $\alpha_i(s) \equiv \alpha_i(u, v, r|s) \in [0, 1]$ and $\alpha_i(s) > 0$ only if $\mathbf{x}(s) \in F_i$. Also, let, for $t \in [0, 1]$

$$\mathbf{\Pi^t}(u, v, r) = \mathbf{z}(1) + t(\mathbf{x}(1) - \mathbf{z}(1)),$$

$$\mathbf{L^i}(u, v, r) = \int_0^1 \alpha_i(t) d|\mathbf{y}|(t), \quad i = 1, ..., N.$$

Finally for $k \in \mathbb{N}_0$, let

$$\Pi_k^t = \mathbf{\Pi^t}(U_{k+1}, X_k, \lambda_{k+1}),\ L_k^i = \mathbf{L^i}(U_{k+1}, X_k, \lambda_{k+1}).$$

For $k \in \mathbb{N}_0$ and $i = 1, ..., N$, define a $\mathcal{M}_F(\mathbb{R}^m)$ valued random variable m_k^i by the relation

$$\langle \psi, m_k^i \rangle = \int_0^1 \mathbf{E}_{X_k}[\psi(\Pi_k^t) L_k^i] dt, \quad \psi \in BM_+(\mathbb{R}^m), \tag{29}$$

where $\mathbf{E}_X[Z]$ denotes $\mathbf{E}[Z|X]$, and $BM_+(\mathbb{R}^m)$ is the space of nonnegative bounded measurable functions on $\mathbb{R}^m$.

For $n \in \mathbb{N}$ and $i = 1, ..., N$, let μ_n^i be a $\mathcal{M}_F(\mathbb{R}^m)$ valued random variable defined as

$$\mu_n^i(A) = \frac{1}{\Lambda_n} \sum_{k=0}^{n-1} m_k^i(A); \quad A \in \mathcal{B}(\mathbb{R}^m).$$

The following lemma relates the above family of random measures with our approximation scheme. Recall the definition of the filtration $\{\mathcal{F}_k\}$ in Section 2.1.

LEMMA 2.6 *For every $f \in C_b^2(\mathbb{R}^m)$, there exists a sequence of real random variables $\{\xi_n^f\}_{n\in\mathbb{N}}$ such that*

$$\frac{1}{\Lambda_n}\sum_{k=0}^{n-1}\mathbf{E}[f(X_{k+1}) - f(X_k)|\mathcal{F}_k] = \sum_{i=1}^{N}\mu_n^i(D_i f) + \nu_n(\mathcal{A}f) + \xi_n^f, \tag{30}$$

and $\sup_n \xi_n^f(\omega) < \infty$ a.s. Furthermore if f has compact support then $\xi_n^f \to 0$ a.s. as $n \to \infty$.

PROOF. Fix $(u, v, r) \in \mathbb{R}^m \times G \times (0, \infty)$. Using the notation introduced above, we have from Taylor's theorem,

$$f(z(1)) - f(v) = \langle \nabla f(v), \eta\rangle + \frac{1}{2}\eta' D^2 f(v)\eta + R_2(v, z(1))$$

where

$$R_2(x, y) = f(y) - f(x) - \langle \nabla f(x), y - x\rangle - \frac{1}{2}(y - x)^T D^2 f(x)(y - x)$$

and

$$\eta \equiv \eta(u, v, r) = b(v)r + \sigma(v)\sqrt{r}u.$$

Define

$$r_2(x, y) = \frac{1}{2}\sup_{t\in(0,1)} ||D^2 f(x + t(y - x)) - D^2 f(x)||,$$

then we have $|R_2(x, y)| \le r_2(x, y)|x - y|^2$.

Also

$$\begin{aligned}
f(\mathbf{x}(1)) - f(\mathbf{z}(1)) &= \int_0^1 \frac{df(\mathbf{z}(1) + t(\mathbf{x}(1) - \mathbf{z}(1)))}{dt}dt \\
&= \int_0^1 \nabla f(\mathbf{z}(1) + t(\mathbf{x}(1) - \mathbf{z}(1))) \cdot (\mathbf{x}(1) - \mathbf{z}(1))dt \\
&= \sum_{i=1}^{N}\int_0^1 \nabla f(\mathbf{z}(1) + t(\mathbf{x}(1) - \mathbf{z}(1)))dt \cdot d_i \int_0^1 \alpha_i(t)d|\mathbf{y}|(t).
\end{aligned}$$

Fix a $k \in \mathbb{N}$ and let $v = X_k$, $u = U_{k+1}$ and $r = \lambda_{k+1}$. Then

$$\mathbf{E}[f(X_{k+1}) - f(X_k)|\mathcal{F}_k] = \mathbf{E}[f(\mathbf{x}(1)) - f(\mathbf{z}(1)) + f(\mathbf{z}(1)) - f(v)|\mathcal{F}_k].$$

From the definition of m_k^i in (29) and observing that $\{X_k\}$ is a Markov chain (with respect to the filtration $\{\mathcal{F}_k\}$) and U_{k+1} is independent of $\mathcal{F}_k$, it follows that

$$\mathbf{E}[f(\mathbf{x}(1)) - f(\mathbf{z}(1))|\mathcal{F}_k] = \sum_{i=1}^{N} m_k^i(D_i f),$$

Using independence of U_{k+1} from $\mathcal{F}_k$ once more,

$$\begin{aligned}
\mathbf{E}[f(\mathbf{z}(1)) - f(v)|\mathcal{F}_k] =& \lambda_{k+1}\langle \nabla f(X_k), b(X_k)\rangle + \frac{1}{2}\lambda_{k+1}\sigma(X_k)' D^2 f(X_k)\sigma(X_k) \\
&+ \frac{1}{2}\lambda_{k+1}^2 b(X_k)' D^2 f(X_k) b(X_k) + \mathbf{E}[R_2(X_k, X_k + \eta_k)|\mathcal{F}_k] \\
=& \lambda_{k+1}\mathcal{A}f(X_k) + \xi^f(k),
\end{aligned}$$

where

$$\xi^f(k) = \frac{1}{2}\lambda_{k+1}^2 b(X_k)' D^2 f(X_k) b(X_k) + \mathbf{E}[R_2(X_k, X_k + \eta_k)|\mathcal{F}_k]$$

and $\eta_k = \eta(U_{k+1}, X_k, \lambda_{k+1})$.

Thus we have

$$\begin{aligned}
&\frac{1}{\Lambda_n}\sum_{k=0}^{n-1}\mathbf{E}[f(X_{k+1}) - f(X_k)|\mathcal{F}_k] \\
&\qquad = \frac{1}{\Lambda_n}\sum_{k=0}^{n-1}\left[\sum_{i=1}^{N} m_k^i(D_i f) + \lambda_{k+1}\mathcal{A}f(X_k) + \xi^f(k)\right] \\
&\qquad = \sum_{i=1}^{N}\mu_n^i(D_i f) + \nu_n(\mathcal{A}f) + \frac{1}{\Lambda_n}\sum_{k=0}^{n-1}\xi^f(k).
\end{aligned}$$

Equality in (30) follows on taking $\xi^f_n = \frac{1}{\Lambda_n}\sum_{k=0}^{n-1}\xi^f(k)$.

We now show that $\sup_n \xi^f_n(\omega) < \infty$ a.s. Write

$$\xi^f(k) = \frac{1}{2}\lambda^2_{k+1} b(X_k)' D^2 f(X_k) b(X_k) + \mathbf{E}[R_2(X_k, X_k+\eta_k)|\mathcal{F}_k] \equiv \xi^f_1(k) + \xi^f_2(k).$$

The term $\frac{1}{\Lambda_n}\sum_{k=0}^{n-1}\xi^f_1(k)$ converges to zero because of the boundedness of b and $D^2 f$. Consider now the contribution from $\xi^f_2(k)$. Let for $p \in \mathbb{R}_+$,

$$h(p) = \frac{1}{2}\sup_{\substack{x_1,x_2\in\mathbb{R}^m\\ |x_1-x_2|\le p}} ||D^2 f(x_2) - D^2 f(x_1)||.$$

Then

$$|R_2(X_k, X_k+\eta_k)| \le h(\eta_k)|\eta_k|^2$$

and so for some $\kappa_1 \in (0,\infty)$,

$$\begin{aligned}|\xi^f_2(k)| &\le \mathbf{E}[h(\eta_k)|\eta_k|^2|\mathcal{F}_k]\\ &\le ||h||_\infty \kappa_1 \lambda_{k+1}.\end{aligned}$$

Thus $\sup_n \xi^f_n(\omega) < \infty$ a.s. This completes the first part of the lemma.

Finally if f in addition has compact support, we have $h(p)\to 0$ as $p \to 0$. Fix $\epsilon > 0$. Since b, σ are bounded, we can find for each $\theta \in (0,\infty)$, $k_\theta \in \mathbb{N}$ such that for every $k \ge k_\theta$,

$$|h(b(x_k)\lambda_{k+1} + \sigma(x_k)\sqrt{\lambda_{k+1}}U_{k+1})|1_{|U_{k+1}|\le\theta} \le \epsilon.$$

Also, for some $l_\eta \in (0,\infty)$, for all $k\in\mathbb{N}$,

$$\begin{aligned}\mathbf{E}[|\eta_k|^2 1_{|U_{k+1}|\ge\theta}|\mathcal{F}_k] &\le l_\eta(\lambda_k^{3/2} + \lambda_k \mathbf{E}[|U_1|^2 1_{|U_1|\ge\theta}]) \quad \text{a.s.},\\ \mathbf{E}[|\eta_k|^2|\mathcal{F}_k] &\le l_\eta \lambda_k \quad \text{a.s.}\end{aligned}$$

Choose $\theta_0\in(0,\infty)$ such that $\mathbf{E}[|U_1|^2 1_{|U_1|\ge\theta_0}] \le \epsilon$. Then

$$\frac{1}{\Lambda_n}\sum_{k=k_{\theta_0}}^{n-1}|\xi^f_2(k)| \le \epsilon l_\eta \frac{1}{\Lambda_n}\sum_{k=k_{\theta_0}}^{n-1}\lambda_k + ||h||_\infty l_\eta(\frac{1}{\Lambda_n}\sum_{k=k_{\theta_0}}^{n-1}\lambda_k^{3/2} + \frac{\epsilon}{\Lambda_n}\sum_{k=k_{\theta_0}}^{n-1}\lambda_k).$$

Thus,

$$\frac{1}{\Lambda_n}\sum_{k=0}^{n-1}|\xi^f_2(k)| \le \frac{1}{\Lambda_n}\sum_{k=0}^{k_{\theta_0}-1}\xi^f_2(k) + \epsilon l_\eta(1+||h||_\infty) + ||h||_\infty l_\eta \frac{1}{\Lambda_n}\sum_{k=k_{\theta_0}}^{n-1}\lambda_k^{3/2}.$$

Sending $n\to\infty$ and then $\epsilon\to 0$, we now see that $\frac{1}{\Lambda_n}\sum_{k=0}^{n-1}|\xi^f_2(k)| \to 0$ as $n\to\infty$. The result follows. □

The following lemma shows that the left side of the expression in (30) converges to 0 as $n\to\infty$.

Lemma 2.7 *For every $f \in C^2_b(G)$,*

$$\frac{1}{\Lambda_n}\sum_{k=0}^{n-1}\mathbf{E}[f(X_{k+1}) - f(X_k)|\mathcal{F}_k] \to 0 \ \textit{a.s.}, \quad \textit{as } n\to\infty.$$

Proof. We can split the sum into two terms:

$$\begin{aligned}&\frac{1}{\Lambda_n}\sum_{k=0}^{n-1}\mathbf{E}[f(X_{k+1}) - f(X_k)|\mathcal{F}_k]\\ &=\frac{1}{\Lambda_n}\sum_{k=0}^{n-1}(\mathbf{E}[f(X_{k+1})|\mathcal{F}_k] - f(X_{k+1})) + \frac{1}{\Lambda_n}\sum_{k=0}^{n-1}(f(X_{k+1}) - f(X_k))\\ &=T_1 + T_2.\end{aligned}$$

Note that,

$$|T_2| = \frac{1}{\Lambda_n}|f(X_n) - f(X_0)| \to 0,$$

as $n \to \infty$, since f is bounded and $\Lambda_n \to \infty$. Also, using Kronecker's Lemma,

$$T_1 = \frac{1}{\Lambda_n}\sum_{k=0}^{n-1}(\mathbf{E}[f(X_{k+1})|\mathcal{F}_k] - f(X_{k+1}))$$

will converge to 0 once the martingale

$$M_n^f := \sum_{k=1}^{n-1}\frac{1}{\Lambda_k}(\mathbf{E}[f(X_{k+1})|\mathcal{F}_k] - f(X_{k+1}))$$

converges a.s. Finally observing that $\mathbf{E}(f(X_{k+1})|\mathcal{F}_k)$ minimizes the L^2 distance from $f(X_{k+1})$ among $\mathcal{F}_k$ measurable square integrable random variables,

$$\begin{aligned}
\mathbf{E}\langle M^f\rangle_\infty &= \sum_{k\geq 1}(\frac{1}{\Lambda_k})^2\mathbf{E}\left(f(X_{k+1}) - \mathbf{E}(f(X_{k+1})|\mathcal{F}_k)\right)^2 \\
&\leq \sum_{k\geq 1}(\frac{1}{\Lambda_k})^2\mathbf{E}\left(f(X_{k+1}) - f(X_k)\right)^2 \\
&\leq ||Df||_\infty \sum_{k\geq 1}(\frac{1}{\Lambda_k})^2\mathbf{E}\left(X_{k+1} - X_k\right)^2 \\
&\leq \kappa_1 \sum_{k\geq 1}\frac{\lambda_{k+1}}{\Lambda_k^2} \\
&< \infty
\end{aligned}$$

for some constant κ_1, where the last inequality follows from the observation that for a positive sequence λ_k, $\sum_{k\geq 1}\lambda_{k+1}/\Lambda_k^2 < \infty$. The lemma follows. □

Next we consider the limit of the first term on the right side of (30). We can regard μ_n^i to be a finite measure on the one point compactificaion of $\mathbb{R}^m$, denoted as $\bar{\mathbb{R}}^m$. In order to show that $\{\mu_n^i\}$ is a.s. a precompact sequence in $\mathcal{M}_F(\bar{\mathbb{R}}^m)$, it suffices to show that $\mu_n^i(\mathbb{R}^m)$ is an a.s. bounded sequence of $\mathbb{R}_+$ valued random variables. This is shown in the following lemma.

Lemma 2.8 *For $i = 1, ..., N$,*

$$\sup_n \mu_n^i(\mathbb{R}^m) < \infty, \quad a.s.$$

Proof. Let $g \in C_b^2(\mathbb{R}^m)$ be as in Lemma 1.1. Then for fixed $(u, v, r) \in \mathbb{R}^m \times G \times (0, \infty)$ and with notation as introduced above Lemma 2.6,

$$g(\mathbf{x}(1)) = g(v) + \int_0^1 [\nabla g(\mathbf{x}(s)) \cdot (b(v)r + \sigma(v)\sqrt{r}u)]ds + \sum_{i=1}^N \int_0^1 d_i \cdot \nabla g(\mathbf{x}(s))\alpha_i(s)d|\mathbf{y}|(s) \tag{31}$$

Since $\alpha_i(s)$ is nonzero only when $\mathbf{x}(s) \in F_i$, and $\langle\nabla g(x), d_i\rangle \geq 1$, for all $x \in F_i$, $i \in \{1, ..., N\}$, we have

$$\begin{aligned}
\sum_{i=1}^N \mathbf{L^i}(v, u, r) &= \sum_{i=1}^N \int_0^1 \alpha_i(s)d|\mathbf{y}|(s) \\
&\leq \sum_{i=1}^N \int_0^1 d_i \cdot \nabla g(\mathbf{x}(s))\alpha_i(s)d|\mathbf{y}|(s) \\
&\leq |g(\mathbf{x}(1)) - g(v)| + ||\nabla g||_\infty |b(v)r + \sigma(v)\sqrt{r}u| \\
&\leq ||\nabla g||_\infty |\mathbf{x}(1) - v| + ||\nabla g||_\infty |b(v)r + \sigma(v)\sqrt{r}u| \\
&\leq ||\nabla g||_\infty (K+1)|b(v)r + \sigma(v)\sqrt{r}u|,
\end{aligned} \tag{32}$$

where the second inequality uses (31), and the last inequality uses the Lipschitz property of the Skorokhod map.

Let $\kappa_1 = ||\nabla g||_\infty (K+1)a_1$, then from (32) we have for $i \in \{1, ..., N\}$,

$$L_k^i \le \kappa_1 \left(\sqrt{\lambda_{k+1}} |U_{k+1}| + \lambda_{k+1} \right). \tag{33}$$

Also note that,

$$\sup_{t\in[0,1]} |x^k(t) - X_k| \le K |b(X_k)\lambda_{k+1} + \sigma(X_k)\sqrt{\lambda_{k+1}} U_{k+1}| \le K a_1 \sqrt{\lambda_{k+1}} |U_{k+1}| + K a_1 \lambda_{k+1}, \tag{34}$$

and for $t \in [0, 1]$,

$$|\Pi_k^t - X_k| \le t|x^k(1) - v| + (1-t)|z^k(1) - v| \le (K+1)a_1\lambda_{k+1} + (K+1)a_1\sqrt{\lambda_{k+1}}|U_{k+1}|, \tag{35}$$

where $x^k(t) = \mathbf{x}(U_{k+1}, X_k, \lambda_{k+1}|t)$, $z^k(t) = \mathbf{z}(U_{k+1}, X_k, \lambda_{k+1}|t)$. Combining (33)-(35) we have that

$$\mathbf{E}_{X_k}(|\Pi_k^t - x^k(s_k^i)| L_k^i) \le (2K+1)a_1\kappa_1 m \lambda_{k+1} + \varphi(\lambda_{k+1})\lambda_{k+1}, \tag{36}$$

where $\varphi : (0, \infty) \to (0, \infty)$ is a bounded function satisfying $\varphi(\alpha) \to 0$ as $\alpha \to 0$.

Next note that L_k^i is not equal to 0 only if there exists $s \in [0, 1]$ such that $\alpha_i^k(s) > 0$, i.e., $x^k(s) \in F_i$, where $\alpha_i^k(t) \equiv \alpha_i(U_{k+1}, X_k, \lambda_{k+1}|t)$. And in that case,

$$D_i g(\Pi_k^t) \ge D_i g(x^k(s)) - ||D^2 g||_\infty |\Pi_k^t - x^k(s)| \ge 1 - ||D^2 g||_\infty |\Pi_k^t - x^k(s)|.$$

Let $A_k^i = \{\omega : \text{there exists } s \in [0, 1] \text{ such that } \alpha_i^k(s) > 0\}$ and

$$s_k^i(\omega) = \begin{cases} \inf\{s \in [0,1] : \alpha_i^k(s) > 0\} & \text{if } \omega \in A_k^i, \\ 1 & \text{if } \omega \notin A_k^i. \end{cases}$$

Then, from (36),

$$\begin{aligned} \mathbf{E}_{X_k}[D_i g(\Pi_k^t) L_k^i 1_{A_k^i}] &\ge \mathbf{E}_{X_k}[L_k^i 1_{A_k^i}] - ||D^2 g||_\infty \mathbf{E}_{X_k}[|\Pi_k^t - x^k(s_k^i)| L_k^i 1_{A_k^i}] \\ &\ge \mathbf{E}_{X_k}[L_k^i] - ||D^2 g||_\infty ((2K+1)a_1\kappa_1 m \lambda_{k+1} + \varphi(\lambda_{k+1})\lambda_{k+1}) \end{aligned}$$

Thus we have

$$\begin{aligned} \langle D_i g, m_k^i \rangle &= \int_0^1 \mathbf{E}_{X_k}[D_i g(\Pi_k^t) L_k^i] dt \\ &= \int_0^1 \mathbf{E}_{X_k}[D_i g(\Pi_k^t) L_k^i 1_{A_k^i}] dt \\ &\ge \langle 1, m_k^i \rangle - ||D^2 g||_\infty \left((2K+1)a_1 m \kappa_1 \lambda_{k+1} + \varphi(\lambda_{k+1})\lambda_{k+1} \right). \end{aligned}$$

Rearranging the terms, we have

$$\langle 1, m_k^i \rangle \le \langle D_i g, m_k^i \rangle + ||D^2 g||_\infty \left((2K+1)a_1 m \kappa_1 \lambda_{k+1} + \varphi(\lambda_{k+1})\lambda_{k+1} \right).$$

Summing over k from 0 to $n-1$ and i from 1 to N, we obtain

$$\sum_{i=1}^N \langle 1, \mu_n^i \rangle \le \sum_{i=1}^N \langle D_i g, \mu_n^i \rangle + N ||D^2 g||_\infty \left((2K+1)a_1\kappa_1 m + |\varphi|_\infty \right). \tag{38}$$

Using Lemma 2.6

$$\sum_{i=1}^N \mu_n^i(D_i g) = \frac{1}{\Lambda_n} \sum_{k=0}^{n-1} \mathbf{E}[g(X_{k+1}) - g(X_k)|\mathcal{F}_k] - \nu_n(\mathcal{A}g) - \xi_n^g.$$

Since $g \in C_b^2(\mathbb{R}^m)$, the second term on the right side is bounded. Also from Lemma 2.7, the first term converges to 0 as $n \to \infty$. Finally from Lemma 2.6, the third term is bounded, a.s.

From this it follows that

$$\sup_n \sum_{i=1}^N \mu_n^i(D_i g) < \infty \quad \text{a.s.}$$

Result follows on using this observation in (38). □

The following lemma will be used to show that for a.e. ω, any limit point of $\mu_n^i(\omega)$ is supported on F_i, $i = 1, ..., N$.

LEMMA 2.9 *Fix $i \in \{1,...,N\}$. Let $\psi \in C_c^2(\mathbb{R}^m)$ be such that $\psi(x) \geq 0$ for all $x \in \mathbb{R}^m$. Suppose that there is a $\epsilon > 0$, such that $\psi(x) = 0$ if $dist(x, F_i) \leq \epsilon$. Then*

$$\int \psi(x)\mu_n^i(dx) \to 0, \ \ a.s. \ \ as \ n \to \infty.$$

PROOF. We have

$$\begin{aligned}
\langle \psi, \mu_n^i \rangle &= \frac{1}{\Lambda_n}\sum_{k=0}^{n-1}\langle \psi, m_k^i\rangle \\
&= \frac{1}{\Lambda_n}\sum_{k=0}^{n-1}\int_0^1 \mathbf{E}_{X_k}[\psi(\Pi_k^t)L_k^i]dt \\
&= \frac{1}{\Lambda_n}\sum_{k=0}^{n-1}\int_0^1 \mathbf{E}_{X_k}\int_0^1 \psi(\Pi_k^t)\alpha_i^k(s)d|y^k|_s dt \\
&\leq \frac{|\psi|_\infty}{\Lambda_n}\sum_{k=0}^{n-1}\mathbf{E}_{X_k}\int_0^1\int_0^1 1_{\{|\Pi_k^t - x^k(s)|>\epsilon\}}\alpha_i^k(s)d|y^k|_s dt,
\end{aligned} \tag{39}$$

where, recall that $\alpha_i^k(s) \equiv \alpha_i(U_{k+1}, X_k, \lambda_{k+1}|s)$ and $x^k(s) = \mathbf{x}(U_{k+1}, X_k, \lambda_{k+1}|s)$, $y^k(s) = \mathbf{y}(U_{k+1}, X_k, \lambda_{k+1}|s)$. The last inequality in the above display follows from noting that $\alpha_i^k(s) > 0$ only when $x^k(s) \in F_i$ and if for such a s, $|\Pi_k^t - x^k(t)| \leq \epsilon$,we have by our choice of ψ that $\psi(\Pi_k^t) = 0$.

Next note that

$$\{(t,s,\omega) : |\Pi_k^t - x^k(s)| > \epsilon\} \subset \{(t,s,\omega) : |x^k(1) - x^k(s)| > \epsilon\} \cup \{(t,s,\omega) : |z^k(1) - x^k(s)| > \epsilon\},$$

where recall that $z^k(t) = \mathbf{z}(U_{k+1}, X_k, \lambda_{k+1}|t)$.

Also, from the Lipschitz property of the Skorokhod map,

$$|x^k(1) - x^k(s)| \leq Ka_1\lambda_{k+1} + Ka_1\sqrt{\lambda_{k+1}}|U_{k+1}|,$$

and

$$|z^k(1) - x^k(s)| \leq |z^k(1) - X_k| + |X_k - x^k(s)| \leq (K+1)a_1\lambda_{k+1} + (K+1)a_1\sqrt{\lambda_{k+1}}|U_{k+1}|.$$

Thus

$$\{\omega : |\Pi_k^t - x^k(s)| > \epsilon \text{ for some } t, s \in [0,1]\} \subset \{\omega : |U_{k+1}(\omega)| \geq p_k\},$$

where $p_k = \frac{\epsilon/((K+1)a_1) - \lambda_{k+1}}{\sqrt{\lambda_{k+1}}}$. Using this observation in (39), we have

$$\begin{aligned}
\langle \psi, \mu_n^i \rangle &\leq \frac{|\psi|_\infty}{\Lambda_n}\sum_{k=0}^{n-1}\mathbf{E}_{X_k}\int_0^1\int_0^1 1_{\{|U|\geq p_k\}}\alpha_i^k(s)d|y^k|_s dt \\
&\leq \frac{|\psi|_\infty}{\Lambda_n}\sum_{k=0}^{n-1}\mathbf{E}_{X_k}\left(1_{\{|U|\geq p_k\}}\int_0^1 \alpha_i^k(s)d|y^k|_s\right) \\
&\leq \frac{|\psi|_\infty}{\Lambda_n}\sum_{k=0}^{n-1}\sqrt{\left(\mathbf{E}_{X_k}\left(\int_0^1 \alpha_i^k(s)d|y^k|_s\right)^2\right)\mathbf{P}(|U| \geq p_k)}.
\end{aligned}$$

From (32) it follows that for some $\kappa_1 \in (0,\infty)$, $\sup_k \mathbf{E}_{X_k}\left(\int_0^1 \alpha_i^k(s)d|y^k|_s\right)^2 \leq \kappa_1$. Also using Condition 1.6, $\mathbf{E}|U|^j < \infty$ for all $j \geq 1$. Choose k_0 large enough so that $\lambda_{k+1} \leq \frac{\epsilon}{2(K+1)a_1}$ for all $k \geq k_0$.

Fix $j > 4$, then

$$\langle \psi, \mu_n^i \rangle \leq \frac{|\psi|_\infty}{\Lambda_n}\sqrt{\kappa_1}k_0 + \frac{|\psi|_\infty}{\Lambda_n}\sum_{k=k_0}^{n-1}\sqrt{\kappa_1}(\mathbf{E}|U|^j)^{1/2}p_k^{-j/2}.$$

The result now follows on observing that for some $\kappa_2 \in (0,\infty)$, $p_k^{-j/2} \leq \kappa_2\lambda_{k+1}^{j/4}$ for all $k \geq k_0$. □

We are now ready to complete the proof of Theorem 1.3.

Proof. [Proof of Theorem 1.3] Fix $f \in C_c^2(G)$. Then such a function can be extended to a function in $C_c^2(\mathbb{R}^m)$. We denote this function once more by f. Then from Lemma 2.6,

$$\frac{1}{\Lambda_n}\sum_{k=0}^{n-1}\mathbf{E}[f(X_{k+1})-f(X_k)|\mathcal{F}_k] = \sum_{i=1}^{N}\mu_n^i(D_i f)+\nu_n(\mathcal{A}f)+\xi_n^f. \tag{40}$$

From Lemmas 2.5, 2.6, 2.7 and 2.8, there exists $\Omega_0 \in \mathcal{F}$ such that $\mathbf{P}(\Omega_0)=1$ and for every $\omega \in \Omega_0$,

- $\{\nu_n(\omega)\}_n$ is precompact in $\mathcal{P}(G)$,
- $\{\mu_n^i(\omega)\}_n$ is precompact in $\mathcal{M}_F(\bar{\mathbb{R}}^m)$, for every $i=1,...,N$,
- Left hand side of (40) converges to 0,
- $\xi_n^f(\omega)$ converges to 0.

Fix a $\omega \in \Omega_0$ and let $\nu_\infty(\omega)$, $\mu_\infty^i(\omega)$, $i=1,...,N$, be a subsequential limit of $\nu_n(\omega)$ and $\mu_n^i(\omega)$, respectively. Then from (40) and the above observations, we have (suppressing ω)

$$\nu_\infty(\mathcal{A}f)+\sum_{i=1}^{N}\mu_\infty^i(D_i f)=0.$$

To complete the proof, in view of Theorem 2.1, it suffices to argue that

$$\int_{\mathbb{R}^m} 1_{F_i^c}(x)\mu_\infty^i(\omega)(dx)=0. \tag{41}$$

By convergence of μ_n^i to μ_∞^i, we have for every ψ as in Lemma 2.9,

$$\int_{\mathbb{R}^m}\psi(x)\mu_\infty^i(\omega)(dx)=0.$$

Therefore

$$\int_{\mathbb{R}^m} 1_{F_i^{\epsilon,r}}(x)\mu_\infty^i(dx)=0 \quad \forall \epsilon, r>0,$$

where $F_i^{\epsilon,r}=\{x\in\mathbb{R}^m|\ \mathrm{dist}(x,F_i)\geq\epsilon \text{ and } |x|\leq r\}$. The equality in (41) now follows on sending $\epsilon\to 0$ and $r\to\infty$. $\square$

2.3 Proof of Theorem 1.4 Recall c from Lemma 2.1 and ϖ from (17). Fix $\zeta \in (0,\varpi c)$. We will prove the theorem with such a choice of ζ. Consider an f as in the statement of the theorem. Then there exists constant κ_1 such that $|f(x)|\leq\kappa_1 e^{\zeta|x|}$ Without loss of generality, we assume $f\geq 0$.

From Theorem 1.3, for any $L>0$, we have

$$\int(f\wedge L)d\nu_n \to \int(f\wedge L)d\nu \quad \text{a.s.}$$

In order to prove the theorem, it suffices to show that

$$\int(f\wedge L)d\nu_n\to\int f d\nu_n,\ \text{and}\ \int(f\wedge L)d\nu\to\int f d\nu,\ \text{as } L\to\infty.$$

First, consider

$$\begin{aligned}\sup_n\left[\int f d\nu_n-\int(f\wedge L)d\nu_n\right] &\leq \sup_n\int 1_{f>L}f d\nu_n\\ &\leq \sup_n\left(\nu_n^{1/p}(f>L)[\nu_n(f^q)]^{1/q}\right),\end{aligned}$$

where $p,q\in(1,\infty)$ are such that $p^{-1}+q^{-1}=1$ and the last inequality follows from Hölder's inequality. Choose $q>1$ such that $\zeta q<\varpi c$, then from Lemma 2.5 we have

$$\sup_n[\nu_n(f^q)]^{1/q}\leq\kappa_1\sup_n[\int e^{\zeta q|x|}\nu_n(dx)]^{1/q}\leq\kappa_1\sup_n\nu_n^{1/q}(V)<\infty, \quad \text{a.s.} \tag{42}$$

Using Markov's Inequality, we have

$$\nu_n^{1/p}(f>L)\leq\frac{\nu_n^{1/p}(f)}{L^{1/p}},$$

which using (42) converges to 0 as L goes to infinity. Combining the above three displays, we have

$$\sup_n \left[\int f d\nu_n - \int (f \wedge L) d\nu_n\right] \to 0, \quad \text{a.s. as } L \to \infty. \tag{43}$$

Also, from Fatou's lemma we have, for a.e. ω,

$$\begin{aligned}\int f d\nu - \int (f\wedge L) d\nu &= \int (f - f\wedge L) d\nu \\ &\leq \liminf_n \int (f - f\wedge L) d\nu_n \\ &\leq \sup_n \int (f - f\wedge L) d\nu_n.\end{aligned}$$

Using (43) the last expression converges to 0 as $L \to \infty$. The result follows.

3. Proof of Theorem 1.5

We begin with a few preliminary lemmas.

Lemma 3.1 *If $\phi \in C^2(G)$, then*

$$\Lambda_n \nu_n(\mathcal{A}\phi) = \sum_{k=1}^n \lambda_k \mathcal{A}\phi(X_{k-1}) = Z_n^{(0)} - (N_n + \sum_{i=1}^4 Z_n^{(i)} + \sum_{i=1}^4 Y_n^{(i)})$$

with

$$\begin{aligned}
Z_n^{(0)} &= \phi(X_n) - \phi(X_0), \\
N_n &= \sum_{k=1}^n \sqrt{\lambda_k}\langle \nabla\phi(X_{k-1}), \sigma(X_{k-1})U_k\rangle, \\
Z_n^{(1)} &= \frac{1}{2}\sum_{k=1}^n \lambda_k^2 b(X_{k-1})^T D^2\phi(X_{k-1}) b(X_{k-1}), \\
Z_n^{(2)} &= \sum_{k=1}^n \lambda_k^{3/2} b(X_{k-1})^T D^2\phi(X_{k-1})\sigma(X_{k-1})U_k, \\
Z_n^{(3)} &= \frac{1}{2}\sum_{k=1}^n \lambda_k[(\sigma(X_{k-1})U_k)^T D^2\phi(X_{k-1})(\sigma(X_{k-1})U_k) \\
&\quad - \mathbf{E}((\sigma(X_{k-1})U_k)^T D^2\phi(X_{k-1})(\sigma(X_{k-1})U_k)|\mathcal{F}_{k-1})], \\
Z_n^{(4)} &= \sum_{k=1}^n R_2(X_{k-1}, X_k),
\end{aligned}$$

and

$$\begin{aligned}
Y_n^{(1)} &= \sum_{k=1}^n \langle \nabla\phi(X_{k-1}), y_{k-1}\rangle, \\
Y_n^{(2)} &= \frac{1}{2}\sum_{k=1}^n y_{k-1}^T D^2\phi(X_{k-1}) y_{k-1}, \\
Y_n^{(3)} &= \sum_{k=1}^n \lambda_k b(X_{k-1})^T D^2\phi(X_{k-1}) y_{k-1}, \\
Y_n^{(4)} &= \sum_{k=1}^n \sqrt{\lambda_k} y_{k-1}^T D^2\phi(X_{k-1})\sigma(X_{k-1})U_k,
\end{aligned}$$

where $R_2(x,y) = \phi(y) - \phi(x) - \langle \nabla\phi(x), y-x\rangle - \frac{1}{2}(y-x)^T D^2\phi(x)(y-x)$, and $y_k = \mathbf{y}(U_{k+1}, X_k, \lambda_{k+1}|1)$.

Proof. Denote $\delta\phi(X_k) = \phi(X_k) - \phi(X_{k-1})$ and $\delta X_k = X_k - X_{k-1}$. We deduce from (8) that

$$\begin{aligned}
\delta\phi(X_k) =& \langle \nabla\phi(X_{k-1}), \delta X_k\rangle + \frac{1}{2}\delta X_k^T D^2\phi(X_{k-1})\delta X_k + R_2(X_{k-1}, X_k) \\
=& \langle \nabla\phi(X_{k-1}), y_{k-1}\rangle + \lambda_k \mathcal{A}\phi(X_{k-1}) + \sqrt{\lambda_k}\langle \nabla\phi(X_{k-1}), \sigma(X_{k-1})U_k\rangle \\
&+ \frac{1}{2}y_{k-1}^T D^2\phi(X_{k-1})y_{k-1} + \frac{1}{2}\lambda_k^2 b(X_{k-1})^T D^2\phi(X_{k-1})b(X_{k-1}) \\
&+ \frac{1}{2}\lambda_k[(\sigma(X_{k-1})U_k)^T D^2\phi(X_{k-1})(\sigma(X_{k-1})U_k) - \mathbf{E}((\sigma(X_{k-1})U_k)^T D^2\phi(X_{k-1})(\sigma(X_{k-1})U_k)|\mathcal{F}_{k-1})] \\
&+ \lambda_k b(X_{k-1})^T D^2\phi(X_{k-1})y_{k-1} + \lambda_k^{3/2} b(X_{k-1})^T D^2\phi(X_{k-1})\sigma(X_{k-1})U_k \\
&+ \sqrt{\lambda_k}y_{k-1}^T D^2\phi(X_{k-1})\sigma(X_{k-1})U_k + R_2(X_{k-1}, X_k).
\end{aligned}$$

The lemma follows by summing the above equality over $k = 1, ..., n$ and rearranging the terms. □

Lemma 3.2 *Let $W : G \to \mathbb{R}$ be a continuous function such that $\sup_{n\in\mathbb{N}} \nu_n(W) < \infty$, a.s. Let $\phi \in C^1(G)$, be such that $\lim_{|x|\to\infty} |\nabla\phi(x)|^2/W(x) = 0$. Then*

$$\frac{1}{\sqrt{\Lambda_n}}\sum_{k=1}^{n}\sqrt{\lambda_k}\langle \nabla\phi(X_{k-1}), \sigma(X_{k-1})U_k\rangle \xrightarrow{\mathcal{L}} \mathcal{N}\left(0, \int_G |\sigma^T\nabla\phi|^2 d\nu\right).$$

Proof. This lemma follows from Theorem 1.3 using the martingale central limit theorem, along the lines of Proposition 2 of [25]. Details are left to the reader. □

Lemma 3.3 *Under the assumptions of Theorem 1.5(b), we have,*

$$\frac{Z_n^{(4)}}{\Lambda_n^{(3/2)}} \xrightarrow{\mathbf{P}} \frac{1}{6}\int_G\int_{\mathbb{R}^m} D^3\phi(x)(\sigma(x)u)^{\otimes 3}\mu(du)\nu(dx),$$

as $n \to \infty$.

Proof. The proof is similar to that of Lemma 10 of [25] except for the treatment of reflection terms. Using the notation above Theorem 1.5 and in Lemma 3.1, we have

$$R_2(x, y) = \frac{1}{6}D^3\phi(x)(y - x)^{\otimes 3} + R_4(x, y), \tag{44}$$

with

$$|R_4(x, y)| \leq \frac{L}{6}|y - x|^4,$$

where L is the Lipschitz constant for $D^3\phi$. Hence

$$R_2(X_{k-1}, X_k) = \frac{1}{6}D^3\phi(X_{k-1})(\delta X_k)^{\otimes 3} + r_k, \tag{45}$$

with

$$|r_k| \leq \frac{L}{6}|\delta X_k|^4 \leq \kappa_1\lambda_k^2(1 + |U_k|^4), \quad k \in \mathbb{N},$$

for some $\kappa_1 \in (0, \infty)$. Since $\mathbf{E}|U_k|^4 := \mu_4 < \infty$ from Condition 1.6, we have

$$\mathbf{E}\sum_{k=1}^{n}|r_k| \leq \kappa_1(1 + \mu_4)\sum_{k=1}^{n}\lambda_k^2.$$

From the assumption $\lim_{n\to\infty}(1/\sqrt{\Lambda_n})\sum_{k=1}^{n}\lambda_k^{3/2} = \tilde{\lambda} \in (0, +\infty]$, we deduce that $\lim_{n\to\infty}\sum_{k=1}^{n}\lambda_k^{3/2} = +\infty$ and

$$\lim_{n\to\infty}\sum_{k=1}^{n}\lambda_k^2/\Lambda_n^{(3/2)} = 0. \tag{46}$$

Therefore,

$$\frac{1}{\Lambda_n^{(3/2)}}\sum_{k=1}^{n} r_k \xrightarrow{L^1} 0. \tag{47}$$

Now consider the first term on the right side of (45).

$$\begin{aligned}D^3\phi(X_{k-1})(\delta X_k)^{\otimes 3} =& D^3\phi(X_{k-1})(\lambda_k b(X_{k-1}) + \sqrt{\lambda_k}\sigma(X_{k-1})U_k + y_{k-1})^{\otimes 3}\\ =& \lambda_k^{3/2} D^3\phi(X_{k-1})(\sqrt{\lambda_k} b(X_{k-1}) + \sigma(X_{k-1})U_k)^{\otimes 3} + f_k^{(1)}(X_{k-1}, U_k)\\ =& \lambda_k^{3/2} D^3\phi(X_{k-1})(\sigma(X_{k-1})U_k)^{\otimes 3} + f_k^{(2)}(X_{k-1}, U_k) + f_k^{(1)}(X_{k-1}, U_k),\end{aligned}$$

where $f_k^{(1)}(X_{k-1}, U_k)$ and $f_k^{(2)}(X_{k-1}, U_k)$ are defined through the second and third equalities, respectively.

Next observe that

- From the assumptions, we have
$$|\lambda_k b(X_{k-1}) + \sqrt{\lambda_k}\sigma(X_{k-1})U_k| \le a_1\sqrt{\lambda_k}(|U_k| + \sqrt{\lambda_0}).$$
- From (28) and (33), we have $y_{k-1} = \sum_{i=1}^N d_i L_{k-1}^i$ and for some $\kappa_2 \in (0, \infty)$,
$$L_{k-1}^i \le \kappa_2 \sqrt{\lambda_k}(|U_k| + 1), \text{ for all } k \in \mathbb{N}$$
- The term L_{k-1}^i is non zero only if there exists $s \in [0, 1]$ such that $x_{k-1}(s) \in F_i$, where $x_{k-1}(s) = \mathbf{x}(U_k, X_{k-1}, \lambda_k|s)$. And in that case, we have from (34), the Lipschitz property of $D^3\phi$ and (11) that, for some $\kappa_3 \in (0, \infty)$,
$$|D^3_{\cdot jk}\phi(X_{k-1}) \cdot d_i| \le \kappa_3\sqrt{\lambda_k}(|U_k| + 1), \quad \forall j, k.$$

Combining these estimates, we see that $\mathbf{E}\sum_{k=1}^n |f_k^{(1)}(X_{k-1}, U_k)| \le \kappa_4 \sum_{k=1}^n \lambda_k^2$. Using (46) we now have

$$\frac{1}{\Lambda_n^{(3/2)}} \sum_{k=1}^n f_k^{(1)}(X_{k-1}, U_k) \xrightarrow{L^1} 0. \tag{48}$$

For the term $f_k^{(2)}(X_{k-1}, U_k)$, using the boundedness of $D^3\phi$, b, and σ, it can be easily checked that $\mathbf{E}|f_k^{(2)}(X_{k-1}, U_k)| \le \kappa_5\lambda_k^2$. Thus

$$\mathbf{E}\sum_{k=1}^n |f_b(X_{k-1}, U_k)| \le \kappa_5 \sum_{k=1}^n \lambda_k^2,$$

and so using (46) once again, we have

$$\frac{1}{\Lambda_n^{(3/2)}} \sum_{k=1}^n f_b(X_{k-1}, U_k) \xrightarrow{\mathbf{P}} 0. \tag{49}$$

Let $\Theta(X_{k-1}, U_k) = D^3\phi(X_{k-1})(\sigma(X_{k-1})U_k)^{\otimes 3}$. Since $\sup_k \mathbf{E}|\Theta(X_{k-1}, U_k)|^2 < \infty$ and $\lim_{n\to\infty} \Lambda_n^{(3)}/(\Lambda_n^{(3/2)})^2 = 0$, we have

$$\frac{1}{\Lambda_n^{(3/2)}} \sum_{k=1}^n \lambda_k^{3/2}[\Theta(X_{k-1}, U_k) - \mathbf{E}(\Theta(X_{k-1}, U_k)|\mathcal{F}_{k-1})] \xrightarrow{L^2} 0. \tag{50}$$

Observe that $\mathbf{E}(\Theta(X_{k-1}, U_k)|\mathcal{F}_{k-1}) = J(X_{k-1})$, where J is given by

$$J(x) := \int_{\mathbb{R}^m} D^3\phi(x)(\sigma(x)u)^{\otimes 3}\mu(du).$$

Since $\Lambda_n^{(3/2)} \to \infty$ as $n \to \infty$, we can apply Theorem 1.3 to the measure $\tilde{\nu}_n = \frac{1}{\Lambda_n^{(3/2)}}\sum_{k=1}^n \lambda_k^{3/2}\delta_{X_{k-1}}$. Since J is continuous and bounded, we have $\lim_{n\to\infty} \tilde{\nu}_n(J) = \int J d\nu$ a.s., and the lemma follows on combining this fact with (44)-(50). □

We are now ready to prove Theorem 1.5.

Proof of Theorem 1.5 The proof is similar as the proof of Theorem 9 of [25], once again the main difference is in the treatment of reflection terms. Using the notation of Lemma 3.1, we first observe that, for any sequence of positive numbers $\{a_n\}_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} a_n = \infty$, we have $Z_n^{(0)}/a_n \to 0$

in probability. This is because, from Lemma 2.4, the sequence $\{X_n\}_{n\in\mathbb{N}}$ is tight, and consequently so is $\{\phi(X_n)\}_{n\in\mathbb{N}}$ as well.

We also derive from the definitions of $Z_n^{(1)}$, $Z_n^{(2)}$ and $Z_n^{(3)}$ the inequalities

$$\mathbf{E}|Z_n^{(1)}| \le \kappa_1 \sum_{k=1}^{n} \lambda_k^2 ||D^2\phi||_\infty, \tag{51}$$

and

$$\mathbf{E}|Z_n^{(i)}|^2 \le \kappa_1 \sum_{k=1}^{n} \lambda_k^2 ||D^2\phi||_\infty^2, \quad i = 2, 3, \tag{52}$$

for some $\kappa_1 \in (0,\infty)$, for all $n \ge 1$.

(a) Now assume that $\lim_{n\to\infty}(1/\sqrt{\Lambda_n})\Lambda_n^{(3/2)} = 0$. We then have $\lim_{n\to\infty}\sum_{k=1}^n \lambda_k^2/\sqrt{\Lambda_n} = 0$, and it follows from (51) that $Z_n^{(1)}/\sqrt{\Lambda_n} \xrightarrow{L^1} 0$. We also deduce from (52), that $Z_n^{(j)}/\sqrt{\Lambda_n} \xrightarrow{L^2} 0$, for $j = 2, 3$. Consider now $Z_n^{(4)}$. Denoting the Lipschitz norm of $D^2\phi$ by L, we have

$$|R_2(X_{k-1}, X_k)| \le \frac{L}{2}|\Delta X_k|^3 \le \frac{L}{2} a_1^3 K^3 (\lambda_k + \sqrt{\lambda_k}|U_k|)^3,$$

where the second inequality follows from the Lipschitz property of the Skorokhod map (Condition 1.1). Thus, there exists $\kappa_2 \in (0,\infty)$ such that, for all $n \ge 1$,

$$\mathbf{E}|Z_n^{(4)}| \le \kappa_2 \sum_{k=1}^{n} \lambda_k^{3/2}, \tag{53}$$

and therefore $Z_n^{(4)}/\sqrt{\Lambda_n} \xrightarrow{L^1} 0$.

We now, consider $Y_n^{(j)}$, for $j = 1, 2, 3, 4$.

$$Y_n^{(1)} = \sum_{k=1}^{n} \langle \nabla\phi(X_{k-1}), y_{k-1}\rangle = \sum_{k=1}^{n} D_i\phi(X_{k-1}) L_{k-1}^i.$$

From (33), we have $|L_{k-1}^i| \le \kappa_3\sqrt{\lambda_k}(|U_k| + 1)$. Also, for any fixed i, L_{k-1}^i is not equal to 0 only if there exists $x \in F_i$, such that $||X_{k-1} - x|| \le a_1 K \lambda_k + a_1 K \sqrt{\lambda_k}|U_k|$; and in that case, using Taylor's theorem and the Lipschitz property of $D^2\phi$, there exists $\kappa_4 \in (0,\infty)$, such that,

$$|D_i\phi(X_{k-1}) - D_i\phi(x) - (X_{k-1} - x)^T D^2\phi(x) d_i| \le \kappa_4 ||X_{k-1} - x||^2.$$

Combining this with (10), we have

$$|D_i\phi(X_{k-1})| \le \kappa_4 ||X_{k-1} - x||^2.$$

Thus we have

$$\mathbf{E}|Y_n^{(1)}| \le \kappa_5 \sum_{k=1}^{n} \lambda_k^{3/2}, \tag{54}$$

for some constant κ_5. Using similar arguments as above, we obtain:

$$\mathbf{E}|Y_n^{(j)}| \le \kappa_5 \sum_{k=1}^{n} \lambda_k^{3/2}, \quad j = 2, 3, 4. \tag{55}$$

Thus we have that $Y_n^{(j)}/\sqrt{\Lambda_n} \xrightarrow{L^1} 0$, for $j = 1, 2, 3, 4$.

From Lemma 2.5, and recalling the definition of V (see (18)) we have that, for every $\zeta \in (0, c\varpi)$,

$$\sup_{n\in\mathbb{N}} \int_G e^{\zeta|x|}\nu_n(dx) < \infty, \quad \text{a.s.}$$

For such a ζ, under the assumption that $\lim_{|x|\to\infty} e^{-\zeta|x|}|\nabla\phi(x)|^2 = 0$, applying Lemma 3.2, we now have

$$\frac{N_n}{\sqrt{\Lambda_n}} \xrightarrow{\mathcal{L}} \mathcal{N}\left(0, \int_G |\sigma^T\nabla\phi|^2 d\nu\right).$$

This completes the proof of part (a).

(b) Assume now that $\lim_{n\to\infty}(1/\sqrt{\Lambda_n})\Lambda_n^{(3/2)} = \tilde{\lambda} \in (0, +\infty]$. We then have that

$$\lim_{n\to\infty} \Lambda_n^{(3/2)} = +\infty \text{ and } \lim_{n\to\infty} \sum_{k=1}^{n} \lambda_k^2/\Lambda_n^{(3/2)} = 0.$$

As before, $Z_n^{(0)}/\Lambda_n^{(3/2)} \xrightarrow{\mathbf{P}} 0$. It follows from (51) that $Z_n^{(1)}/\Lambda_n^{(3/2)} \xrightarrow{L^1} 0$, and from (52) that $Z_n^{(j)}/\Lambda_n^{(3/2)} \xrightarrow{L^2} 0$, for $j = 2, 3$.

Under the assumptions of part (b) (i.e. that $D^3\phi$ is bounded, Lipschitz and (11) holds), we have, using similar arguments as in part (a), for some $\kappa_6 \in (0, \infty)$,

$$\mathbf{E}|Y_n^{(j)}| \leq \kappa_6 \sum_{k=1}^{n} \lambda_k^2, \quad j = 1, ..., 4;\ n \geq 1. \tag{56}$$

Thus we have that $Y_n^{(j)}/\Lambda_n^{(3/2)} \xrightarrow{L^1} 0$, for $j = 1, 2, 3, 4$.

Applying Lemma 3.2 once again, we have, for ϕ satisfying $\lim_{|x|\to\infty} e^{-\zeta|x|}|\nabla\phi(x)|^2 = 0$,

$$\frac{N_n}{\sqrt{\Lambda_n}} \xrightarrow{\mathcal{L}} \mathcal{N}\left(0, \int_G |\sigma^T \nabla\phi|^2 d\nu\right). \tag{57}$$

Also from Lemma 3.3

$$\frac{Z_n^{(4)}}{\Lambda_n^{(3/2)}} \xrightarrow{\mathbf{P}} \frac{1}{6} \int_G \int_{\mathbb{R}^m} D^3\phi(x)(\sigma(x)u)^{\otimes 3}\mu(du)\nu(dx) = -\tilde{m}. \tag{58}$$

Now, if $\tilde{\lambda} < +\infty$, we have from the above observations that $Z_n^{(j)}/\sqrt{\Lambda_n} \xrightarrow{\mathbf{P}} 0$, for $j = 0, 1, 2, 3$, $Y_n^{(j)}/\sqrt{\Lambda_n} \xrightarrow{\mathbf{P}} 0$, for $j = 1, 2, 3, 4$ and

$$\frac{Z_n^{(4)}}{\sqrt{\Lambda_n}} \xrightarrow{\mathbf{P}} -\tilde{\lambda}\tilde{m}. \tag{59}$$

The statement in (12) now follows on combining this with (57).

Finally, if $\tilde{\lambda} = +\infty$, we have $Z_n^{(j)}/\Lambda_n^{(3/2)} \xrightarrow{\mathbf{P}} 0$, for $j = 0, 1, 2, 3$, $Y_n^{(j)}/\Lambda_n^{(3/2)} \xrightarrow{\mathbf{P}} 0$, for $j = 1, 2, 3, 4$ and $N_n/\Lambda_n^{(3/2)} \xrightarrow{\mathbf{P}} 0$, and (13) follows from (58). This completes the proof of Theorem 1.5. □

4. Numerical Results

4.1 Evaluation of the Euler Time Step. A key step in simulating the sequence $\{X_k\}$ in (8) is the evaluation of $\mathcal{S}(X_k, Y_{k+1} - X_k)$, where $\mathcal{S} : G \times \mathbb{R}^m \to G$ is the time-1 Skorokhod map defined in (7). In this section we describe a procedure for computing $\mathcal{S}(x, v)$ that uses well known relationships between Skorokhod problems and linear complementary problems (LCP). We restrict ourselves to a setting where $N = m$ and $G = \mathbb{R}_+^m$. We begin by recalling the basic formulation of the LCP (see [10]). For $j \in \mathbb{N}$, a $j \times j$ matrix R and a j-dimensional vector θ, the LCP for (R, θ) is to find vectors $u, v \in \mathbb{R}^j$ such that

$$\begin{cases} u \geq 0, v \geq 0; \\ v = \theta + Ru; \\ u \cdot v = 0. \end{cases}$$

It is well known (see [16] and [6]) that with $R = [d_1, ..., d_m]$, under Condition 1.1, for every $\theta \in \mathbb{R}^m$, the LCP for (R, θ) admits a unique solution $(u, v) \equiv (\mathcal{L}_m^1(R, \theta), \mathcal{L}_m^2(R, \theta))$, and furthermore $\mathcal{L}_m^2(R, \theta) = \mathcal{S}(0, \theta)$. Thus the evaluation of $\mathcal{S}(0, \theta)$ reduces to solving the above LCP for which numerous algorithms are available. In the examples considered in the current work we used a quadratic programming algorithm. Evaluation of $\mathcal{S}(x, \theta)$ for $x \neq 0$ can be carried out using a localization procedure as follows.

Fix $x \in G$ and let $J = \text{In}(x) = \{j \in \{1, ..., m\} | \langle x, e_j\rangle = 0\}$. Let $P_J = \{z \in \mathbb{R}^m | \langle z, e_j\rangle = 0,\ \forall j \in J^c\}$. Let $\pi_J : \mathbb{R}^m \to P_J$ be the orthogonal projection:

$$\pi_J(z) = z - \sum_{j \in J^c} \langle z, e_j\rangle e_j.$$

Suppose that $|J| = p$ and $J = \{i_1, ..., i_p\}$. Define a $p \times p$ matrix R_J be the relation $R_J(k,l) = (\pi_J d_{i_l})_{i_k}$, for $k,l = 1, ...p$. Let $u_J, v_J \in \mathbb{R}^p$ be the solution of LCP for $(R_J, \pi_J\theta)$, i.e., $(u_J, v_J) = (\mathcal{L}_p^1(R_J, \pi_J\theta), \mathcal{L}_p^2(R_J, \pi_J\theta))$. Once again unique solvability of LCP for $(R_J, \pi_J\theta)$ is assured from Condition 1.1. Denote $u_J = (\eta_1, ..., \eta_p)$ and define $x_1(t) = x + \theta t + t\sum_{j=1}^p \eta_j d_{i_j}$. Let

$$\tau_1 = \inf\{t \geq 0 | \mathrm{In}(x_1(t)) \neq \mathrm{In}(x)\}.$$

We define $\tau_1 = \infty$ if the above set is empty. Then $\Gamma(x+\theta i)(t) = x_1(t)$ for all $t < \tau_1$. If $\tau_1 < \infty$ set the initial point to be $x_1 = x_1(\tau_1)$ and define the trajectory $\{x_2(t)\}_{t\geq 0}$ in a similar way as $\{x_1(t)\}$ by replacing x with x_1. Set $\tau_2 = \inf\{t \geq 0 | \mathrm{In}(x_2(t)) \neq \mathrm{In}(x_1)\}$. Then

$$\Gamma(x+\theta i)(\tau_1 + t) = \Gamma(x_1 + \theta i)(t) = x_2(t), \text{ for all } t < \tau_2.$$

Define now recursively trajectory $\{x_j(t)\}$ with time points τ_j and end points $x_j(\tau_j)$, $j = 3, 4, ...$ Let j_0 be such that $\sum_{i=1}^{j_0} \tau_i < 1 \leq \sum_{i=1}^{j_0+1} \tau_i$. Then

$$\mathcal{S}(x,\theta) = \Gamma(x+\theta i)(1) = \Gamma(x_{j_0} + \theta i)(1 - \sum_{i=1}^{j_0} \tau_i).$$

Thus the evaluation of $\mathcal{S}(x,\theta)$ can be carried out by recursively solving a sequence of LCP problems.

One difficulty in implementing the above scheme is the possibility that $\sum_{i=1}^{\infty} \tau_i \leq 1$. However using regularity property of the Skorokhod map, we see that this occurs only when $\mathcal{S}(x,\theta)$ is zero. Thus in the practical implementation of the algorithm we fix a finite threshold L and carry out the above recursive procedure at most L times and set $\mathcal{S}(x,\theta) = 0$ if $\sum_{i=1}^{L} \tau_i < 1$.

4.2 Results.

4.2.1 A 3-d Example with Product Form Stationary Distribution. Let $m = 3$ and suppose that the reflection matrix is of the form $R = I + Q$, where I is the identity matrix, and Q is given as

$$Q = \begin{bmatrix} 0 & 0.1 & -0.2 \\ -0.1 & 0 & 0 \\ 0.2 & 0 & 0 \end{bmatrix}.$$

It can be checked that the spectral radius of Q is less than 1, and so Conditions 1.1 and 1.2 hold. Take the drift function $b(x) = [-1/2, -1/2, -1/2]^T$ and $\sigma(x) = I$, $x \in \mathbb{R}_+^3$. The stationary distribution ν for this example is of product form (see [21]): $\exp(1.1667) \otimes \exp(1.0938) \otimes \exp(0.8537)$, where $\exp(\mu)$ is the exponential distribution with parameter μ. In implementing the above numerical scheme, we set our initial point to $X_0 = [1,1,1]^T$, and simulate $\{X_k\}_{k=1}^n$ defined by equation (8), taking $U_k \sim \mathcal{N}(0, I)$, $\lambda_k = 1/\sqrt{k}$ and $n = 10^7$. Figure 1 shows the comparison between the exact distribution with the first-coordinate marginal of the measure ν_n.

4.2.2 Effect of Choice of $\{\lambda_k\}$. Consider a two-dimensional SRBM with covariance matrix $\sigma(x) = I$, drift vector $b(x) = [-1, 0]^T$ and reflection matrix

$$R = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

This example was considered in [11]. We consider the first moment of the x_1-coordinate. The exact value for this moment is known to be 0.5. We consider $\lambda_k = k^{-\alpha}$ and examine the influence of the choice of α on the numerical performance. The results are given in Figure 2. We find that $\alpha = 0.5$ gives the best numerical convergence.

4.2.3 An 8-d symmetric SRBM. A SRBM is said to be symmetric if its covariance matrix Γ, drift vector μ and reflection matrix R are symmetric in the following sense: $\Gamma_{ij} = \Gamma_{ji} = \rho$ for $1 \leq i < j \leq d$, $\mu_i = -1$ for $1 \leq i \leq d$ and $R_{ij} = R_{ji} = -r$ for $1 \leq i < j \leq d$, where $r \geq 0$. The positiveness of Γ implies $-1/(d-1) < \rho < 1$ and the completely-$\mathcal{S}$ condition of R implies $r(d-1) < 1$. In this case, It is known (see [11]) that, the first moment of each of the component is the same, and is given by the following formula

$$m_1 = \frac{1 - (d-2)r + (d-1)r\rho}{2(1+r)}.$$

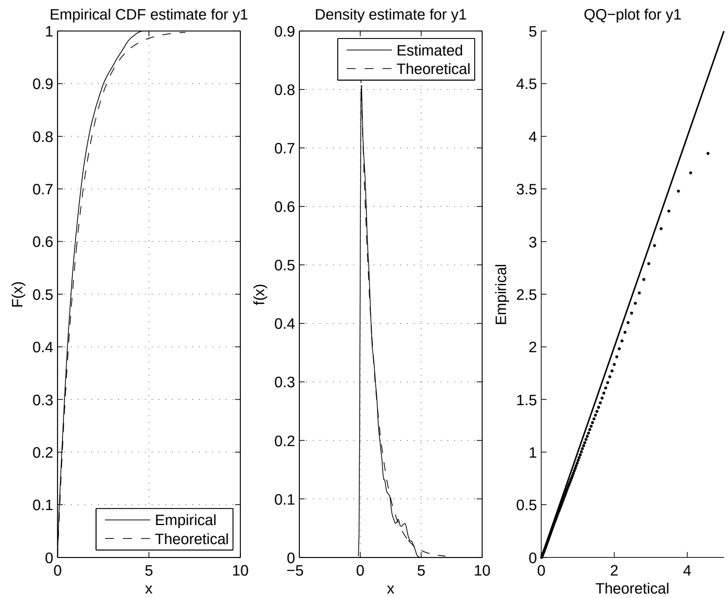

Figure 1: Comparison between the nnumerically computed distribution with exact distribution. The left figure shows the comparison between the empirical cumulative distribution function (cdf) and the exact cdf. The middle figure makes a comparison between the estimated density function and the exact density function. And the right figure is the qq-plot of the empirical quantiles versus the exact quantiles.

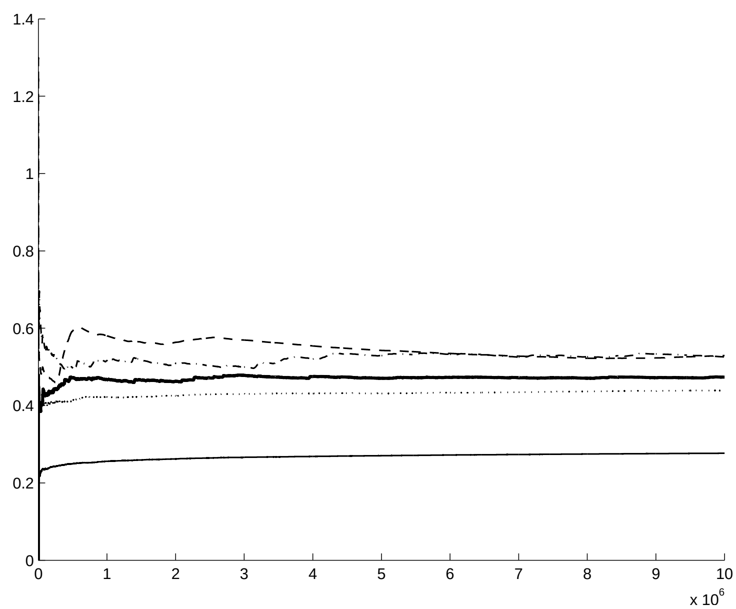

Figure 2: We consider time step sequence $\lambda_n = n^{-\alpha}$ with different choice of α and study the influence of α on numerical convergence. The thin solid line, the dotted line, the thick solid line, the dash-dot line, and the dashed line correspond to α =0.1, 0.3, 0.5, 0.7, and 0.9 respectively. The x-axis shows the value of n while the y-axis corresponds to $\int x_1 \nu_n(dx)$.

Here we take $d = 8$. Then the conditions on the data yield $-1/7 < \rho < 1$ and $0 \leq r < 1/7$. Letting ρ range through $\{-0.1, -0.05, 0, 0.2, 0.9\}$, and r take value 0.1, we obtain estimates of m_1 using algorithm in this work. We take $\lambda_k = k^{-\alpha}$, $\alpha = 0.5$ and $n = 10^7$. The results are shown in Table 1. The results show that as the correlation coefficient ρ approaches 1, the performance of the algorithm deteriorates.

Table 1: Estimates for m_1 when $d = 8$.

ρ	-0.1	-0.05	0	0.2	0.9
Estimated Val.	0.131	0.137	0.163	0.414	3.205
True Val.	0.150	0.166	0.182	0.246	0.468

Appendix A. Appendix

Lemma A.1 *Let U be a random variable with bounded support. Suppose that $\mathbf{E}U = 0$. Then there exists $\alpha \in (0, \infty)$, such that*

$$\mathbf{E}e^{\lambda U} \le e^{\alpha \lambda^2} \text{ for all } \lambda \in \mathbb{R}.$$

Proof. Without lots of generality we assume that $|U| \le 1$.

Using the convexity of the function $e^{\lambda x}$, we have

$$e^{\lambda U} \le \frac{U+1}{2} e^{\lambda} + \frac{1-U}{2} e^{-\lambda}.$$

Taking expectations in the above inequality and using Taylor's expansion, we have

$$\mathbf{E}e^{\lambda U} \le \frac{e^{\lambda} + e^{-\lambda}}{2} \le e^{\frac{\lambda^2}{2}}.$$

The lemma then follows on taking $\alpha = \frac{1}{2}$. □

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