# SECOND ORDER VARIATIONAL HEURISTICS FOR THE MONGE PROBLEM ON COMPACT MANIFOLDS

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# Variational heuristics for the Monge problem on compact manifolds*

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**Abstract**

We consider Monge's optimal transport problem posed on compact manifolds (possibly with boundary) for a lower semi-continuous cost function $c$. When all data are smooth and the given measures, positive, we restrict the total cost $\mathcal{C}$ to diffeomorphisms. If a diffeomorphism is stationary for $\mathcal{C}$, we know that it admits a potential function. If it realizes a local minimum of $\mathcal{C}$, we prove that the $c$-Hessian of its potential function must be non-negative, positive if the cost function $c$ is non degenerate. If $c$ is generating non-degenerate, we reduce the existence of a local minimizer of $\mathcal{C}$ to that of an elliptic solution of the Monge–Ampère equation expressing the measure transport; moreover, the local minimizer is unique. It is global, thus solving Monge's problem, provided $c$ is superdifferentiable with respect to one of its arguments.

## Introduction

The solution of Monge's problem [16] in optimal transportation theory, with a general cost function, has been applied to many questions in various domains tentatively listed in the survey paper [11], including in cosmology [4]. The book [20] offers a modern account on the theory (see also [5, 10, 11]).

In case data are smooth, manifolds compact, measures positive, maps one-to-one and the solution of Monge's problem unique, the question of the *smoothness* of that solution was addressed in the landmark paper [13]. In that case, restricting Monge's problem to diffeomorphisms becomes a natural *ansatz*. Doing so, the use of differential geometry and the calculus of variations enables one to bypass the general optimal transportation approach and figure out directly some basic features of the solution map. Such a variational heuristics goes back to [1] and was elaborated stepwise in [9, 19, 4, 18, 7] (see also [4]). In the present note, we take a new step in that elaboration.

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Specifically, working in the $C^\infty$ category[1], we are given a couple of compact connected diffeomorphic manifolds each equipped with a probability volume measure, $(M,\mu)$ and $(P,\varpi)$. The manifold $M$ (resp. $P$), either has no boundary or it is the closure of a domain contained in some larger manifold. We consider a domain $\Omega \subset M \times P$ such that, for each $(m_0,p_0) \in M \times P$, the subset $\Omega^P_{m_0} = \big\{p \in P, (m_0,p) \in \Omega\big\}$ is a domain of *full measure* in $(P,\varpi)$ and similarly for $\Omega^M_{p_0} = \big\{m \in M, (m,p_0) \in \Omega\big\}$ in $(M,\mu)$; no smoothness assumption bears on the boundary of $\Omega$. We denote by $\mathrm{Diff}_{\mu,\varpi}(\Omega)$ the subset of diffeomorphisms from $M$ to $P$ which push $\mu$ to $\varpi$, with graph lying in $\Omega$ (an obvious item missing in [7]). The pushing condition means for a Borel map $\phi : M \to P$ that, for any continuous function $h : P \to \mathbb{R}$, the following equality holds:

$$\int_P h \, d\varpi = \int_M (h \circ \phi) \, d\mu, \tag{1}$$

a property commonly denoted by $\phi_\# \mu = \varpi$. When it is satisfied by a diffeomorphism $\varphi : M \to P$, one may use the change of variable $p = \varphi(m)$ in the left-hand integral of (1) and infer pointwise, in any couple $(x,y)$ of source and target charts, keeping abusively the notation $y = \varphi(x)$ for the local expression of $\varphi$, the so-called *Jacobian equation*, namely:

$$\frac{d\varpi}{dy}(\varphi(x)) \left|\det\left(\frac{\partial \varphi}{\partial x}\right)(x)\right| = \frac{d\mu}{dx}(x), \tag{2}$$

where $\frac{d\mu}{dx}$ stands for the Radon–Nikodym derivative of the push-forward measure $x_\#\mu$ with respect to the Lebesgue measure $dx$ of the chart $x$, and similarly for $\frac{d\varpi}{dy}$. The existence of diffeomorphisms pushing $\mu$ to $\varpi$ is well-known [17, 3, 6] but we have to *assume* henceforth that the graph of at least one diffeomorphism of the sort actually lies in $\Omega$. Under that assumption, $\mathrm{Diff}_{\mu,\varpi}(\Omega)$ is non empty and so is $\mathrm{Diff}_{\varpi,\mu}(\widetilde{\Omega})$, setting $\widetilde{\Omega} = \big\{(p,m) \in P \times M, (m,p) \in \Omega\big\}$. We view $\mathrm{Diff}_{\mu,\varpi}(\Omega)$ and $\mathrm{Diff}_{\varpi,\mu}(\widetilde{\Omega})$ as open manifolds respectively modeled on the Fréchet manifolds $\mathrm{Diff}_\varpi(P) := \mathrm{Diff}_{\varpi,\varpi}(P \times P)$ and $\mathrm{Diff}_\mu(M)$ (see details in [7, 8]). Finally, we consider a function $c : \Omega \to \mathbb{R}$, called the *cost function*[2], together with the total cost functional

$$\phi \in \mathrm{Diff}_{\mu,\varpi}(\Omega) \to \mathcal{C}(\phi) = \int_M c(m,\phi(m)) \, d\mu$$

and its counterpart $\psi \in \mathrm{Diff}_{\varpi,\mu}(\widetilde{\Omega}) \to \widetilde{\mathcal{C}}(\psi) = \int_P c(\psi(p),p) \, d\varpi$, which satisfy the identity: $\mathcal{C}(\varphi) \equiv \widetilde{\mathcal{C}}(\varphi^{-1})$. We will occasionally require additional

[1] so, all objects are smooth and maps, smooth up to the boundary (if any), unless otherwise specified

[2] thus, locally smooth in $\Omega$

conditions on the cost function (anytime we do, it will be explicit) among the following ones:

(3a) $c$ is lower semi continuous on $M \times P$;

(3b) $c(.,p)$ is superdifferentiable on $M$ for $\varpi$-almost all $p \in P$;

(3c) $\det(\mathrm{d}_M \mathrm{d}_P c) \neq 0$ on $\Omega$;

(3d) $\forall m_0 \in M,$ the map $p \in \Omega^P_{m_0} \to -\mathrm{d}_M c(m_0, p) \in T^*_{m_0} M$ is one-to-one and so is the map $m \in \Omega^M_{p_0} \to -\mathrm{d}_P c(m, p_0) \in T^*_{p_0} P$, $\forall p_0 \in P$.

Here, dealing with a two point function, we have set $\mathrm{d}_M$ (resp. $\mathrm{d}_P$) for the exterior derivative with respect to the argument in the manifold $M$ (resp. $P$). We refer to [20, Chapter 10] (see also [14, p.598]) for an account on the notion of superdifferentiability. As it will be clear from the proof of Proposition 1.2 below, the results of this paper would hold as well with condition (3b) replaced by the symmetric one for $c(m,.)$ instead. Condition (3c) is a *non-degeneracy* condition labelled as (A2) in [13], while (3d) is a *generating* condition often called bi-twist [12, 20] (when the smoothness of the inverse maps is further assumed, it becomes condition (A1) of [13]). A typical example of cost function for which *all* the conditions (3) are fulfilled is given by $M = P$ equipped with a Riemannian metric and $c$ is (half) the squared distance (the so-called Brenier–McCann cost function) [14]; if so, $p \in \Omega^P_m$ means that $p$ is not a *cut point* of $m$ and the inverse maps determined by (3d) are given by the *exponential* map.

Assuming (3a), we consider the *restricted Monge problem*, namely the question: can we find $\varphi \in \mathrm{Diff}_{\mu,\varpi}(\Omega)$ satisfying $\mathcal{C}(\varphi) = \inf_{\phi \in \mathrm{Diff}_{\mu,\varpi}(\Omega)} \mathcal{C}(\phi)$ ? Extending the total cost functional $\mathcal{C}$ to the set $B_{\mu,\varpi}$ of Borel maps from $M$ to $P$ pushing $\mu$ to $\varpi$, *Monge's problem* itself reads: can we find $\phi \in B_{\mu,\varpi}$ realizing the $\inf_{B_{\mu,\varpi}} \mathcal{C}$ ?

The outline of the paper is as follows: in Section 1, we recall what was obtained in [7] by writing the stationary condition for the total cost $\mathcal{C}$ and we relate it to Monge's problem; in Section 2, we state the new results which can be obtained by expressing the minimum condition for $\mathcal{C}$; Section 3 contains the corresponding proofs.

## 1 Preliminary results

In [7], writing down the Euler equation of the functional $\mathcal{C}$, we obtained the following result:

**Proposition 1.1** *If $\varphi \in \mathrm{Diff}_{\mu,\varpi}(\Omega)$ is stationary for $\mathcal{C}$, so is $\varphi^{-1}$ for $\widetilde{\mathcal{C}}$, and there exists two functions $f : M \to \mathbb{R}$, $\tilde{f} : P \to \mathbb{R}$, defined up to addition of constants, such that each point of the graph of $\varphi$ is stationary for the two* *point real function:*

$$(m,p)\in\Omega\to F(m,p)=c(m,p)+f(m)+\tilde f(p).$$

Let us call the function $f$ (resp. $\tilde f$) so determined (up to constant addition), the $c$-potential of the diffeomorphism $\varphi$ (resp. $\varphi^{-1}$). In [7], we assumed condition (3d), but it is not required for the proof of Proposition 1.1, indeed solely based on the Helmholtz lemma. For the reader's convenience, let us indicate the argument (see [7] for details). We write $\delta\mathcal{C}=0$ with $\delta\mathcal{C}=\int_M \mathrm{d}_P c(m,\varphi(m))(\delta\varphi(\varphi(m))\ d\mu$, where $\delta\varphi$ stands for a variation of the transporting diffeomorphism $\varphi$ which keeps it on the manifold $\mathrm{Diff}_{\mu,\varpi}(\Omega)$, that is, a vector field of a special kind on $P$, evaluated at the image point $\varphi(m)$. Specifically, such a vector field $V$ on $P$ should be: first of all *tangential* to the boundary of $P$, if any, so that its flow send $P$ to itself (without crossing $\partial P$); moreover, its flow should preserve the volume measure $\varpi$ or, equivalently, $V$ should satisfy: $\mathrm{div}_\varpi V=0$. In other words, the tangent space to $\mathrm{Diff}_{\mu,\varpi}(\Omega)$ at $\varphi$ is spanned by the tangential vectors of the form $V\circ\varphi$ with $V\in\ker\mathrm{div}_\varpi$. Here, the symbol $\mathrm{div}_\varpi$ denotes the divergence operator defined by the identity: $\int_P h\,\mathrm{div}_\varpi V\ d\varpi\equiv\int_P \mathrm{d}h(V)\ d\varpi$ valid for each function $h:P\to\mathbb{R}$ and each vector field $V$ on $P$ (tangential, as said[3]). Recalling (1), we thus find:

$$\forall V\in\ker\mathrm{div}_\varpi,\ \int_P \mathrm{d}_P c(\varphi^{-1}(p),p)(V(p))\ d\varpi=0.$$

Arguing likewise on $\widetilde{\mathcal{C}}$, we further get:

$$\forall U\in\ker\mathrm{div}_\mu,\ \int_M \mathrm{d}_M c(m,\varphi(m))(U(m))\ d\mu=0.$$

The conclusion of Proposition 1.1 now readily follows from Helmholtz lemma, which we recall (for a proof, see [7, Appendix]):

**Lemma 1.1 (Helmholtz)** *Let $(N,\nu)$ be a measured manifold as above. A 1-form $\alpha$ on $N$ satisfies: $\int_N \alpha(Z)\ d\nu=0$ for each vector field $Z\in\ker\mathrm{div}_\nu$ (tangential to $\partial N$ if $\partial N\neq\emptyset$) if and only if $\alpha$ is exact.*

The outcome of Proposition 1.1 for the Monge problem is known; it may be described as follows:

**Proposition 1.2** *If* (3a) (3b) (3d) *hold, a diffeomorphism $\varphi\in\mathrm{Diff}_{\mu,\varpi}(\Omega)$ is stationary for $\mathcal{C}$ if and only if it solves Monge's problem; moreover, if so, the c-potential of $\varphi$ is c-convex.*

[3] otherwise, a boundary integral should occur, of course

**Proof.** The 'if' part is obvious, let us prove the 'only if' one with an argument of [14]; the meaning of the last statement of the proposition will be cleared up on the way. Letting $\varphi \in \mathrm{Diff}_{\mu,\varpi}(\Omega)$ be stationary for $\mathcal{C}$ and using Proposition 1.1, consider the function $f^c$ given by:

$$\forall p \in P,\ f^c(p) = F\big(\varphi^{-1}(p), p\big) - \inf_M \big(c(m,p) + f(m)\big)\,,$$

called the $c$-transform [20] (or supremal convolution [14]) of $f$, up to the addition of the *constant* term $F\big(\varphi^{-1}(p), p\big)$. From (3a), for each $p_0 \in P$, the infimum appearing in the right-hand side is assumed at some point $m_0 \in M$. The latter satisfies: $\forall m \in M$, $c(m,p_0)+f(m)+f^c(p_0) \geqslant c(m_0,p_0)+f(m_0)+f^c(p_0)$, or else: $\forall m \in M$, $c(m,p_0) \geqslant c(m_0,p_0) - \big(f(m)-f(m_0)\big)$, which shows that the function $m \in M \to c(m,p_0)$ is *subdifferentiable* at $m_0$. By (3b), it is thus differentiable at $m_0$ with $d_M c(m_0,p_0) = -df(m_0)$. From (3d) combined with Proposition 1.1, we get $m_0 = \varphi^{-1}(p_0)$ hence $(p_0,m_0) \in \widetilde{\Omega}$ and $f^c(p_0) = \tilde{f}(p_0)$; since $p_0$ is arbitrary, we obtain: $f^c = \tilde{f}$. From the latter and the definition of $f^c$, we infer: $\forall m \in M$, $F(m,.) \geqslant F(m,\varphi(m))$, from what we readily conclude that $f = (\tilde{f})^c$ with $(\tilde{f})^c$ given by:

$$\forall m \in M,\ (\tilde{f})^c(m) = F(m,\varphi(m)) - \inf_P \big(c(m,p) + \tilde{f}(p)\big)\,.$$

So $f = (f^c)^c$, a property of $f$ called $c$-convexity [20, 11]. Besides, for each map $\phi \in B_{\mu,\varpi}$, integrating on $M$ the inequality $F(m,\phi(m)) \geqslant F(m,\varphi(m))$ satisfied $\mu$-almost everywhere yields $\mathcal{C}(\phi) \geqslant \mathcal{C}(\varphi)$ by using (1), which shows that $\varphi$ solves, indeed, the Monge problem $\square$

***Remark* 1.1** If, in Proposition 1.2, the manifold $M$ has no boundary and we strengthen (3b) by assuming that the map $m \in M \to c(m,p)$ is differentiable for $\varpi$-almost all $p \in P$, then either $\mathcal{C}$ has no stationary point or Monge's problem is trivial. Indeed, if so, letting $\varphi \in \mathrm{Diff}_{\mu,\varpi}(\Omega)$ be stationary for $\mathcal{C}$, condition (3d) implies that, for $\varpi$-almost all $p \in P$, the equation $p = \varphi(m)$ holds at any stationary point $m$ of the function $F(.,p)$. In particular, it holds at the *extrema* of that function. By Proposition 1.1, the function $F(.,p)$ must be constant, equal to $F\big(\varphi^{-1}(p),p\big)$ which is independent of $p \in P$. We infer that $F$ is constant on $M \times P$ hence, recalling (1), that the total cost $\mathcal{C}$ itself must be constant.

## 2 Statement of new results

No one pursued the variational heuristics beyond Proposition 1.1 probably due to Proposition 1.2. But if we drop the conditions (3a) (3b) (3d), noting that the variational heuristics presented so far is incomplete because no local *minimum* condition is expressed yet for the total cost $\mathcal{C}$, becomes a timely observation. It is our aim in the present note to write down that minimum condition and to derive from it further properties of minimizing diffeomorphisms. Before stating our results, we require a notion of $c$-Hessian.

**Definition 2.1** *Let $\Phi : M \to P$ be a map whose graph lies in $\Omega$ and $h : M \to \mathbb{R}$ a function related to $\Phi$ by the equation $\mathrm{d}_M c(m, \Phi(m)) + \mathrm{d}h(m) = 0$ on $M$. The* $(c, \Phi)$-Hessian *of $h$ is the covariant symmetric 2-tensor on $M$, denoted by $\mathrm{Hess}_{c,\Phi}(f)$, intrinsically*[4] *defined, in any couple of source and target charts $(x, y)$, by:*

$$\mathrm{Hess}_{c,\Phi}(h)(x_0) := \frac{\partial^2}{\partial x^i \partial x^j} \left[ c(x, y_0) + h(x) \right] \text{ at } x = x_0,$$

*where, if $x_0 = x(m_0)$, we have set $y_0 = y(\Phi(m_0))$.*

If $\varphi \in \mathrm{Diff}_{\mu,\varpi}(\Omega)$ is stationary for the total cost $\mathcal{C}$, Proposition 1.1 shows that the couple $(\varphi, f)$, with $f$ the $c$-potential of $\varphi$, fulfills the assumption of Definition 2.1. In that case, for simplicity, we will simply speak of the $c$-Hessian of $f$ and denote it by $\mathrm{Hess}_c(f)$. We would define likewise the $c$-Hessian of the $c$-potential $\tilde{f}$ of $\varphi^{-1}$ by the local expression (sticking to the notations used in the preceding definition):

$$\mathrm{Hess}_c(\tilde{f})(y_0) := \frac{\partial^2}{\partial y^i \partial y^j} \left[ c(x_0, y) + \tilde{f}(y) \right] \text{ at } y = y_0,$$

where, if $y_0 = y(p_0), x_0 = x\left(\varphi^{-1}(p_0)\right)$. We are in position to state our first result:

**Theorem 2.1** *Let $\varphi \in \mathrm{Diff}_{\mu,\varpi}(\Omega)$ be stationary for the total cost $\mathcal{C}$. The following properties are equivalent (still setting $f$ for the $c$-potential of $\varphi$ and $\tilde{f}$ for that of $\varphi^{-1}$):*

*(i) the second variation of $\mathcal{C}$ at $\varphi$ is non negative;*

*(ii) $\forall U \in \ker \mathrm{div}_\mu, \int_M \mathrm{Hess}_c(f)(U, U)\, d\mu \geqslant 0$;*

*(iii) $\forall V \in \ker \mathrm{div}_\varpi, \int_P \mathrm{Hess}_c(\tilde{f})(V, V)\, d\varpi \geqslant 0$.*

Assuming that $\mathcal{C}$ admits a local minimum at $\varphi$, we will infer the *pointwise* non negativity of the $c$-Hessians of $f$ and $\tilde{f}$ from the conclusions $(ii)(iii)$ of Theorem 2.1, arguing by contradiction with suitable localized divergence free vector fields. The resulting statement goes as follows:

**Corollary 2.1** *If the total cost $\mathcal{C}$ admits a local minimum at $\varphi \in \mathrm{Diff}_{\mu,\varpi}(\Omega)$, the c-Hessian of the c-potential of $\varphi$ is non negative, and so is that of $\varphi^{-1}$.*

[4] the Hessian of a function at a stationary point is intrinsic [15, pp.4–5]

At this stage, let us record a related result, usually derived from the optimality of the mass transfer plan $(I \times \varphi)_{\#}\mu$ by means of Kantorovich relaxation and duality [11], according to which the Hessian of the function $F$ considered in Proposition 1.1 is non negative at each point of the graph of $\varphi$. Specifically, since $F$ is stationary on that graph, its Hessian is intrinsically defined there [15, pp.4–5], and we can quickly derive from Corollary 2.1 the

**Corollary 2.2** *If the total cost $\mathcal{C}$ admits a local minimum at $\varphi \in \mathrm{Diff}_{\mu,\varpi}(\Omega)$, the Hessian of $F$ is non negative at each point of the graph of $\varphi$.*

**Proof.** Observe that, anytime a symmetric block matrix $H$ reads $\left(\begin{smallmatrix} A & C \\ {}^tC & B \end{smallmatrix}\right)$ with matrices $A, B, C$ of equal size, $A$ and $B$ non negative and $\det(H) = 0$, the matrix $H$ itself must be non negative. Indeed, the equation $\det(H - \lambda I) = 0$ reads $\det(A - \lambda I)\det(B - \lambda I) = \det(A)\det(B)$; if a negative eigenvalue $\lambda$ satisfied it, each left hand determinant being strictly larger than the corresponding right hand one, we would reach a contradiction. Under the assumption of Corollary 2.1, at each point of the graph of $\varphi$ in $M \times P$, in any couple of charts $(x, y)$, the matrix of the Hessian of $F$ is easily seen to fulfill the above conditions. So it must be non negative $\square$

From now on, condition (3c) is assumed; let us figure out its impact on the preceding results. If (3c) holds, the implicit function theorem provides, for each $(m_0, p_0) \in \Omega$, setting $\alpha_0 = -\mathrm{d}_M c(m_0, p_0)$, neighborhoods $\mathcal{U}_0$ of $p_0$ in $P$ and $\mathcal{W}_0$ of $(m_0, \alpha_0)$ in $T^*M$, and a map $\mathrm{E}_c^{(m_0,p_0)} : \mathcal{W}_0 \to \mathcal{U}_0$ such that:

$$\alpha \equiv -\mathrm{d}_M c\left(m, \mathrm{E}_c^{(m_0,p_0)}(m, \alpha)\right).$$

Following [12], maps like $\mathrm{E}_c^{(m_0,p_0)}$ may be called local $c$-exponential maps. Observe that, for any $(m_1, p_1)$ close enough to $(m_0, p_0)$ in $\Omega$ such that, setting $\alpha_1 = -\mathrm{d}_M c(m_1, p_1)$, the point $(m_1, \alpha_1)$ lies in $\mathcal{W}_0$, the maps $\mathrm{E}_c^{(m_0,p_0)}$ and $\mathrm{E}_c^{(m_1,p_1)}$ coincide on $\mathcal{W}_0 \cap \mathcal{W}_1$. If $M$ is simply connected, one can thus extend uniquely the map $\mathrm{E}_c^{(m_0,p_0)}$ to a neighborhood $\mathcal{W}^{(m_0,\alpha_0)}$ of a *global* section of $T^*M$ passing through $\alpha_0$ at $m_0$ and call it the $(m_0, p_0)$-determination of the $c$-exponential map on $M$. With these notions at hand, Corollary 2.1 can be strengthened as follows:

**Corollary 2.3** *If condition* (3c) *holds and $\mathcal{C}$ admits a local minimum at $\varphi \in \mathrm{Diff}_{\mu,\varpi}(\Omega)$, the $c$-Hessian of the $c$-potential $f$ of $\varphi$ must be positive definite, and so must be that of $\varphi^{-1}$; in particular, the local minimum of $\mathcal{C}$ must be strict. Moreover, the Jacobian equation* (2) *satisfied by $\varphi$, written locally in terms of $f$* via *a local exponential map, reads as a Monge–Ampère equation of* elliptic *type.*

***Remark* 2.1** Under the assumption of Corollary 2.3, we can improve Corollary 2.2 by noting that, for each $(m_0, p_0) \in M \times P$ lying in the graph $\Gamma_\varphi$ of $\varphi$, the restriction of $\mathrm{Hess}(F)(m_0,p_0)$ to the subspace $T_{m_0}M\times\{0\}$ of $T_{(m_0,p_0)}\Omega$ coincides with $\mathrm{Hess}_c(f)(m_0)$, hence it admits $n$ positive eigenvalues (each repeated with its multiplicity); by Proposition 1.1, the $n$ remaining eigenvalues of $\mathrm{Hess}(F)(m_0,p_0)$ vanish. Using the compactness of $M$, we infer the existence of a neighborhood $\mathcal{N}_\varphi$ of $\Gamma_\varphi$ in $\Omega$ such that $F > F\big|_{\Gamma_\varphi}$ on $\mathcal{N}_\varphi\setminus\Gamma_\varphi$. This local result, obtained under (3c), should be compared to the global one, namely $F\big|_{\Gamma_\varphi} = \inf\limits_{M\times P} F$, derived from the conditions (3) but (3c), in the course of the proof of Proposition 1.2.

Finally, if the conditions (3c) (3d) hold, all local $c$-exponential maps coincide with the one (just denoted by $\mathrm{E}_c$) globally defined by the generating condition. If so, we can improve the preceding results and, ultimately, reduce the solution of Monge's problem to the construction of an elliptic solution of a Monge–Ampère equation. Specifically, we will prove:

**Theorem 2.2** *If the cost function satisfies* (3c) (3d)*, there exists at most one local minimizer* $\varphi\in\mathrm{Diff}_{\mu,\varpi}(\Omega)$ *of the total cost functional* $\mathcal{C}$*. If it exists, it should read* $m\mapsto\varphi(m)=\mathrm{E}_c(m,df(m))$ *for some c-potential f with* $\mathrm{Hess}_c(f)$ *positive definite on M and f elliptic solution of the Monge–Ampère equation:*

$$(4)\qquad \frac{d\varpi}{dy}(\varphi(x))\,\det\big(\mathrm{Hess}_c(f)(x)\big)=\left|\det\left(\frac{\partial^2 c}{\partial y^k\partial x^i}(x,y)\right)_{y=\varphi(x)}\right|\frac{d\mu}{dx}(x)$$

*Conversely, if f solves* (4) *with*[5]*,* $\forall m\in M$, $\varphi(m)=\mathrm{E}_c(m,df(m))\in\Omega^P_m$ *and* $\mathrm{Hess}_c(f)$ *non negative at one point, the map* $\varphi: M\to M$ *must lie in* $\mathrm{Diff}_{\mu,\varpi}(\Omega)$ *and be a local minimizer of the total cost* $\mathcal{C}$*. Furthermore, if* (3a) (3b) *hold, f is c-convex and the diffeomorphism* $\varphi$ *actually solves the Monge problem associated to* $\mathcal{C}$*.*

The next section contains successively the proofs of Theorem 2.1, of its two corollaries, and of Theorem 2.2 (except for its last part, straightforward here from Proposition 1.2).

# 3 Proofs of the new results

## 3.1 Proof of Theorem 2.1

Let $\varphi\in\mathrm{Diff}_{\mu,\varpi}(\Omega)$ be stationary for $\mathcal{C}$ and, for $t$ a small real parameter, let $t\mapsto\varphi_t\in\mathrm{Diff}_{\mu,\varpi}(\Omega)$ be an arbitrary path such that $\varphi_0=\varphi$. We can uniquely write $\varphi_t=\xi_t\circ\varphi$ with $t\mapsto\xi_t\in\mathrm{Diff}_\varpi(P)$ such that $\xi_0=$ the identity of $P$. In particular, $\dot{\xi}_0$ is a tangential vector field on $P$ lying in $\ker\mathrm{div}_\varpi$ (setting

[5] the first condition is sometimes called *stay-away* (away, from the singular locus of $c$)

as usual $\dot\xi_0(p) = \frac{\partial}{\partial t}\xi_t(p)\big|_{t=0}$). We will prove the equivalence $(i) \iff (ii)$ by establishing the equality:

$$\frac{d^2}{dt^2}\mathcal{C}(\varphi_t)\big|_{t=0} = \int_P \mathrm{Hess}_c(\tilde f)(\dot\xi_0, \dot\xi_0)\ d\varpi. \tag{5}$$

Using arbitrary source and target charts $x, y$, and the Einstein summation convention, one routinely finds: $\frac{d}{dt}\mathcal{C}(\varphi_t) = \int_M \frac{\partial c}{\partial y^i}(x,\varphi_t(x))\frac{\partial \varphi_t^i}{\partial t}(x)\ d\mu(x)$ where $d\mu(x) = \frac{d\mu}{dx}(x)dx$, then the second variation expression:

$$\frac{d^2}{dt^2}\mathcal{C}(\varphi_t) = \int_M \left\{\frac{\partial c}{\partial y^i}(x,\varphi_t(x))\frac{\partial^2 \varphi_t^i}{\partial t^2}(x) + \frac{\partial^2 c}{\partial y^i \partial y^j}(x,\varphi_t(x))\frac{\partial \varphi_t^i}{\partial t}(x)\frac{\partial \varphi_t^i}{\partial t}(x)\right\} d\mu(x).$$

Here, to spare the unfamiliar reader, we did not use a global linear connection (on $P$) to compute the second derivatives which occur in the integrand. Doing so, we must be careful, as explained in the following remark.

***Remark* 3.1** Taken separately, the local scalar terms $\frac{\partial c}{\partial y^i}(x,\varphi_t(x))\frac{\partial^2 \varphi_t^i}{\partial t^2}(x)$ and $\frac{\partial^2 c}{\partial y^i \partial y^j}(x,\varphi_t(x))\frac{\partial \varphi_t^i}{\partial t}(x)\frac{\partial \varphi_t^i}{\partial t}(x)$ are *not* invariant under a change of charts, unlike their sum, indeed equal to the global real function: $m \in M \to \frac{\partial^2}{\partial t^2}c(m,\varphi_t(m))$. Therefore these terms *cannot be integrated separately* (unless the deformation $\xi_t$ is supported in the domain of a single chart of $P$, of course). Still, in each couple of source and target charts, splitting the local expression $\frac{\partial^2}{\partial t^2}c(x,\varphi_t(x))$ of the global function into the above non invariant terms can be done in a *unique* way, by using the canonical flat connection of the $P$ chart. Anytime we will have to integrate on a manifold a global real function splitting uniquely into a sum of non invariant local terms, we will stress that the integral should not be split by putting the sum between braces, as done above. As long as they are written in the *same charts*, the addition of two such sums $\{a_1 + a_2\} + \{b_1 + b_2\}$ may, of course, be written between a single pair of braces $\{a_1 + a_2 + b_1 + b_2\}$.

Back to our proof, using $\varphi_t = \xi_t \circ \varphi$, recalling (1) and setting $t = 0$, we infer:

$$\frac{d^2}{dt^2}\mathcal{C}(\varphi_t)\big|_{t=0} = \int_P \left\{\ddot\xi_0^i \frac{\partial c}{\partial y^i}(\varphi^{-1}(y), y) + \dot\xi_0^i\dot\xi_0^j \frac{\partial^2 c}{\partial y^i \partial y^j}(\varphi^{-1}(y), y)\right\} d\varpi(y), \tag{6}$$

where $d\varpi(y) = \frac{d\varpi}{dy}(y)dy$ and $\ddot\xi_0^i = \frac{\partial^2}{\partial t^2}\xi_t^i(y)|_{t=0}$. To proceed further, we note that the integral $\int_P \tilde f\left(\xi_t(y)\right)\ d\varpi(y)$ is independent of $t$. Differentiating it twice with respect to $t$ at $t = 0$, we get:

$$0 = \int_P \left\{ \ddot{\xi}_0^i \frac{\partial \tilde{f}}{\partial y^i}(y) + \dot{\xi}_0^i \dot{\xi}_0^j \frac{\partial^2 \tilde{f}}{\partial y^i \partial y^j}(y) \right\} d\varpi(y).$$

Adding this vanishing integral to the right-hand side of (6), applying the last part of Remark 3.1 and recalling the stationary point equation:

$$\frac{\partial c}{\partial y^i}(\varphi^{-1}(y), y) + \frac{\partial \tilde{f}}{\partial y^i}(y) \equiv 0,$$

derived at once from Proposition 1.1, we obtain (5) as desired.

A similar argument would yield:

$$\frac{d^2}{dt^2} \widetilde{\mathcal{C}}(\varphi_t^{-1})\big|_{t=0} = \int_M \mathrm{Hess}_c(f)(\dot{\zeta}_0, \dot{\zeta}_0)\ d\mu,$$

with the vector field $\dot{\zeta}_0 \in \ker \mathrm{div}_\mu$ obtained by writing $\varphi_t^{-1} = \zeta_t \circ \varphi^{-1}$ for a unique path $t \mapsto \zeta_t \in \mathrm{Diff}_\mu(M)$. It would imply the other equivalence $(i) \iff (iii)$, since $\widetilde{\mathcal{C}}(\varphi^{-1}) \equiv \mathcal{C}(\varphi)$ is a local minimum of $\widetilde{\mathcal{C}}$ as well. The proof of Theorem 2.1 is complete.

### 3.2 Proof of Corollary 2.1

**Strategy** Let $\varphi \in \mathrm{Diff}_{\mu,\varpi}(\Omega)$ realize a local minimum of the total cost $\mathcal{C}$ and let $f : M \to \mathbb{R}$ denote its $c$-potential, as provided by Proposition 1.1. Arguing by contradiction, we suppose the existence of a point $m_0 \in M$ such that the quadratic form associated to the symmetric bilinear one $\mathrm{Hess}_c(f)(m_0) : T_{m_0}M \times T_{m_0}M \to \mathbb{R}$ can take *negative* values. We will contradict property $(ii)$ of Theorem 2.1 by constructing a vector field $U \in \ker \mathrm{div}_\mu$ supported near $m_0$ such that $\int_M \mathrm{Hess}_c(f)(U,U)\ d\mu < 0$. A similar argument would hold for $\varphi^{-1}$, of course.
We will proceed stepwise, choosing a good chart at $m_0$, constructing the vector field $U$ in that chart and evaluating the above integral; we set $n = \dim M$.

**Choice of a chart** We pick any chart $y$ of $P$ at $\varphi(m_0)$ but a special chart $x$ of $M$ centered at $m_0$, namely a chart which pushes the measure $d\mu$ to the canonical Lebesgue measure $dx$. The existence of such $\mu$-adapted charts, to call them so, is well-known [3, 6] and timely, here, to transform the $\mathrm{div}_\mu$ operator on $M$ into the usual div operator of $\mathbb{R}^n$ (up to sign) that is, the divergence operator associated to the measure $dx$ (simply denoted by div below). Since the orthogonal group $O(n)$ preserves the measure $dx$, we may further choose the chart $x$ such that the matrix $H_{ij}(0)$ of $\mathrm{Hess}_c(f)(m_0)$ is diagonal, with eigenvalues $\lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_n$ (each repeated with its multiplicity). Under our assumption: $\lambda_1 < 0$. Since the unimodular group $SL(n,\mathbb{R})$ preserves the measure $dx$, we may rescale the chart $x$ in order to have: $\lambda_1 \leqslant -3$ and, $\forall i \in \{2,\dots,n\},\ \lambda_i \leqslant \frac{1}{2}$. Let the chart $x$ be fixed so and let $x \mapsto H_{ij}(f)(x)$ denote the local expression of the map $m \mapsto \mathrm{Hess}_c(f)(m)$. The inequality: $\forall v \in \mathbb{R}^n,\ H_{ij}(f)(0)v^iv^j \leqslant -3(v^1)^2 + \frac{1}{2}\sum_{i=2}^n (v^i)^2$, combined with the continuity of the map $(x,v) \mapsto H_{ij}(f)(x)v^iv^j$ as $(x,v)$ varies near $x=0$ with $v$ of length 1 (say), implies the existence of a real $\varepsilon > 0$ such that:

$$(7)\ \forall (x,v) \in \mathbb{R}^n \times \mathbb{R}^n,\ \max_{1\leqslant i\leqslant n} |x^i| \leqslant \varepsilon \Rightarrow H_{ij}(f)(x)v^iv^j \leqslant -2(v^1)^2 + \sum_{i=2}^n (v^i)^2.$$

**Construction of a divergence free vector field** The vector field on $\mathbb{R}^n$ given by $w(x) = x^1\frac{\partial}{\partial x^2} - x^2\frac{\partial}{\partial x^1}$ satisfies $\mathrm{div}(w) = 0$. The flow of $w$ preserves any function $h : \mathbb{R}^n \to \mathbb{R}$ factoring through a function $H : [0,\infty)\times\mathbb{R}^{n-2} \to \mathbb{R}$ as: $h(x) = H(\sqrt{(x^1)^2+(x^2)^2}, x^3, \dots, x^n)$. Any such function $h$ thus satisfies $\mathrm{div}(hw) = 0$.

Let us fix a cut-off function $\alpha : [0,\infty) \to [0,1]$ equal to 1 on $[0,\varepsilon/2]$, vanishing on $[\varepsilon/\sqrt{2},\infty)$, decreasing in-between, and consider the function $h : \mathbb{R}^n \to \mathbb{R}$ given by:

$$h(x) = \alpha\left(\sqrt{(x^1)^2+(x^2)^2}\right)\prod_{i=3}^n \alpha\left(|x^i|\right).$$

We know that $\mathrm{div}(hw) = 0$ and that the vector field $hw$ is supported in the open box $B_\varepsilon^n = \left\{x \in \mathbb{R}^n, \max_{1\leqslant i\leqslant n}|x^i| < \varepsilon\right\}$. Since the chart $x$ is $\mu$-adapted, we may view $hw$ as the expression in that chart of a $\mathrm{div}_\mu$ free vector field $U$ in $M$ supported in the inverse image $x^{-1}(B_\varepsilon^n) \subset M$.

**Calculation of an integral** Let us consider the integral $\int_M \mathrm{Hess}_c(f)(U,U)d\mu$ which is equal to: $\int_{B_\varepsilon^n} h^2(x)H_{ij}(f)(x)w^iw^j dx$. From (7), it is bounded above by:

$$\left(2\int_0^\infty \alpha^2(\rho)d\rho\right)^{n-2}\int_{B_\varepsilon^2}\alpha^2\left(\sqrt{(x^1)^2+(x^2)^2}\right)\left(-2(x^2)^2+(x^1)^2\right)dx^1dx^2.$$

Note that the function $(x^1,x^2) \to \alpha\left(\sqrt{(x^1)^2+(x^2)^2}\right)$ vanishes outside the Euclidean ball of radius $\varepsilon$ centered at 0; using polar coordinates $(r,\theta)$ in $\mathbb{R}^2\setminus\{0\}$, we thus find that the last integral is equal to:

$$\int_0^\varepsilon \alpha^2(r)r^3dr \times \int_0^{2\pi}\left(\cos^2\theta - 2\sin^2\theta\right)d\theta \equiv -\pi\int_0^\varepsilon \alpha^2(r)r^3dr.$$

We conclude that $\int_M \mathrm{Hess}_c(f)(U,U)d\mu$ is bounded above by a *negative* real, namely by $-\pi\left(2\int_0^\infty \alpha^2(\rho)d\rho\right)^{n-2}\int_0^\varepsilon \alpha^2(r)r^3dr$, contradicting property *(ii)* of Theorem 2.1, as desired.

### 3.3 Proof of Corollary 2.3

Let again $\varphi \in \mathrm{Diff}_{\mu,\varpi}(\Omega)$ realize a local minimum of the total cost $\mathcal{C}$. Fix an arbitrary point $m_0 \in M$ and take a $\mu$-adapted chart $x$ of $M$ centered at $m_0$ and a $\varpi$-adapted chart $y$ of $P$ centered at $p_0 = \varphi(m_0)$. From (3c) and the definition of $\mathrm{E}_c^{(m_0,p_0)}$, the map $x \mapsto y = \varphi(x)$ is defined near $x = 0$ by the equation:

$$\frac{\partial f}{\partial x^i}(x) + \left(\frac{\partial c}{\partial x^i}(x,y)\right)_{y=\varphi(x)} = 0. \tag{8}$$

Differentiating the latter yields (sticking to the notation $\mathrm{Hess}_c(f) = H_{ij}(f)dx^i \otimes dx^j$):

$$H_{ij}(f)(x) = -\left(\frac{\partial^2 c}{\partial y^k \partial x^i}(x,y)\right)_{y=\varphi(x)} \frac{\partial \varphi^k}{\partial x^j}(x).$$

Taking determinants, recalling that the symmetric matrix $H_{ij}(f)(x)$ is non negative (by Corollary 2.1) and using (2), we obtain a local Monge–Ampère equation satisfied by $f$, namely:

$$\det\left(H_{ij}(f)(x)\right) = \left|\det\left(\frac{\partial^2 c}{\partial y^k \partial x^i}(x,y)\right)_{y=\varphi(x)}\right|,$$

where $y = \varphi(x)$ is given by (8). From (3c), the right-hand side of this equation nowhere vanishes; so the matrix $H_{ij}(f)(x)$ must be positive definite. So must be the $c$-Hessian of $f$ throughout the manifold $M$, since the point $m_0$ is arbitrary. Finally, as is well-known, the positive definiteness just obtained implies, indeed, the ellipticity of the Monge–Ampère equation; from (5), it also shows that $\mathcal{C}(\psi) > \mathcal{C}(\varphi)$ for any $\psi \neq \varphi$ close enough to $\varphi$ in $\mathrm{Diff}_{\mu,\varpi}(\Omega)$. A similar argument would hold for $\varphi^{-1}$.

### 3.4 Proof of Theorem 2.2

Let $\varphi \in \mathrm{Diff}_{\mu,\varpi}(\Omega)$ be a local minimizer of $\mathcal{C}$. By Proposition 1.1 and (2), it must admit a $c$-potential $f : M \to \mathbb{R}$ solving equation (4). Furthermore, from Corollary 2.3 and its proof, the $c$-Hessian of $f$ must be positive definite, so $f$ is an elliptic solution of (4). As such, it is unique up to addition of a constant, as readily shown by a maximum principle argument [2, p.97].

Conversely, let $f$ solve (4) with $\varphi(m) \equiv \mathrm{E}_c(m, df(m))$, the graph of $\varphi$ lying in $\Omega$ and $\mathrm{Hess}_c(f)$ non negative at some point $m_0 \in M$. From (4) read at $m_0$ and (3c), we see that $\mathrm{Hess}_c(f)(m_0)$ must be positive definite; the positive definiteness of $\mathrm{Hess}_c(f)$ now spreads to the whole of $M$ due to (4). At each $m \in M$, using adapted charts at $m$ and $\varphi(m)$ as above, the definition of the map $\varphi$ yields (8) from what we infer, by differentiation and combination with (4), that (2) holds; so $\varphi$ pushes $\mu$ to $\varpi$. Since the latter are volume measures, the map $\varphi : M \to P$ must be *onto*[6] as it is easy to check using (1) and, by the inverse function theorem, it must be a local diffeomorphism, thus a covering map. If $M$ is orientable, since $\mu$ and $\varpi$ have equal total mass, a degree argument shows that $\varphi$ must be 1-sheeted hence, indeed, a diffeomorphism. In the non orientable case, one can argue similarly using the orientation covers of $M$ and $P$, and infer again that $\varphi$ must be a diffeomorphism (exercise). So $\varphi \in \mathrm{Diff}_{\mu,\varpi}(\Omega)$. Finally, from the definition of $\varphi$ and the proof of Proposition 1.1, we know that $\varphi^{-1} \in \mathrm{Diff}_{\varpi,\mu}(\widetilde{\Omega})$ is stationary for $\widetilde{\mathcal{C}}$. From the positive definiteness of $\mathrm{Hess}_c(f)$ combined with the analogue of (5) for $\widetilde{\mathcal{C}}$ at $\varphi^{-1}$, we see that $\widetilde{\mathcal{C}}$ admits at $\varphi^{-1}$ a strict local minimum. Since $\mathcal{C}(\varphi) \equiv \widetilde{\mathcal{C}}(\varphi^{-1})$, so does $\mathcal{C}$ at $\varphi$.

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